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ON FOURIER TRANSFORMS

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ABSTRACT. If $f(x)$ and $g(x)$ satisfy the equations

$$g(x) = \frac{d}{dx}\int_0^\infty \frac{1}{t} f(t)k_1(xt)\,dt, \qquad f(x) = \frac{d}{dx}\int_0^\infty \frac{1}{t} g(t)k_1(xt)\,dt,$$

then we call f and g a pair of k_1-transforms, where

$$k_1 = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{K(s)}{1-s} x^{1-s}\,ds.$$

In this paper alternative sets of conditions are established for f and g to be k_1-transform provided $K(s)$ is decomposable in a special way. These conditions involve simpler functions, which replace the kernel $k_1(x)$. Results are proved for the function spaces L^2. The necessary and sufficient conditions are established for the two functions to be self-reciprocal. Conditions are given for generating pairs of transforms for a given kernel. Two examples are given at the end to illustrate the methods and the advantage of the results.

1. Introduction. Iterations of the Laplace transforms [9] are well known and are of the form,

$$g(x) = \int_0^\infty e^{-xt} f(t)\,dt, \quad f(x) = \int_0^\infty e^{-xt} h(t)\,dt, \quad g(x) = \int_0^\infty \frac{h(t)}{x+t}\,dt. \tag{1}$$

Charles Fox [3] calls this system a chain transform of order 3. He generalizes this to the chain transform of order n by the equations

$$g_{p+1}(x) = \int_0^\infty \frac{1}{t} r_p(x/t) g_p(t)\,dt, \qquad g_q(x) = \int_0^\infty \frac{1}{t}\, l_q(x/t) g_{q+1}(t)\,dt$$

and

$$g_{n+1}(x) = g_1(x),$$

where p and q run through first n integers. He points out that in order to prove this, all the kernels l_q and r_p must be known along with one of $g_i(x)$ functions, cf. [7].

We shall consider the iteration of Laplace transforms from a different point of view. Let us consider the following formal result. Let

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(2) $$g(x) = \int_0^\infty \phi(t) l(xt)\,dt \quad \text{and} \quad f(x) = \int_0^\infty \frac{1}{t}\phi\left(\frac{1}{t}\right) m(xt)\,dt.$$

If $K(s) = L(s)/M(1-s)$, then $g(x) = \int_0^\infty f(t)k(xt)\,dt$. Here $K(s)$, $L(s)$ and $M(s)$ denote the Mellin transforms of $k(x)$, $l(x)$ and $m(x)$ respectively. The system of equations (1), becomes a special case of this if we put

$$\frac{1}{x}\phi\left(\frac{1}{x}\right) = h(x), \quad m(x) = e^{-x}, \quad l(x) = \frac{1}{1+x} \quad \text{and} \quad k(x) = e^{-x}.$$

If we assume further that $L(s)L(1-s) = M(s)M(1-s)$, then it follows that $K(s)K(1-s) = 1$ and $k(x)$ is then a Fourier kernel. Thus the system of equations (2), implies that

$$g(x) = \int_0^\infty f(t)k(xt)\,dt \quad \text{and} \quad f(x) = \int_0^\infty g(t)k(xt)\,dt.$$

In other words for a pair of functions $f(x)$ and $g(x)$ to be Fourier transforms of each other with respect to the kernel $k(x)$, it is necessary that they satisfy the equations (2) for some $\phi(x)$. Thus we have an alternative way of expressing the relationship between $f(x)$ and $g(x)$. For example, let

(3) $$g(x) = \int_0^\infty e^{-\frac{1}{2}x^2t^2}\phi(t)\,dt \quad \text{and} \quad f(x) = \int_0^\infty e^{-\frac{1}{2}x^2t^2}\cdot\frac{1}{t}\phi\left(\frac{1}{t}\right)dt$$

for some $\phi(x)$. Then [4],

$$g(x) = 2\pi\int_0^\infty f(t)\cos xt\,dt \quad \text{and} \quad f(x) = 2\pi\int_0^\infty g(t)\cos xt\,dt,$$

since the Mellin transform of $\cos x$ is given by $L(s)/L(1-s)$ where $L(s)$ is the Mellin transform of $e^{-x^2/2}$. The advantage of this set of conditions for a pair of functions, is that the integrals involve simpler and elementary functions, thus replacing a somewhat unwieldy function $k(x)$. In most cases the integrals in these new conditions can simply be read from the tables. Also, numerous sets of pairs of transforms for a given kernel can be generated by varying $\phi(x)$ in equation (2). For instance putting $\phi(x) = xe^{-x^2/2}$ in (3), we get a well-known pair of cosine transforms e^{-x}, $1/(1+x^2)$ [1, p. 146, (28)].

In this paper we shall establish results dealing with these new conditions for a pair of Fourier transforms, for the function class $L^2(0,\infty)$.

2. **Preliminaries.** (i) *Notations.* We shall use the following notations:

(1) $(1/2\pi i)\int_{\frac{1}{2}}$ for $\text{l.i.m.}_{T\to\infty}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}$, where l.i.m. stands for limit in the mean.

(2) $L^p(0,\infty)$ will be the Lebesgue integrable class, $p \geq 1$.

(3) J_ν, K_ν, Y_ν are the usual Bessel functions of order ν.

(ii) *Known results.* (A) Let $f(x) \in L^2(0,\infty)$, then $F(s) = \int_0^\infty f(x)x^{s-1}\,dx$ exists in the mean square sense and $F(s) \in L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. If $F(s) \in L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$, then $f(x) = (1/2\pi i)\int_{\frac{1}{2}} F(s)x^{-s}\,ds$, and $f(x) \in L^2(0,\infty)$. We call $F(s)$ the Mellin transform of $f(x)$.

(B) **The Parseval theorem.** *Let* $f(x)$ *and* $g(x) \in L^2(0, \infty)$; *then*

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)G(1-s)\,ds,$$

where $F(s)$ *and* $G(s)$ *are the Mellin transforms of* $f(x)$ *and* $g(x)$, *respectively.*

(C) *Let* $K(s)$ *be any function satisfying* $|K(s)| = 1$, $K(s)K(1-s) = 1$ *on* $s = \frac{1}{2} + it$, *all real t. Let* $k_1(x) = (1/2\pi i)\int_{\frac{1}{2}} K(s)x^{1-s}/(1-s)\,ds$, *and* $f(x) \in L^2(0, \infty)$, *then*

$$g(x) = \frac{d}{dx}\int_0^\infty f(t)k_1(xt)\frac{dt}{t}, \quad a.e., \; x > 0$$

and $g(x) \in L^2(0, \infty)$. *Further*

$$f(x) = \frac{d}{dx}\int_0^\infty g(t)k_1(xt)\,\frac{dt}{t} \quad a.e.$$

We shall call $f(x)$ and $g(x)$ k_1-transforms of each other. These are known results, and can be obtained as special cases or simple modifications of results in Titchmarsh [8].

3. **Properties in the class** L^2. We shall now prove two theorems for transforms of L^2-class.

Theorem 1. *Let* $L(s)$ *and* $M(s)$ *be such that* (i) $L(s) \neq 0$ *and is bounded on the line* $s = \frac{1}{2} + it$, $-\infty < t < \infty$, *and* (ii) $L(s)L(1-s) = M(s)M(1-s)$. *Let there exist a function* $\phi(x) \in L^2(0, \infty)$. *Define functions* $f(x)$ *and* $g(x)$ *by the equations* (iii) $G(s) = L(s)\Phi(1-s)$, (iv) $F(s) = M(s)$, $\Phi(s)$, *where* $\Phi(s)$, $F(s)$ *and* $G(s)$ *are the Mellin transforms of* $\phi(x)$, $f(x)$ *and* $g(x)$ *respectively. Then* $f(x)$ *and* $g(x) \in L^2(0, \infty)$ *and*

$$(4) \qquad \int_0^x g(t)\,dt = \int_0^\infty \frac{1}{t}\, l_1(xt)\phi(t)\,dt, \qquad \int_0^x f(t)\,dt = \int_0^\infty \frac{1}{t^2}\, m_1(xt)\phi(t^{-1})\,dt$$

for all $x > 0$, *where*

$$(5) \qquad l_1(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}} \frac{L(s)}{1-s}\, x^{1-s}\,ds, \qquad m_1(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}} \frac{M(s)}{1-s}\, x^{1-s}\,ds.$$

Further if (v) $K(s) = L(s)/M(1-s)$, *then* $f(x)$ *and* $g(x)$ *are* k_1*-transforms of each other as in the result* (C).

Proof. Since $\phi(x) \in L^2(0, \infty)$, therefore its Mellin transform $\Phi(s) \in L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, due to the result (A). Now from the conditions (i) and (iii), we deduce that $G(s) \in L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, hence $g(x)$, whose Mellin transform is $G(s)$ must belong to $L^2(0, \infty)$, as required. Note that from (i) and (ii), we conclude that $M(s) \neq 0$ and is bounded on the line $s = \frac{1}{2} + it$, and therefore using (iv), as before, we can show that $F(s) \in L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and consequently $f(x) \in L^2(0, \infty)$. The Mellin transform of $h(t) = 1$ $(t < x)$ and $h(t) = 0$ $(t > x)$ is $H(s) = x^s/s$. On multiplying the equation in (iii) by $H(1-s)$ and integrating we obtain

$$\frac{1}{2\pi i}\int_{½} G(s)\,\frac{x^{1-s}}{1-s}\,ds = \frac{1}{2\pi i}\int_{½} L(s)\Phi(1-s)\,\frac{x^{1-s}}{1-s}\,ds.$$

Now from (i), we obtain that $(L(s)/(1-s)) = O(t^{-1})$, therefore $L(s)/(1-s) \in L^2(½ - i\infty, ½ + i\infty)$ and so from (5), $l_1(x)/x$, whose Mellin transform is $L(s)/(1-s)$, also $\in L^2(0, \infty)$. Thus using the result (B), we have from above,

$$\int_0^x g(t)\,dt = \int_0^\infty \frac{1}{t}\, l_1(xt)\phi(t)\,dt,$$

as required. Similarly starting from the equation in (iv) and multiplying it by $H(1-s)$, and using the result (B), we obtain

$$\int_0^x f(t)\,dt = \int_0^\infty \frac{1}{t^2}\, m_1(xt)\phi(t^{-1})\,dt.$$

Now replace s by $1-s$ in (iv) and divide into (iii) to eliminate $\Phi(1-s)$, to get

$$G(s) = F(1-s)L(s)/M(1-s) = F(1-s)K(s), \tag{6}$$

using (v). From (6), we obtain the following equation

$$\frac{1}{2\pi i}\int_{½} G(s)\,\frac{x^{1-s}}{1-s}\,ds = \frac{1}{2\pi i}\int_{½} F(1-s)K(s)\,\frac{x^{1-s}}{1-s}\,ds. \tag{7}$$

By using (i) and (v), we have $|K(s)/(1-s)| = O(t^{-1})$, therefore $K(s)/(1-s) \in L^2(½ - i\infty, ½ + i\infty)$ and consequently $k_1(x)/x \in L^2(0, \infty)$. And by the result (B), equation (7) yields

$$\int_0^x g(t)\,dt = \int_0^\infty \frac{1}{t}\, f(t)k_1(xt)\,dt. \tag{8}$$

Next, replace s by $1-s$ in (iii) and divide into (iv) to obtain

$$F(s) = G(1-s)M(s)/L(1-s) = G(1-s)K(s).$$

As before, this equation gives rise to the equation

$$\frac{1}{2\pi i}\int_{½} F(s)\,\frac{x^{1-s}}{1-s}\,ds = \frac{1}{2\pi i}\int_{½} G(1-s)K(s)\,\frac{x^{1-s}}{1-s}\,ds,$$

which yields, using the result (B), the equation

$$\int_0^x f(t)\,dt = \int_0^\infty \frac{1}{t}\, g(t)k_1(xt)\,dt. \tag{9}$$

Hence from (8) and (9) we conclude that $f(x)$ and $g(x)$ are k_1-transforms of each other.

Next we shall consider some special cases of Theorem 1.

A most general solution of the equations $L(s)L(1-s) = M(s)M(1-s)$ and $|L(½ + it)| = |M(½ - it)|$ is $M(s) = a(s)b(s)$ and $L(s) = a(s)b(1-s)$.

The two extreme cases are when $a(s) \equiv 1$ and $b(s) \equiv 1$.

Case 1. Let $a(s) \equiv 1$. Then $L(s) = M(1-s)$ and $K(s) = 1$, whence $k_1(x) = (2\pi i)^{-1}\int_{½} x^{1-s}/(1-s)\,ds$. Now (9) gives

$$\int_0^x f(t)\,dt = \int_0^\infty \frac{1}{t}\,k_1(xt)g(t)\,dt$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{G(1-s)}{1-s}\,x^{1-s}\,ds = \int_0^x \frac{1}{t}\,g(t^{-1})\,dt,$$

by Parseval's theorem for Mellin transforms. Hence $f(x) = x^{-1}g(x^{-1})$ a.e. Similarly (8) gives $g(x) = x^{-1}f(x^{-1})$ a.e., which are clearly consequences of one another [8, p. 218].

Case 2. Let $h(s) \equiv 1$. Then $L(s) = M(s)$ and $K(s) = L(s)/L(1-s)$. Thus

Corollary 1. *Let $l_1(x)$ and $\phi(x)$ satisfy the conditions of Theorem* 1. *Define $f(x)$ and $g(x)$, both $\in L^2(0,\infty)$, by* (4). *Then $f(x)$ and $g(x)$ are a pair of k_1-transforms, with*

$$k_1(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}} \frac{K(s)}{1-s}\,x^{1-s}\,ds \quad and \quad K(s) = \frac{L(s)}{L(1-s)}.$$

If we choose $\phi(x)$ to be such that $\phi(x) = x^{-1}\phi(x^{-1})$, the equations (4) *give us a self-reciprocal function with respect to the kernel $k_1(x)$* [8].

Next we shall prove a converse of Theorem 1.

Theorem 2. *Let $f(x)$ and $g(x)$ be k_1-transforms of each other according to the result* (C). *Let* (i) $K(s) = M(s)/L(1-s)$ *where* (ii) $L(1-s)$ *is bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$. If* (iii) $\Phi(s) = G(1-s)/L(1-s)$, *then*

$$\int_0^x f(t)\,dt = \int_0^\infty \frac{1}{t^2}\,m_1(xt)\phi(t^{-1})\,dt, \qquad \int_0^x g(t)\,dt = \int_0^\infty \frac{1}{t}\,l_1(xt)\phi(t)\,dt,$$

where $\phi(x)$ and $g(x)$ are the functions whose Mellin transforms are $\Phi(s)$ and $G(s)$, and $l_1(x)$, $m_1(x)$ are defined by (5) *above.*

Proof. Since $f(x)$ and $g(x)$ are k_1-transforms of each other therefore, $\int_0^x f(t)\,dt = \int_0^\infty t^{-1}k_1(xt)g(t)\,dt$. Define a function $h(t) = 1$ $(t < x)$ and $h(t) = 0$ $(t > x)$; its Mellin transform $H(s) = x^s/x$. Now from above $\int_0^\infty f(t)h(t)\,dt = \int_0^\infty t^{-1}k_1(xt)g(t)\,dt$. We have seen earlier that $k_1(x)/x \in L^2(0,\infty)$. Also $h(x) \in L^2(0,\infty)$. Therefore by the result (B), the last equation yields,

$$\frac{1}{2\pi i}\int_{\frac{1}{2}} F(s)\,\frac{x^{1-s}}{1-s}\,ds = \frac{1}{2\pi i}\int_{\frac{1}{2}} K(s)G(1-s)\,\frac{x^{1-s}}{1-s}\,ds,$$

where, from the rusult (C), $K(s)/(1-s)$ is the Mellin transform of $k_1(t)/t$ and $F(s)$ denotes the Mellin transform of $f(x)$. Thus

$$F(s) = \frac{M(s)G(1-s)}{L(1-s)} = \Phi(s)M(s), \quad \text{a.e.}$$

and therefore $F(s)H(1-s) = \Phi(s)M(s)H(1-s)$ which yields

$$(10) \qquad \frac{1}{2\pi i}\int_{\frac{1}{2}} F(s)\,\frac{x^{1-s}}{1-s}\,ds = \frac{1}{2\pi i}\int_{\frac{1}{2}} \Phi(s)M(s)\,\frac{x^{1-s}}{1-s}\,ds.$$

Now $K(s) = M(s)/L(1-s)$ and $K(s)\,K(1-s) = 1$, therefore $L(s)L(1-s) = M(s)M(1-s)$. From (ii) we conclude that $M(s)$ is bounded on $s = \frac{1}{2} + it$, $-\infty < t < \infty$ and so $|M(s)/(1-s)| = O(t^{-1})$. Thus $M(s)/(1-s) \in L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ which implies that $m_1(x)/x$ whose Mellin transform is $M(s)/(1-s)$ by (5), $\in L^2(0, \infty)$. Also from (iii) we conclude that $\Phi(s) \in L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, therefore $\phi(x)$, whose Mellin transform is $\Phi(s)$, $\in L^2(0, \infty)$. It is an easy matter to deduce that $\Phi(1-s)$ is the Mellin transform of $t^{-1}\phi(t^{-1})$. By the result (B), equation (10) gives

$$\int_0^\infty f(t)h(t)\,dt = \int_0^\infty t^{-1}\phi(t^{-1})t^{-1}m_1(xt)\,dt,$$

or

$$\int_0^x f(t)dt = \int_0^\infty t^{-2}\phi(t^{-1})m_1(xt)\,dt,$$

as required.

Next from (iii) we deduce that $G(s) = \Phi(1-s)L(s)$. By multiplying the above equation by $H(1-s)$, integrating along the line $\frac{1}{2} + it$, $-\infty < t < \infty$, and applying the result (B), we conclude, as before, that $\int_0^x g(t)\,dt = \int_0^\infty t^{-1}\phi(t)l_1(xt)\,dt$, and hence the theorem.

One can obtain results analogous to those proved in Theorems 1 and 2, but in the nonintegrated form, under suitable conditions. We shall see below a formal derivation of one such result, from Theorem 1.

Suppose we let $l_1(x) = \int_0^x l(t)\,dt$ and $m_1(x) = \int_0^x m(t)\,dt$. By differentiating the equations (4) and (5), of Theorem 1, we obtain formally that

$$g(x) = \int_0^\infty \phi(t)l(xt)dt, \qquad f(x) = \int_0^\infty t^{-1}\phi(t^{-1})m(xt)\,dt, \tag{11}$$

where $l(x)$ and $m(x)$ would be in some sense of the form

$$l(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}} L(s)x^{-s}\,ds \quad \text{and} \quad m(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}} M(s)x^{-s}\,ds.$$

Further if $K(s) = L(s)/M(1-s)$, then by differentiating equations (8) and (9), we obtain formally, that

$$g(x) = \int_0^\infty f(t)k(xt)\,dt \quad \text{and} \quad f(x) = \int_0^\infty g(t)k(xt)\,dt,$$

where the kernel $k(x)$ is given by $k(x) = (2\pi i)^{-1}\int_{\frac{1}{2}} K(s)x^{-s}\,ds$. That is, $f(x)$ and $g(x)$ are k-transforms of each other.

5. Examples. (i) If $k(x) = \pi J_0(2\pi x^{\frac{1}{2}})$, then $l(x) = 2\cos 2\pi x$, $m(x) = \sin \frac{1}{2}\pi x$.
(ii) If $k(x) = x^{\frac{1}{2}}J_\nu(x)$, $\nu \geq -\frac{1}{2}$, then $l(x) = x^{\nu+\frac{1}{2}}e^{\frac{1}{2}x^2} = m(x)$.

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BIBLIOGRAPHY

1. A. Erdélyi et al., *Tables of integral transforms*. Vol. 1, McGraw-Hill, New York, 1954. MR 15, 868.

2. ———, *Tables of integral transforms*. Vol. 2, McGraw-Hill, New York, 1954. MR 16, 468.

3. C. Fox, *Chain transforms*, Proc. Amer. Math. Soc. 5 (1954), 677–688. MR 16, 127.

4. A. P. Guinand, *Reciprocal convergence classes for Fourier series and integrals*, Canad. J. Math. 13 (1961), 19–36. MR 23 #A472.

5. C. Nasim, *On the summation formula of Voronoi*, Trans. Amer. Math. Soc. 163 (1972), 35–45.

6. T. L. Pearson, *Note on the Hardy-Landau summation formula*, Canad. Math. Bull. 8 (1965), 717–720. MR 33 #2616.

7. O. P. Sharma, *The H-functions as kernels in chain transforms*, Proc. Nat. Inst. Sci. India Part A 34 (1968), 320–325. MR 40 #3206.

8. E. C. Titchmarsh, *Introduction to the theory of Fourier Integrals*, Clarendon Press, Oxford, 1937.

9. D. V. Widder, *The Laplace transform*, Princeton Math. Series, vol. 6, Princeton Univ. Press, Princeton, N. J., 1941. MR 3, 232.

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