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THE FREDHOLM SPECTRUM OF THE SUM AND PRODUCT OF TWO OPERATORS

BY

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ABSTRACT. Let $C(X)$ denote the set of closed operators with dense domain on a Banach space X, and $L(X)$ the set of all bounded linear operators on X. Let $\Phi(X)$ denote the set of all Fredholm operators on X, and $\sigma_\Phi(A)$ the set of all complex numbers λ such that $(\lambda - A) \notin \Phi(X)$. In this paper we establish conditions under which $\sigma_\Phi(A+B) \subseteq \sigma_\Phi(A) + \sigma_\Phi(B)$, $\sigma_\Phi(\overline{BA}) \subseteq \sigma_\Phi(A) \cdot \sigma_\Phi(B)$, and $\sigma_\Phi(AB) \subseteq \sigma_\Phi(A)\sigma_\Phi(B)$.

In this paper we will use the operational calculus developed in [3] to establish a property of the Fredholm spectrum of the sum and product of two operators.

Definition 1. A closed operator A from a Banach space X to a Banach space Y is called a Fredholm operator if:

(1) the domain of A, $D(A)$, is dense in X.

(2) $\alpha(A) = \dim [N(A)] < \infty$.

(3) $R(A)$, the range of A, is closed in Y.

(4) $\beta(A)$, the codimension of $R(A)$ in Y, is finite.

It is shown in [1, Lemma 332] that condition (4) implies condition (3). A discussion of Fredholm operators can be found in [2].

We denote the set of Fredholm operators from X to Y by $\Phi(X)$.

Definition 2. $\lambda \in \Phi_A$ if and only if $(\lambda - A) \in \Phi(X)$.

Definition 3. $\lambda \in \sigma_\Phi(A)$ if and only if $\lambda \notin \Phi_A$.

Definition 4. A bounded operator B will be called a quasi-inverse of the closed operator A if:

(1) $R(B) \subset D(A)$ and $AB = I + K_1$, $K_1 \in \mathcal{K}(X)$.

(2) $BA = I + K_2$, $K_2 \in \mathcal{K}(X)$.

$\mathcal{K}(X)$ denotes the set of all compact operators on X.

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By [2, Theorem 2.9], Φ_A is open and is thus the union of a disjoint collection of connected open sets. Each such set, $\Phi_i(A)$, will be called a component of Φ_A.

Let $C(X)$ denote the set of closed operators on X with dense domain.

Suppose $A \in C(X)$ with Φ_A not empty, and let $\lambda \in \Phi_A$. In [3], a quasi-inverse of $\lambda - A$, $R'_\lambda(A)$, was constructed in the following way. In each $\Phi_i(A)$, a fixed point, λ_i, is chosen in a prescribed manner. There exist subspaces, X_i and Y_i such that $X = N(\lambda_i - A) \oplus X_i$, X_i is closed, and $X = Y_i \oplus R(\lambda_i - A)$, $\dim Y_i = \beta(\lambda_i - A)$.

Let F_{1i} be the projection of X onto $N(\lambda_i - A)$ along X_i, and let F_{2i} be the projection of X onto Y_i along $R(\lambda_i - A)$. F_{1i} and F_{2i} are bounded finite rank operators. $(\lambda_i - A)|_{D(A)\cap X_i}$ has a bounded inverse, A_i; $A_i: R(\lambda_i - A) \xrightarrow{\text{onto}} D(A) \cap X_i$.

Let the operator T_i be defined by: $T_i x = A_i(I - F_{2i})x$. T_i is a quasi-inverse of $(\lambda_i - A)$.

$R'_\lambda(A)$ is then defined by $R'_\lambda(A) = T_i[(\lambda - \lambda_i)T_i + I]^{-1}$ when $\lambda \in \Phi_i(A)$ and $-1/(\lambda - \lambda_i) \in \rho(T_i)$. In [3, Theorems 2 and 5, §2], $R'_\lambda(A)$ is shown to be a quasi-inverse of $(\lambda - A)$ defined and analytic for all $\lambda \in \Phi_A$ except for at most an isolated set, $\Phi^0(A)$, having no accumulation point in Φ_A.

Lemma 1.1. *Let n be a positive integer and $A \in C(X)$ such that Φ_A is not empty. Then for each $\lambda \in \{\Phi_A \backslash \Phi^0(A)\}$, there exists a subspace V_λ, dense in X and depending on λ, such that $\forall x \in V_\lambda$, $R'_\lambda(A)x \in D(A^n)$.*

Proof. Let $\lambda \in \{\Phi_A \backslash \Phi^0(A)\}$. By [2, Theorem 2.5], $D(A^n) = D[(\lambda - A)^n]$ is dense in X for all n. Therefore, $T_i^{-1}[D(A^n)] \cap R(A) = D(A^{n-1}) \cap R(A)$ is dense in $R(A)$. By $T_i^{-1}[D(A^n)]$ we mean $\{x | T_i x \in D(A^n)\}$. Let Y_i be the complement of $R(A)$ used in the construction of T_i. Since $T_i: Y_i \to 0 \in D(A^n)$ and $X = R(A) + Y_i$, we have $T_i^{-1}[D(A^n)] = \{T_i^{-1}[D(A^n)] \cap R(A)\} \oplus Y_i$ is dense in X. Therefore, $V_\lambda = [(\lambda - \lambda_i)T_i + I]\{T_i^{-1}[D(A^n)]\}$ is dense in X because $[(\lambda - \lambda_i)T_i + I]$ is invertible. Q.E.D.

We denote the set of all bounded operators on X by $L(X)$.

Lemma 1.2. *Let $A \in \Phi(X)$, $B \in L(X)$, and $K \in \mathcal{K}(X)$. Suppose $AB|_V = K|_V$ where V is a dense subspace of X. Then $B \in \mathcal{K}(X)$.*

Proof. There exists $A_0 \in L(X)$ such that $A_0 A = I - K_1$, $K_1 \in \mathcal{K}(X)$.

$$A_0 AB|_V = A_0 K|_V, \qquad (I - K_1)B|_V = K_2|_V, \qquad B|_V = (K_1 B + K_2)|_V.$$

Since B and $(K_1 B + K_2)$ are bounded, and V is dense, we have $B = K_1 B + K_2$ by continuity. Q.E.D.

Lemma 1.3. *Let $B \in L(X)$, $A \in C(X)$, $\mu \in \{\Phi_B \backslash \Phi^0(B)\}$ and $\lambda \in \{\Phi_A \backslash \Phi^0(A)\}$. Let there exist a positive integer n and a compact operator K_1, such that*

$B: D(A^n) \to D(A)$ and $ABx = BAx + K_1x$, $\forall x \in D(A^n)$. Then there exists a compact operator K, depending analytically on λ and μ, such that

$$R'_\lambda(A)R'_\mu(B) = R'_\mu(B)R'_\lambda(A) + K.$$

Proof. By Lemma 1.1 there exists a subspace V_λ, dense in X, such that $\forall x \in V_\lambda$, $R'_\lambda(A)x \in D(A^n)$. Let $x \in V_\lambda$.

$$\begin{aligned}(\lambda - A)BR'_\lambda(A)x &= [B(\lambda - A) - K_1]R'_\lambda(A)x\\ &= [B(I - K_2) + K_3]x\\ &= Bx + K_4x;\end{aligned}$$

$$(\lambda - A)R'_\lambda(A)Bx = (I - K_2)Bx = Bx - K_5x.$$

Therefore, $(\lambda - A)[BR'_\lambda(A) - R'_\lambda(A)B]x = K_6x$. Since this equality holds for all $x \in V_\lambda$, we have by Lemma 1.2, that $BR'_\lambda(A) - R'_\lambda(A)B_\lambda = K'$, and $BR'_\lambda(A) = R'_\lambda(A)B + K'$, K', $K_i \in \mathcal{K}(X)$ for $i = 1, 2, \cdots, 6$.

$$\begin{aligned}(\mu - B)[R'_\mu(B)R'_\lambda(A) - R'_\lambda(A)R'_\mu(B)] &= (I - K_7)R'_\lambda(A) - (\mu - B)R'_\lambda(A)R'_\mu(B)\\ &= R'_\lambda(A) + K_8 - R'_\lambda(A)(I - K_7) + K_9 = K_{10}.\end{aligned}$$

Therefore, $R'_\mu(B)R'_\lambda(A) - R'_\lambda(A)R'_\mu(B) = K$, and $R'_\mu(B)R'_\lambda(A) = R'_\lambda(A)R'_\mu(B) + K$, K, $K_i \in \mathcal{K}(X)$ for $i = 7, 8, 9, 10$. The analyticity of K in λ and μ follows from the analyticity of $R'_\mu(B)$ and $R'_\lambda(A)$.

Theorem 1. *Let $B \in L(X)$ and $A \in C(X)$. Suppose there exist a positive integer n and a compact operator K such that B: $D(A^n) \to D(A)$ and $BAx = ABx + Kx$ for all $x \in D(A^n)$. Then $\sigma_\Phi(A + B) \subseteq \sigma_\Phi(A) + \sigma_\Phi(B)$. If $\sigma_\Phi(A)$ is empty, we interpret $\sigma_\Phi(A) + \sigma_\Phi(B)$ to be the empty set.*

Proof. If $\sigma_\Phi(A) + \sigma_\Phi(B)$ is the entire complex plane, then the theorem is trivially ture. We therefore assume that $\sigma_\Phi(A) + \sigma_\Phi(B)$ is not the entire plane.

Let γ be a fixed point not contained in $\sigma_\Phi(A) + \sigma_\Phi(B)$. We shall show that $\gamma \in \Phi_{(A+B)}$. If $\lambda \in \sigma_\Phi(B)$, then $(\gamma - \lambda) \in \Phi_A$. Since $\sigma_\Phi(A)$ is closed and $\sigma_\Phi(B)$ is compact, there exists an open set $U \supset \sigma_\Phi(B)$ such that $B(U)$, the boundary of U, is bounded, and when $\lambda \in U$, $(\gamma - \lambda) \in \Phi_A$. Let $A_1 = \gamma - A$. $(\gamma - \lambda) \in \Phi_A$ if and only if $\lambda \in \Phi_{A_1}$. Therefore, $\sigma_\Phi(B) \subset U \subset \Phi_{A_1}$. There exists a bounded Cauchy domain D such that $\sigma_\Phi(B) \subset D \subset U$. See [4, Theorem 3.3]. Since $\Phi^0(A_1)$ does not accumulate in Φ_{A_1} and $\Phi^0(B)$ does not accumulate in Φ_B, D can be chosen so that $R'_\lambda(A_1)$ and $R'_\lambda(B)$ are analytic on $B(D)$, the boundary of D.

Define the operators S_1 and S_2 by

$$S_1 = \frac{-1}{2\pi i} \int_{+B(D)} R'_\lambda(A_1)R'_\lambda(B)\, d\lambda,$$

and

$$S_2 = \frac{-1}{2\pi i} \int_{+B(D)} R'_\lambda(B) R'_\lambda(A_1)\, d\lambda.$$

$R'_\lambda(A_1)$ is of the form $TC(\lambda)$, where $C(\lambda)$ is bounded operator valued analytic function of λ and T is a fixed bounded operator such that $T: X \to D(A_1) = D(A)$. Therefore, $S_1: X \to D(A)$.

We will now show that there exist compact operators K_1 and K_2 such that $(\gamma - B - A)S_1 = I + K_1$ and $S_2(\gamma - B - A) = (I + K_2)|_{D(A)}$.

$$\gamma - B - A = (\gamma - \lambda - A) + (\lambda - B) = -(\lambda - A_1) + (\lambda - B).$$

$$(\gamma - B - A)S_1 = \frac{-1}{2\pi i} \int_{+B(D)} -(\lambda - A_1) R'_\lambda(A_1) R'_\lambda(B)\, d\lambda$$

$$= \frac{-1}{2\pi i} \int_{+B(D)} (\lambda - B) R'_\lambda(A_1) R'_\lambda(B)\, d\lambda.$$

Since $(\lambda - A_1)R'_\lambda(A_1)$ is of the form $I + F(\lambda)$ where $F(\lambda)$ is a bounded finite rank operator depending analytically on λ and $(2\pi i)^{-1}\int_{+B(D)} R'_\lambda(B)\,d\lambda$ is of the form $I + K_3$, see [3, Theorem 13, §2], the first integral is of the form $I + K_4$. $K_3, K_4 \in \mathcal{K}(X)$.

By Lemma 1.3, there exists a compact operator $K(\lambda)$, depending analytically on λ, such that $R'_\lambda(A_1)R'_\lambda(B) = R'_\lambda(B)R'_\lambda(A_1) + K(\lambda)$. Therefore, the second integral is equal to

$$\frac{-1}{2\pi i} \int_{+B(D)} (\lambda - B) R'_\lambda(B) R'_\lambda(A_1)\, d\lambda - \frac{1}{2\pi i} \int_{+B(D)} (\lambda - B) K(\lambda)\, d\lambda.$$

The first of these two integrals equals $\int_{+B(D)} R'_\lambda(A_1)d\lambda + K_5$, $K_5 \in \mathcal{K}(X)$. As in [3, proof of Theorem 10, §2], $\int_{+B(D)} R'_\lambda(A)\,d\lambda$ is compact. Since $(2\pi i)^{-1}\int_{+B(D)} (\lambda - B)K(\lambda)\,d\lambda$ is also compact, we have that $(\gamma - B - A)S_1 = I + K_1$, $K_1 \in \mathcal{K}(X)$.

By a similar argument we have that $S_2(\gamma - B - A) = I + K_2$, $K_2 \in \mathcal{K}(X)$. Therefore, by [2, Lemma 2.4], $(\gamma - B - A) \in \Phi(X)$ and, thus, $\gamma \in \Phi_{A+B}$. Therefore, $\sigma_\Phi(A + B) \subseteq \sigma_\Phi(A) + \sigma_\Phi(B)$. This completes the proof of the theorem.

Theorem 2. *Let $A \in C(X)$ and $B \in L(X) \cap \Phi(X)$. Let $B: D(A) \to D(A)$ and let there exist a compact operator K such that $BAx = ABx + Kx$, $\forall x \in D(A)$. Then BA is preclosed and*

(1) $\sigma_\Phi(\overline{BA}) \subseteq \sigma_\Phi(A)\sigma_\Phi(B)$, *and*

(2) $\sigma_\Phi(AB) \subseteq \sigma_\Phi(A)\sigma_\Phi(B)$,

Proof. Since $(AB)|_{D(A)}$ is preclosed and K is bounded, we have that BA is preclosed.

Let $\mathbf{C}$ denote the set of all complex numbers. Since $0 \notin \sigma_\Phi(B)$ and $\sigma_\Phi(B)$

is not empty, we have that if $\sigma_\Phi(A) = \mathbf{C}$ then $\sigma_\Phi(B)\sigma_\Phi(A) = \mathbf{C}$, and the lemma is established. We therefore assume that $\sigma_\Phi(A) \neq \mathbf{C}$.

Suppose γ is a fixed point not in $\sigma_\Phi(B)\sigma_\Phi(A)$. We proceed to show that $\gamma \in \Phi(\overline{BA})$.

Since $\sigma_\Phi(A)$ is closed, $\sigma_\Phi(B)$ is compact, and $0 \notin \sigma_\Phi(B)$, there exists an open set $U \supset \sigma_\Phi(B)$ such that $0 \notin U$, $B(U)$, the boundary of U, is bounded, and $(\gamma - \mu A) \in \Phi(X)$ $\forall \mu \in U$. Let D be a bounded Cauchy domain such that $\sigma_\Phi(B) \subset D \subseteq U$. $(\gamma - \mu A) = \mu\gamma(1/\mu - (1/\gamma)A) = (\gamma/\lambda)(\lambda - (1/\gamma)A)$, where $\lambda = 1/\mu$.

Let D' be the image of D under the map $\lambda = 1/\mu$. Let $A_1 = (1/\gamma)A$. Since $\forall \mu \in \overline{D}$, $1/\mu \in \Phi_{A_1}$, and since $R'_\lambda(A_1)$ is analytic in λ throughout Φ_{A_1}, except for at most an isolated set having no accomulation point in Φ_{A_1}, we can assume that $R'_\lambda(A_1)$ is analytic in λ on $B(D')$. *Note.* $R'_\lambda(A_1)$ is of the form $TC(\lambda)$ where $C(\lambda)$ is an analytic operator valued function of λ and T is a bounded operator such that $R(T) \subseteq D(A_1) = D(A)$. Let

$$S_1 = \frac{1}{2\pi i}\int_{+B(D')} \frac{-1}{\gamma\lambda} R'_\lambda(A_1)R'_{1/\lambda}(B)\, d\lambda$$

and

$$S_2 = \frac{1}{2\pi i}\int_{+B(D')} \frac{-1}{\gamma\lambda} R'_{1/\lambda}(B)R'_\lambda(A_1)\, d\lambda.$$

Since $R(S_1) \subseteq D(A)$, $(\gamma - \overline{BA})S_1$ is defined, and $(\gamma - \overline{BA})S_1 = (\gamma - BA)S_1$.

$$\begin{aligned}\gamma - BA &= \gamma - B\gamma A_1 = \gamma(I - BA_1)\\ &= \gamma B(\lambda - A_1) - \gamma\lambda B + \gamma I = \gamma B(\lambda - A_1) + \gamma(I - \lambda B).\end{aligned}$$

Therefore,

$$\begin{aligned}(\gamma - \overline{BA})S_1 &= \frac{1}{2\pi i}\int_{+B(D')} \frac{-1}{\lambda} B(\lambda - A_1)R'_\lambda(A_1)R'_{1/\lambda}(B)\, d\lambda\\ &\quad - \frac{1}{2\pi i}\int_{+B(D')} (1/\lambda - B)R'_\lambda(A_1)R'_{1/\lambda}(B)\, d\lambda\\ &= \frac{1}{2\pi i}\int_{+B(D')} \frac{-1}{\lambda} B[I + K_1(\lambda)]R'_{1/\lambda}(B)\, d\lambda\\ &\quad - \frac{1}{2\pi i}\int_{+B(D')} (1/\lambda - B)[R'_{1/\lambda}(B)R'_\lambda(A_1) + K_2(\lambda)]\, d\lambda\\ &= \frac{1}{2\pi i}\int_{+B(D')} \frac{-1}{\lambda} BR'_{1/\lambda}(B)\, d\lambda + K_3\\ &\quad - \frac{1}{2\pi i}\int_{+B(D')} [I + K_4(\lambda)]R'_\lambda(A_1)\, d\lambda + K_5\\ &= \frac{1}{2\pi i}\int_{+B(D)} \frac{1}{\mu} BR'_\mu(B)\, d\mu + K_3 - \frac{1}{2\pi i}\int_{+B(D')} R'_\lambda(A_1)\, d\lambda + K_6.\end{aligned}$$

Since $0 \notin D$, we have

$$\frac{1}{2\pi i}\int_{+B(D)}\frac{1}{\mu}BR'_{\mu}(B)\,d\mu = B\frac{1}{2\pi i}\int_{+B(D)}\frac{1}{\mu}R'_{\mu}(B)\,d\mu = I + K_7$$

by [3, Theorems 14, 9 and 13, §2].

Since $R'_{\lambda}(A_1)$ is analytic in D' except for at most a finite number of points, we have that $(2\pi i)^{-1}\int_{+B(D')}R'_{\lambda}(A_1)\,d\lambda = K_8$ by [3, Lemma 7.4, §2]. Therefore, $(\gamma - \overline{BA})S_1 = I + K'$, K', K_i, $K_i(\lambda) \in \mathcal{K}(X)$ for $i = 1, 2, \cdots, 8$.

Claim. $D(\overline{BA}) \subseteq D(AB)$ and $\overline{BA}x = ABx + Kx$ $\forall x \in D(\overline{BA})$.

Proof. Let $x \in D(\overline{BA})$. Then there exists a sequence, $\{x_n\}_{n=1}^{\infty}$, such that $x_n \in D(A)$, $x_n \to x$ as $n \to \infty$, and $BAx_n \to \overline{BA}x$ as $n \to \infty$. Therefore $\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} BAx_n - \lim_{n\to\infty} Kx_n = \overline{BA}x - Kx$. Therefore, $Bx \in D(A)$ and $ABx = \lim_{n\to\infty} ABx_n = \overline{BA}x - Kx$.

$$\gamma - \overline{BA} = \overline{\gamma B(\lambda - A_1)} + \gamma(I - \lambda B) = \gamma(\lambda - A_1)B + \gamma(I - \lambda B) + K_9.$$

$$\begin{aligned}
S_2(\gamma - \overline{BA}) &= \frac{1}{2\pi i}\int_{+B(D')}\frac{-1}{\gamma\lambda}R'_{1/\lambda}(B)R'_{\lambda}(A_1)\,d\lambda[\gamma(\lambda - A_1)B + \gamma(I - \lambda B) + K_9] \\
&= \frac{1}{2\pi i}\int_{+B(D')}\frac{-1}{\lambda}R'_{1/\lambda}(B)[I + K_{10}(\lambda)]B\,d\lambda \\
&\quad + \frac{1}{2\pi i}\int_{+B(D')}\frac{-1}{\gamma\lambda}[R'_{\lambda}(A_1)R'_{1/\lambda}(B) - K_2(\lambda)][\gamma(I - \lambda B) + K_9]\,d\lambda \\
&= \left[\frac{1}{2\pi i}\int_{+B(D)}\frac{1}{\mu}R'_{\mu}(B)\,d\mu\right]B + K_{11} \\
&\quad + \frac{1}{2\pi i}\int_{+B(D')}(-1)R'_{\lambda}(A_1)R'_{1/\lambda}(B)(1/\lambda - B)\,d\lambda + K_{12} \\
&= I + K_{13} + \frac{1}{2\pi i}\int_{+B(D')}(-1)R'_{\lambda}(A_1)[I + K_{14}(\lambda)]\,d\lambda \\
&= I + K_{15},
\end{aligned}$$

K_i, $K_i(\lambda) \in \mathcal{K}(x)$ for $i = 9, 10, \cdots, 14$.

Therefore $(\gamma - \overline{BA}) \in \Phi(X)$ by [2, Lemma 2.4]. This completes the proof that $\sigma_{\Phi}(\overline{BA}) \subseteq \sigma_{\Phi}(B)\sigma_{\Phi}(A)$.

To show that $\sigma_{\Phi}(AB) \subseteq \sigma_{\Phi}(B)\sigma_{\Phi}(A)$ we proceed to show that $(\gamma - AB) \in \Phi(X)$. Let S_1 and S_2 be as above. Since $R(S_1) \subseteq D(A)$ and $ABx = BAx - Kx$, $\forall x \in D(A)$, we have that $(\gamma - AB)S_1 = (\gamma - BA)S_1 + K_{15} = I + K' + K_{15} = I + K_{16}$.

$$\begin{aligned}
S_2(\gamma - AB) &= S_2[\gamma(\lambda - A_1)B + \gamma(I - \lambda B)] \\
&= I + K_{17}, \qquad K_{15}, K_{16}, K_{17} \in \mathcal{K}(X).
\end{aligned}$$

Therefore, by [2, Lemma 2.4], $(\gamma - AB) \in \Phi(X)$. This completes the proof that $\sigma_{\Phi}(AB) \subseteq \sigma_{\Phi}(B)\sigma_{\Phi}(A)$.

Corollary 2.1. *Let $A \in C(X)$ and $B \in L(X)$. Assume $0 \notin \sigma(B)$. Let $B: D(A) \to D(A)$ and let there exist a compact operator, K, such that $BAx = ABx + Kx$, $\forall x \in D(A)$. Then BA and $AB|_{D(A)}$ are closed and*

(1) $\sigma_\Phi(AB|_{D(A)}) = \sigma_\Phi(BA) \subseteq \sigma_\Phi(B)\sigma_\Phi(A)$, *and*

(2) $\sigma_\Phi(AB) \subseteq \sigma_\Phi(B)\sigma_\Phi(A)$.

Proof. Since B is invertible and A is closed, BA is closed. We proceed to show that $AB|_{D(A)}$ is closed.

Let $x_n \in D(A)$, $x_n \to x$ and $ABx_n \to y$. Since B is bounded and A is closed, we have $y = ABx$. We have only to show that $x \in D(A)$. $BAx_n = (ABx_n + Kx_n)$ converges to some vector, z. Therefore, $Ax_n \to B^{-1}z$, and, since A is closed, $x \in D(A)$. This shows that $AB|_{D(A)}$ is closed.

Since $BA = AB|_{D(A)} + K$, we have by [2, Theorem 2.8] that $\sigma_\Phi(AB|_{D(A)}) = \sigma_\Phi(BA)$. The remainder of the corollary follows from Theorem 2.

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