

## A PROBABILISTIC APPROACH TO $H^p(R^d)$ (1)

BY

D. STROOCK AND S.R.S. VARADHAN

**ABSTRACT.** The relationship between  $H^p(R^d)$ ,  $1 \leq p < \infty$ , and the integrability of certain functionals of Brownian motion is established using the connection between probabilistic and analytic notions of functions with bounded mean oscillation. An application of this relationship is given in the derivation of an interpolation theorem for operators taking  $H^1(R^d)$  to  $L^1(R^d)$ .

**Introduction.** The purpose of this paper is to give a "soft" analytic treatment of the connection between Brownian motion and the theory of  $H^p(R^d)$ ,  $1 \leq p \leq \infty$ . The original relationship between these two subjects was discovered by Burkholder, Gundy, and Silverstein [2] for  $S^1$ . Moreover, we have learned that Burkholder and Gundy have recently extended their earlier result to  $R^d$ . The route they take is more refined than ours and establishes the connection for all  $0 < p < \infty$ , whereas our technique restricts us to  $p \geq 1$ . Nonetheless, our method avoids the difficult estimates on which theirs turns and has some nice dividends of its own.

There are three contexts in which we discuss  $H^p(R^d)$ . The first of these (cf. §1) is described in terms of Wiener martingales having certain integrability properties. This space is denoted by  $\mathcal{H}^p(\Omega)$ . Second, we talk about those harmonic functions in  $R_+^{d+1}$  which become members of  $\mathcal{H}^p(\Omega)$  when evaluated along Wiener paths. This space is denoted by  $\mathcal{H}^p(R_+^{d+1})$ . Finally, in §2 we discuss  $H^p(R^d)$  itself, described in terms of Riesz transforms. Theorem (3.3) and its corollaries establish  $H^p(R^d)$  as the boundary values of  $\mathcal{H}^p(R_+^{d+1})$ , and Theorem (3.1) shows that  $H^p(R^d)$  arises as the result of "conditioning" members of  $\mathcal{H}^p(\Omega)$  with respect to the first place that a Wiener path exits from  $R_+^{d+1}$ .

§4 is concerned with a Marcinkiewicz-type interpolation theorem (cf. Theorem (4.1)) whose proof takes advantage of the above relationships between the various ways of defining  $H^p(R^d)$ . As a consequence of this theorem, we are able to show that  $H^p(R^d) = L^p(R^d)$  for  $1 < p < \infty$ .

It has been pointed out to us by E. M. Stein that essentially the same interpolation theorem was discovered much earlier by S. Igari [7] using entirely different methods.

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**1. Probabilistic background.** Let  $\Omega = C([0, \infty), R^{d+1})$  <sup>(2)</sup> and let  $z(t, \omega)$  denote the position of the path  $\omega$  at time  $t$ . We will use  $x(t, \omega)$  for the first  $d$ -coordinates of  $z(t, \omega)$  and  $y(t, \omega)$  for the  $(d + 1)$ st coordinate. Let  $\mathfrak{N}_t = \mathfrak{B}\{z(s): 0 \leq s \leq t\}$ , for  $t \geq 0$ , and put  $\mathfrak{N} = \sigma(\cup_{t \geq 0} \mathfrak{N}_t)$ . Given  $z = (x, y) \in R^{d+1}$ , we will denote by  $P_z$  the Wiener measure on  $\langle \Omega, \mathfrak{N} \rangle$  starting at  $z$ . That is,  $P_z(z(0) = z) = 1$  and

$$P_z(z(t + s) \in \Gamma \mid \mathfrak{N}_t) = \frac{1}{(2\pi t)^{(d+1)/2}} \int_{\Gamma} \exp(-|\xi - z(s)|^2/2t) d\xi \quad (\text{a.s., } P_z)$$

for all  $s, t \geq 0$  and  $\Gamma \in \mathfrak{B}[R^{d+1}]$ .

Let  $\zeta = \inf\{t \geq 0: y(t) \leq 0\}$ . Given  $z \in R_+^{d+1} = \{(x, y) \in R^{d+1}: y > 0\}$ , we will call the triple  $\langle \eta(t), \mathfrak{N}_t, P_z \rangle$  a *continuous local martingale* on  $[0, \zeta)$  if  $\eta$  is a real valued, measurable function on  $\{(t, \omega) \in [0, \infty) \times \Omega: \zeta(\omega) > t\}$ ,  $\eta(\cdot, \omega)$  is continuous on  $[0, \zeta(\omega))$  for  $P_z$ -almost all  $\omega$ , and there exists a sequence of stopping times  $\tau_n < \zeta$  such that  $\tau_n \nearrow \zeta$  (a.s.,  $P_z$ ) and  $\langle \eta(t \wedge \tau_n), \mathfrak{N}_t, P_z \rangle$  is a bounded martingale for each  $n$ . If  $\sup_n E_z[|\eta(\tau_n)|] < \infty$ , then

$$\eta(\zeta) = \lim_{t \nearrow \zeta} \eta(t)$$

exists for  $P_z$ -almost all  $\omega$ . Moreover, if  $\{\eta(\tau_n)\}_{n \geq 1}$  is uniformly  $P_z$ -integrable, then  $\langle \eta(t \wedge \zeta), \mathfrak{N}_t, P_z \rangle$  is a uniformly  $P_z$ -integrable, continuous martingale.

Examples of continuous local martingales on  $[0, \zeta)$  are plentiful. For instance, if  $h \in C^2(R_+^{d+1})$ , then  $\langle h(z(t)) - \int_0^t \Delta h(z(s)) ds, \mathfrak{N}_t, P_z \rangle$  is a continuous local martingale on  $[0, \zeta)$  for all  $z \in R_+^{d+1}$ . This can be easily seen by combining the basic property defining  $P_z$  with the Doob stopping time theorem applied to  $\tau_n = \inf\{t \geq 0: y(t) \notin (1/n, n) \text{ or } |x(t)| \geq n\} \wedge n$ . See [12] for details.

The continuous version of the Doob decomposition theorem, proved by Meyer (cf. M. Rao [9]), says that if  $\langle \eta(t), \mathfrak{N}_t, P_z \rangle$  is a continuous local martingale on  $[0, \zeta)$ , then there is a measurable function  $\Psi_\eta: [0, \infty) \times \Omega \rightarrow R$  such that  $\Psi_\eta(0, \cdot) = 0$ ,  $\Psi_\eta(\cdot, \omega)$  is continuous and nondecreasing on  $[0, \zeta(\omega))$  for  $P_z$ -almost all  $\omega$ ,  $\Psi_\eta(t, \cdot)$  is  $\mathfrak{N}_t$ -measurable for all  $t \geq 0$ , and  $\langle \eta^2(t) - \Psi_\eta(t), \mathfrak{N}_t, P_z \rangle$  is a continuous local martingale on  $[0, \zeta)$ . Moreover,  $\Psi_\eta(t \wedge \zeta)$  is unique up to a set of  $P_z$ -measure zero. In the example cited above,  $\Psi_\eta(t \wedge \zeta) = \int_0^{t \wedge \zeta} |\nabla h(z(s))|^2 ds$ .

The Burkholder-Gundy inequality says that if  $0 < p < \infty$ , then there exist universal constants  $0 < a_p \leq A_p < \infty$  such that

$$(1.1) \quad a_p E_z[|\Psi_\eta(\zeta)|^{p/2}] \leq E_z \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)|^p \right] \leq A_p E_z[|\Psi_\eta(\zeta)|^{p/2}]$$

for all continuous local martingales  $\langle \eta(t), \mathfrak{N}_t, P_z \rangle$  on  $[0, \zeta)$ . A nice proof of (1.1) can be found in the paper of Gettoor and Sharpe [5].

Let  $1 \leq p < \infty$ . We will say that  $\eta \in \mathcal{H}^p(\Omega)$  if  $\langle \eta(t) - \eta(0), \mathfrak{N}_t, P_z \rangle$  is a continuous local martingale on  $[0, \zeta)$  for all  $z \in R_+^{d+1}$  and

<sup>(2)</sup>  $C([0, \infty), R^{d+1})$  is the space of continuous functions on  $[0, \infty)$  with values in  $R^{d+1}$ .

$$(1.2) \quad \|\eta\|_{\mathcal{BC}^p(\Omega)} \equiv \left( \sup_{y>0} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)|^p \right] dx \right)^{1/p} < \infty.$$

Notice that if  $\eta \in \mathcal{C}^p(\Omega)$  then, for each  $y > 0$ ,  $\langle \eta(t \wedge \zeta) - \eta(0), \mathfrak{N}_t, P_{x,y} \rangle$  is a continuous uniformly integrable martingale for almost every  $x \in R^d$ . There is a subspace of  $\mathcal{C}^p(\Omega)$  which will be of particular interest to us. Namely, let  $\mathcal{H}^p(R^{d+1}_+)$  be the space of harmonic functions  $u$  on  $R^{d+1}_+$  such that

$$(1.3) \quad \|u\|_{\mathcal{BC}^p(R^{d+1}_+)} \equiv \left( \sup_{y>0} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |u(z(t))|^p \right] dx \right)^{1/p} < \infty.$$

Obviously  $u \in \mathcal{H}^p(R^{d+1}_+)$  implies  $\eta_u \in \mathcal{C}^p(\Omega)$  and  $\|\eta_u\|_{\mathcal{BC}^p(\Omega)} \leq 2\|u\|_{\mathcal{BC}^p(R^{d+1}_+)}$ , where  $\eta_u(t) = u(z(t \wedge \zeta))$ .

There is one more class of martingales with which we will have to deal. We will say that  $\theta \in \mathfrak{B.N.O.}(\Omega)$  if  $\langle \theta(t \wedge \zeta) - \theta(0), \mathfrak{N}_t, P_z \rangle$  is a continuous square integrable martingale for every  $z \in R^{d+1}_+$  and

$$(1.4) \quad \|\theta\|_{\mathfrak{B.N.O.}(\Omega)} \equiv \sup_{z:t \geq 0} \text{ess sup } E_z [(\theta(\zeta) - \theta(t \wedge \zeta))^2 | \mathfrak{N}_t]^{1/2} < \infty.$$

As we will see later, the notion of  $\mathfrak{B.N.O.}$  martingales is a natural probabilistic outgrowth of the analytic notion of a function of bounded mean oscillation. The importance of this class of martingales to us is contained in the following theorem.

**Theorem (1.1).** *Let  $\eta \in \mathcal{C}^1(\Omega) \cap \mathcal{C}^2(\Omega)$  and  $\theta \in \mathfrak{B.N.O.}(\Omega)$ . Then for  $z \in R^{d+1}_+$ :*

$$|E_z[(\eta(\zeta) - \eta(0))\theta(\zeta)]| \leq 2^{1/2} \|\theta\|_{\mathfrak{B.N.O.}(\Omega)} E_z[(\Psi_\eta(\zeta))^{1/2}].$$

*In particular, there is a universal constant  $F$  such that*

$$\begin{aligned} \int_{R^d} |E_{x,y}[(\eta(\zeta) - \eta(0))\theta(\zeta)]| dx &\leq F \|\theta\|_{\mathfrak{B.N.O.}(\Omega)} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)| \right] dx \\ &\leq F \|\eta\|_{\mathcal{BC}^1(\Omega)} \|\theta\|_{\mathfrak{B.N.O.}(\Omega)}. \end{aligned}$$

A proof of this theorem may be found in [5]. It is the inequality of Fefferman in the present context.

**2. Analytic background.** For each  $1 \leq j \leq d$ , define  $R_j$  on  $L^2(R^d)$  by the relation  $(R_j f)^\wedge(\xi) = i(\xi_j/|\xi|)\hat{f}(\xi)$ . Clearly  $R_j$  is a bounded, translation invariant, skew-adjoint operator on  $L^2(R^d)$  into itself. One can also express the action of  $R_j$  by the use of principal value integrals. Namely,

$$(2.1) \quad R_j f(x) = \lim_{\epsilon \searrow 0} R_j^{(\epsilon)} * f(x),$$

where  $R_j^{(\epsilon)}(x) = a_d(x_j/|x|^{d+1})\mathfrak{K}_{[\epsilon,\infty)}(|x|)$ . The convergence takes place in  $L^2(R^d)$ .

These operators have an extensive theory and are known as the *Riesz transformations*. The book of Stein and Weiss [11] is an excellent reference for this material.

For each  $1 \leq p < \infty$ , let  $\|\cdot\|_{H^p(R^d)}$  be defined on  $L^p(R^d) \cap L^2(R^d)$  by

$$\|f\|_{H^p(R^d)} = \sum_0^d \|R_j f\|_{L^p(R^d)},$$

where  $R_0$  is the identity. Denote by  $H^p(R^d)$  the completion with respect to  $\|\cdot\|_{H^p(R^d)}$  of the class of  $f \in L^p(R^d) \cap L^2(R^d)$  such that  $\|f\|_{H^p(R^d)} < \infty$ . Observe that  $\|\cdot\|_{H^2(R^d)}$  is commensurate with  $\|\cdot\|_{L^2(R^d)}$  and that  $H^2(R^d) = L^2(R^d)$ . We will see later that this is also the case for all  $1 < p < \infty$ , but it is false for  $p = 1$ . Obviously, the  $R_j$  determine unique bounded operators from  $H^p(R^d)$  to  $L^p(R^d)$ . We will use  $R_j$  to denote these operators as well. Note that  $R_j$  is determined on  $H^1(R^d)$  by the relation  $(R_j f)^\wedge(\xi) = i(\xi_j/|\xi|)\hat{f}(\xi)$ .

**Lemma (2.1).** *The space  $H^1(R^d)$  coincides with the class of  $f \in L^1(R^d)$  such that, for each  $1 \leq j \leq d$ ,  $i(\xi_j/|\xi|)\hat{f}(\xi)$  is the Fourier transform of a function in  $L^1(R^d)$ .*

It is immediately apparent from this lemma why  $H^1(R^d)$  is smaller than  $L^1(R^d)$ . Indeed, if  $f \in H^1(R^d)$ , then  $(\xi_j/|\xi|)\hat{f}(\xi)$  is continuous, since it is the Fourier transform of an  $L^1(R^d)$ -function. Hence a necessary condition for  $f \in H^1(R^d)$  is that  $\int_{R^d} f(x) dx = \hat{f}(0) = 0$ .

There is another class of functions with which we will be concerned; namely, the John-Nirenberg class of functions having bounded mean oscillation. A locally integrable function  $\phi$  on  $R^d$  will be said to have bounded mean oscillation if

$$(2.2) \quad \|\phi\|_{\text{B.M.O.}(R^d)} \equiv \sup_Q \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q| dx < \infty, \quad (3)$$

where  $Q$  runs over the cubes contained in  $R^d$ . The space of all such functions is denoted by  $\text{B.M.O.}(R^d)$ . Actually, it is convenient to think of functions in  $\text{B.M.O.}(R^d)$  as being defined only up to additive constants. Then one can easily check that  $\|\cdot\|_{\text{B.M.O.}(R^d)}$  is a complete norm on  $\text{B.M.O.}(R^d)$ . One of the nice facts about  $\text{B.M.O.}(R^d)$  is that many maps which do not map  $L^\infty(R^d)$  into itself continuously map  $L^\infty(R^d)$  continuously into  $\text{B.M.O.}(R^d)$ . The basic result in this direction is the following (cf. Fefferman and Stein [3]).

**Lemma (2.2).** *Let  $K \in L^1(R^d)$  be given and suppose  $B$  is a constant satisfying  $\|\hat{K}\|_{L^\infty(R^d)} \leq B$  and*

$$\sup_{x \in R^d} \int_{|\xi| \geq 2|x|} |K(x - \xi) - K(-\xi)| d\xi \leq B.$$

*Then there is a constant  $C$  depending only on  $d$  and  $B$  such that  $\|K * \phi\|_{\text{B.M.O.}(R^d)} \leq C\|\phi\|_{L^\infty(R^d)}$ ,  $\phi \in L^\infty(R^d)$ .*

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(3) Here and in what follows, if  $f$  is a function on  $R^d$  and  $S \subseteq R^d$ , then  $f_S$  denotes that mean value of  $f$  on  $S$ .

Following Fefferman and Stein [3], one can show from Lemma (2.2) that the Riesz transforms determine a continuous map of  $L^\infty(R^d)$  into  $B.M.O.(R^d)$ . Indeed, the mapping with which they work is given by:

$$(2.3) \quad \lim_{\epsilon \searrow 0} \int_{R^d} (R_j^{(\epsilon)}(x - \xi) - R_j^{(1)}(-\xi))\phi(\xi) d\xi.$$

Notice that the definition here differs from that in (2.1), but the difference is contained in an additive constant and so does not matter in  $B.M.O.(R^d)$ .

An immediate consequence of the preceding is the next lemma.

**Lemma (2.3).** *Let  $f \in L^2(R^d)$  and assume that*

$$\left| \int_{R^d} f(x)\phi(x) dx \right| \leq A \|\phi\|_{B.M.O.(R^d)}$$

for all  $\phi \in L^2(R^d) \cap B.M.O.(R^d)$ . Then  $f \in H^1(R^d)$  and  $\|f\|_{H^1(R^d)} \leq CA$ , where  $C$  depends only on  $d$ .

We next want to give a characterization of  $B.M.O.(R^d)$  which will enable us to relate this class of functions to  $\mathfrak{B.N.O.}(\Omega)$ . Essential for this is the following inequality of John and Nirenberg [8].

**Theorem (2.1).** *Suppose  $\phi \in B.M.O.(R^d)$  and that  $\|\phi\|_{B.M.O.(R^d)} \leq 2^{-(d+1)}$ . Then for all cubes  $Q$ ,  $(1/|Q|) \int_Q \exp[\alpha|\phi(x) - \phi_Q|] dx \leq 2e^{4\alpha}/(2 - e^{4\alpha})$  for  $\alpha < \frac{1}{4} \log 2$ . In particular, there exist universal constants  $B_p$ ,  $1 \leq p < \infty$ , such that*

$$\left( \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|^p dx \right)^{1/p} \leq B_p \|\phi\|_{B.M.O.(R^d)}.$$

A nice probabilistic derivation of the above appears in Garsia [4]. Using this result we are now able to prove the characterization of  $B.M.O.(R^d)$  which we will need. This characterization was first discovered by Gundy for  $S^1$ . The proof we give below is based on an idea due to Garsia.

**Theorem (2.2).** *If  $\phi \in B.M.O.(R^d)$ , then  $\phi(x)/(1 + |x|^{d+1}) \in L^1(R^d)$  and*

$$\sup_{y>0; x \in R^d} [p_y * (\phi - u_\phi(x, y))^2](x) \leq C \|\phi\|_{B.M.O.(R^d)}^2, \quad (*)$$

where  $C$  depends only on  $d$ . Conversely, if  $\phi(x)/(1 + |x|^{d+1}) \in L^1(R^d)$ , then

$$\|\phi\|_{B.M.O.(R^d)}^2 \leq C' \sup_{y>0; x \in R^d} [p_y * (\phi - u_\phi(x, y))^2](x),$$

where  $C'$  depends only on  $d$ .

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(\*) Here and in what follows, if  $f$  is a function on  $R^d$ , then  $u_f(x, y) = p_y * f(x)$  is its harmonic extension to  $R_+^{d+1}$ , where  $p_y(x) = b_d(y/(y^2 + |x|^2))^{(d+1)/2}$ .

**Proof.** We first observe that there are positive constants  $a$  and  $A$  depending only on  $d$  such that for all  $p \geq 1$ :

$$\begin{aligned} \frac{1}{2} \left( \sup_B \frac{a}{|B|} \int_B |\phi(x) - \phi_B|^p dx \right)^{1/p} &\leq \left( \sup_Q \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|^p dx \right)^{1/p} \\ &\leq 2 \left( \sup_B \frac{A}{|B|} \int_B |\phi(x) - \phi_B|^p dx \right)^{1/p} \end{aligned}$$

where  $B$  runs over the balls in  $R^d$ . Combining this with Theorem (2.1), we see in particular that

$$\sup_B \frac{1}{|B|} \int_B |\phi(x) - \phi_B|^p dx \leq C_p \|\phi\|_{\text{B.M.O.}(R^d)}^p$$

for  $1 \leq p < \infty$ .

We next note that

$$p_y(x) = \int_0^\infty \frac{1}{|B(0,r)|} \mathfrak{X}_{B(0,r)}(x) \rho_y(r) dr$$

where

$$\rho_y(r) = c_d \frac{y r^{d+1}}{(r^2 + y^2)^{(d+3)/2}}$$

and  $c_d$  is a positive constant. Integrating with respect to  $x$ , one sees that  $\int_0^\infty \rho_y(r) dr = 1$ .

Now suppose  $\phi \in \text{B.M.O.}(R^d)$ . In [3] it is shown that  $\phi(x)/(1 + |x|^{d+1}) \in L^1(R^d)$ . In particular,  $u_\phi(x, y)$  is well defined. Using the preceding paragraph, we have:

$$\begin{aligned} [p_y * (\phi - u_\phi(x, y))^2](x) &= p_y * \phi^2(x) - (p_y * \phi(x))^2 \\ &= \int_0^\infty \rho_y(r) (\phi^2)_{B(x,r)} dr - \left( \int_0^\infty \rho_y(r) \phi_{B(x,r)} dr \right)^2 \\ &= \int_0^\infty \rho_y(r) ((\phi - \phi_{B(x,r)})^2)_{B(x,r)} dr \\ &\quad + \int_0^\infty \rho_y(r) \left( \phi_{B(x,r)} - \int_0^\infty \rho_y(s) \phi_{B(x,s)} ds \right)^2 dr \\ &\leq \int_0^\infty \rho_y(r) ((\phi - \phi_{B(x,r)})^2)_{B(x,r)} dr \\ &\quad + \int_0^\infty \rho_y(s) ds \int_0^\infty \rho_y(r) (\phi_{B(x,s)} - \phi_{B(x,r)})^2 dr. \end{aligned}$$

We know that  $(|\phi - \phi_{B(x,r)}|)_{B(x,r)} \leq C_1 \|\phi\|_{\text{B.M.O.}(R^d)}^2$ . Moreover,  $0 < r < s$ ,

$$\begin{aligned} |\phi_{B(x,s)} - \phi_{B(x,r)}|^{2d+2} &= |(\phi - \phi_{B(x,s)})_{B(x,r)}|^{2d+2} \\ &\leq \frac{1}{|B(x,r)|} \int_{B(x,s)} |\phi(\xi) - \phi_{B(x,s)}|^{2d+2} d\xi \\ &= \left(\frac{s}{r}\right)^d \frac{1}{|B(x,s)|} \int_{B(x,s)} |\phi(\xi) - \phi_{B(x,s)}|^{2d+2} d\xi \\ &\leq C_2 \left(\frac{s}{r}\right)^d \|\phi\|_{\text{B.M.O.}(R^d)}^{2d+2}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^\infty \rho_y(s) ds \int_0^s \rho_y(r) (\phi_{B(x,s)} - \phi_{B(x,r)})^2 dr \\ \leq C_2^{1/(d+1)} \|\phi\|_{\text{B.M.O.}(R^d)}^2 \int_0^\infty \rho_y(s) ds \int_0^s \rho_y(r) \left(\frac{s}{r}\right)^{d/(d+1)} dr. \end{aligned}$$

Note that

$$\int_0^\infty \rho_y(s) ds \int_0^s \rho_y(r) \left(\frac{s}{r}\right)^{d/(d+1)} dr = \int_0^\infty \rho_1(s) ds \int_0^s \rho_1(r) \left(\frac{s}{r}\right)^{d/(d+1)} dr < \infty,$$

and therefore the first assertion is proved.

To prove the converse statement, let  $B = B(x, r)$  be given. Note that

$$\frac{r}{(r^2 + |x|^2)^{(d+1)/2}} = \frac{1}{r^d} \frac{1}{(|x|^2/r^2 + 1)^{(d+1)/2}} \geq \frac{1}{2^{(d+1)/2} r^d}$$

for  $|x| \leq r$ . Hence

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} (\phi(\xi) - \phi_{B(x,r)})^2 d\xi &\leq \frac{1}{|B(x,r)|} \int_{B(x,r)} (\phi(\xi) - u_\phi(x,r))^2 d\xi \\ &\leq C \int_{R^d} |\phi(\xi) - u_\phi(x,r)|^2 p_r(x - \xi) d\xi. \quad \text{Q.E.D.} \end{aligned}$$

We conclude this section with an important theorem due to Stein and Weiss. Let  $f \in L^2(R^d)$  and define  $u_j(\cdot, y) = p_j * R_j f$  for  $0 \leq j \leq d$  and  $y > 0$ . Then it is easy to check that the  $u_j$  satisfy the generalized Cauchy-Riemann equations of M. Riesz (cf. p. 91 of Stein and Weiss [11]). Hence by Theorem (4.14) of [8],  $(\sum_0^d u_j^2)^{\alpha/2}$  is subharmonic in  $R^{d+1}$  for  $\alpha \geq (d-1)/d$ . We will not go through the proof of this fact but will only mention that it can be checked by hand when  $d = 1$  and  $\alpha = \frac{1}{2}$ . The general proof involves a clever application of elementary matrix algebra.

3. A projection theorem.

**Lemma (3.1).** *Suppose  $\phi \in \text{B.M.O.}(R^d)$  and let  $\theta_\phi(t) = u_\phi(z(t))$ ,  $0 < t < \zeta$ . Then  $\theta_\phi \in \mathfrak{B}\mathfrak{M}\mathfrak{O}(\Omega)$  <sup>(5)</sup> and there is a universal constant  $C$  depending only on  $d$  such that*

$$\|\theta_\phi\|_{\mathfrak{B}\mathfrak{M}\mathfrak{O}(\Omega)} \leq C\|\phi\|_{\text{B.M.O.}(R^d)}.$$

**Proof.** Observe that for  $z \in R_+^{d+1}$ :

$$\begin{aligned} E_z[(\theta_\phi(\zeta) - \theta_\phi(t \wedge \zeta))^2 | \mathfrak{N}_t] &= \mathfrak{X}_{\{t > 0\}} E_{x(t)}[(\phi(x(\zeta)) - u_\phi(z(0)))^2] \\ &= \mathfrak{X}_{\{t > 0\}} p_{y(t)} * [(\phi - u_\phi(z(t)))^2](x(t)). \end{aligned}$$

Hence the lemma follows from Theorem (2.2). Q.E.D.

**Lemma (3.2).** *If  $\eta \in \mathfrak{C}^1(\Omega) \cap \mathfrak{C}^2(\Omega)$  and  $\phi \in \text{B.M.O.}(R^d)$ , then*

$$\begin{aligned} \int_{R^d} |E_{x,y}[(\eta(\zeta) - \eta(0))\phi(x(\zeta))]| dx \\ \leq CF\|\phi\|_{\text{B.M.O.}(R^d)} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)| \right] dx, \end{aligned}$$

where  $C$  is the constant in Lemma 3.1 and  $F$  is the constant in Theorem 1.1.

**Theorem (3.1).** *For each  $1 \leq p < \infty$  and  $y > 0$  there is a continuous linear map  $\Pi_y: \mathfrak{C}^p(\Omega) \rightarrow L^p(R^d)$ , such that  $\|\Pi_y \eta\|_{L^p(R^d)} \leq \|\eta\|_{\mathfrak{B}\mathfrak{C}^p(\Omega)}$ , given by*

$$(3.1) \quad \int_{R^d} (\Pi_y \eta)(x)\phi(x) dx = \int_{R^d} E_{x,y}[(\eta(\zeta) - \eta(0))\phi(x(\zeta))] dx$$

for all  $\phi \in L^q(R^d)$ ,  $1/p + 1/q = 1$ . Moreover, if  $\eta \in \mathfrak{C}^1(\Omega)$ , then  $\Pi_y \eta \in H^1(R^d)$  and there is a universal constant  $C_\phi$  depending only on  $d$ , such that

$$(3.2) \quad \|\Pi_y \eta\|_{H^1(R^d)} \leq C_d \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)| \right] dx.$$

Finally, if  $\eta$  is bounded, then  $\Pi_y \eta$  is dominated by twice the bound on  $\eta$ .

**Proof.** When  $1 < p < \infty$  the assertion is trivial. Indeed:

$$\begin{aligned} \left| \int_{R^d} E_{x,y}[(\eta(\zeta) - \eta(0))\phi(x(\zeta))] dx \right| &\leq \|\eta\|_{\mathfrak{B}\mathfrak{C}^p(\Omega)} \left( \int_{R^d} E_{x,y}[|\phi(x(\zeta))|^q] dx \right)^{1/q} \\ &= \|\eta\|_{\mathfrak{B}\mathfrak{C}^p(\Omega)} \left( \int_{R^d} dx \int_{R^d} p_y(x - \xi) |\phi(\xi)|^q d\xi \right)^{1/q} \\ &= \|\eta\|_{\mathfrak{B}\mathfrak{C}^p(\Omega)} \|\phi\|_{L^q(R^d)}, \end{aligned}$$

<sup>(5)</sup> Note that  $\langle u_\phi(z(t)), \mathfrak{N}_t, P_z \rangle$  is a continuous local martingale on  $[0, \zeta]$ , since  $\Delta u_\phi = 0$ . In particular,  $u_\phi(z) = E_z[\phi(x(\zeta))]$ .

and therefore the right-hand side of (3.1) determines a continuous linear functional on  $L^q(R^d)$ .

To treat the case when  $p = 1$ , we first suppose that  $\eta \in \mathfrak{C}^1(\Omega) \cap \mathfrak{C}^2(\Omega)$ . Then  $\Pi, \eta \in L^2(R^d)$ . Moreover, if  $\phi \in L^2(R^d) \cap \text{B.M.O.}(R^d)$ , then by Lemma (3.2):

$$\left| \int_{R^d} (\Pi, \eta)(x) \phi(x) dx \right| \leq CF \left( \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)| \right] dx \right) \|\phi\|_{\text{B.M.O.}(R^d)}.$$

Hence, by Lemma (2.3),  $\Pi, \eta \in H^1(R^d)$  and (3.2) obtains. Given a general  $\eta \in \mathfrak{C}^1(\Omega)$ , define  $\eta^{(\alpha)}(t) = \eta(t \wedge \tau_\alpha)$  for  $0 \leq t < \zeta$ , where  $\tau_\alpha = \inf\{t \geq 0: |\eta(t \wedge \zeta) - \eta(0)| \geq \alpha\}$ . Clearly  $\eta^{(\alpha)} \in \mathfrak{C}^1(\Omega) \cap \mathfrak{C}^2(\Omega)$  for all  $\alpha > 0$ . Also,

$$\begin{aligned} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta^{(\alpha)}(t)| \right] dx \\ \leq 2 \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)|; \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)| \geq \alpha \right] dx \rightarrow 0 \end{aligned}$$

as  $\alpha \rightarrow \infty$ . From this and the preceding it follows that  $\Pi, \eta^{(\alpha)}$  tends in  $H^1(R^d)$  to a  $\Pi, \eta$  satisfying (3.1) and (3.2). Q.E.D.

**Lemma (3.3).** *Let  $1 \leq p \leq \infty$  and suppose  $u \in \mathfrak{C}^p(R_+^{d+1})$ . Given  $h > 0$ , let  $w(x, y) = u(x, y + h)$  in  $R_+^{d+1}$ . Then  $w \in \mathfrak{C}^p(R_+^{d+1})$  and  $\|w\|_{\mathfrak{C}^p(R_+^{d+1})} \leq \|u\|_{\mathfrak{C}^p(R_+^{d+1})}$ .*

**Proof.** Observe that

$$\begin{aligned} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |w(z(t))|^p \right] dx &= \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |u(x(t), y(t) + h)|^p \right] dx \\ &= \int_{R^d} E_{x,y+h} \left[ \sup_{0 \leq t < \zeta_h} |u(z(t))|^p \right] dx \\ &\leq \int_{R^d} E_{x,y+h} \left[ \sup_{0 \leq t < \zeta} |u(z(t))|^p \right] dx \\ &\leq \|u\|_{\mathfrak{C}^p(R_+^{d+1})}^p, \end{aligned}$$

where  $\zeta_h = \inf\{t \geq 0: y(t) \leq h\}$ . Q.E.D.

**Theorem (3.2).** *Let  $1 \leq p < \infty$  and  $u \in \mathfrak{C}^p(R_+^{d+1})$ . Set  $\eta_u(t) = u(z(t))$ ,  $0 \leq t < \zeta$ . Then  $\eta_u \in \mathfrak{C}^p(\Omega)$  and the function  $f = u(\cdot, 2h) + \Pi_h \eta_u$  is independent of  $h > 0$ . Moreover,  $u = u_f$  and  $\|f\|_{L^p(R^d)} \leq \|u\|_{\mathfrak{C}^p(R_+^{d+1})}$ . Finally, if  $p = 1$ , then  $f \in H^1(R^d)$  and  $\|f\|_{H^1(R^d)} \leq C \|u\|_{\mathfrak{C}^1(R_+^{d+1})}$ , where  $C$  depends only on  $d$ .*

**Proof.** Let  $h > 0$  be given and set  $f = u(\cdot, 2h) + \Pi_h \eta_u$ . Clearly  $f \in L^p(R^d)$ . Moreover, if  $\phi \in L^q(R^d)$ , then

$$\begin{aligned} \int_{R^d} f(x) \phi(x) dx &= \int_{R^d} u(x, 2h) \phi(x) dx + \int_{R^d} E_{x,h} [(\eta_u(\zeta) - \eta_u(0)) \phi(x(\zeta))] dx \\ &= \int_{R^d} E_{x,h} [\eta_u(\zeta) \phi(x(\zeta))] dx, \end{aligned}$$

since

$$\int_{R^d} E_{x,h}[\eta_u(0)\phi(x(\xi))] dx = \int_{R^d} u(x,h)u_\phi(x,h) dx = \int_{R^d} u(x,2h)\phi(x) dx. \tag{6}$$

Define  $\xi_n = \inf\{t \geq 0: y(t) \leq h/n\}$ . Then

$$\begin{aligned} \int_{R^d} E_{x,h}[\eta_u(\xi)\phi(x(\xi))] dx &= \lim_{n \rightarrow \infty} \int_{R^d} E_{x,h}[u(x(\xi_n), h/n)u_\phi(x(\xi_n), h/n)] dx \\ &= \lim_{n \rightarrow \infty} \int_{R^d} u(x, 2h/n)\phi(x) dx. \end{aligned}$$

Taking  $\phi(x) = p_y(x^0 - x)$ , we obtain

$$\begin{aligned} u_f(x^0, y) &= \int_{R^d} f(x)p_y(x^0 - x) dx = \lim_{n \rightarrow \infty} \int_{R^d} u(x, 2h/n)p_y(x^0 - x) dx \\ &= \lim_{n \rightarrow \infty} u(x^0, y + 2h/n) = u(x^0, y). \end{aligned}$$

This proves that  $u = u_f$  and, in particular, that  $u(\cdot, 2h) + \Pi_h \eta_u$  is independent of  $h > 0$ . Obviously, from  $u = u_f$  it follows that  $\|f\|_{L^p(R^d)} \leq \|u\|_{\mathcal{BC}^p(R^{d+1})}$ ,  $f = \lim_{y \searrow 0} p_y * f = \lim_{y \searrow 0} u_f$ .

Now suppose that  $p = 1$ . Since  $f = u(\cdot, 2) + \Pi_1 \eta_w$  and  $\Pi_1 \eta_w \in H^1(R^{d+1})$  satisfies (3.2), it suffices to prove that  $u(\cdot, 2) \in H^1(R^{d+1})$  and  $\|u(\cdot, 2)\|_{H^1(R^d)} \leq C\|u\|_{\mathcal{BC}^1(R^{d+1})}$ . Let  $g = u(\cdot, 2)$  and  $w = u_g$ . Then  $w(x, y) = u(x, y + 2)$  and therefore  $\|w\|_{\mathcal{BC}^1(R^{d+1})} \leq \|u\|_{\mathcal{BC}^1(R^{d+1})}$ . Also we have just seen that  $g = w(\cdot, 2h) + \Pi_h \eta_w$  for all  $h > 0$ . Clearly  $g \in L^2(R^d)$ . Given  $\phi \in L^2(R^d) \cap \text{B.M.O.}(R^d)$ , we have

$$\begin{aligned} \left| \int_{R^d} g(x)\phi(x) dx \right| &= \left| \int_{R^d} w(x, 2h)\phi(x) dx + \int_{R^d} E_{x,h}[(\eta_w(\xi) - \eta_w(0))\phi(x(\xi))] dx \right| \\ &\leq \|w(\cdot, 2h)\|_{L^2(R^d)}\|\phi\|_{L^2(R^d)} + C\|u\|_{\mathcal{BC}^1(R^{d+1})}\|\phi\|_{\text{B.M.O.}(R^d)}. \end{aligned}$$

Since  $\|w(\cdot, 2h)\|_{L^2(R^d)} \rightarrow 0$  as  $h \rightarrow \infty$ , we see from Lemma (2.3) that  $g \in H^1(R^d)$  and  $\|g\|_{H^1(R^d)} \leq C'\|u\|_{\mathcal{BC}^1(R^{d+1})}$ . Q.E.D.

**Theorem (3.3).** *If  $1 < p < \infty$  and  $f \in L^p(R^d)$ , then  $u_f \in \mathcal{IC}^p(R^{d+1})$  and  $\|u_f\|_{\mathcal{BC}^p(R^{d+1})} \leq C_p\|f\|_{L^p(R^d)}$ , where  $C_p$  depends only on  $p$ . If  $f \in H^1(R^d)$  and  $u_j = u_{R_j, \beta}$ ,  $0 \leq j \leq d$ , then  $u_j \in \mathcal{IC}^1(R^{d+1})$  and  $\|u_j\|_{\mathcal{BC}^1(R^{d+1})} \leq C\|f\|_{H^1(R^d)}$ , where  $C$  depends only on  $d$ .*

**Proof.** First suppose  $f \in L^p(R^d)$  and  $1 < p < \infty$ . Then, by Doob's inequality,

$$E_z \left[ \sup_{0 \leq t \leq \xi} |u_f(z(t))|^p \right] \leq C_p^p E_z [|f(x(\xi))|^p]$$

(6) We have used the fact that convolution with  $p_y$  is selfadjoint.

for all  $z \in R^{d+1}$ . Hence

$$\int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |u_f(z(t))|^p \right] dx < C_p^p \int_{R^d} p_y * |f|^p(x) dx = C_p^p \|f\|_{L^p(R^d)}^p.$$

Next suppose  $f \in H^1(R^d) \cap L^2(R^d)$  and set  $u_j = u_{R_j f}$ ,  $0 \leq j \leq d$ . By the theorem of Stein and Weiss mentioned at the end of §2,  $F = (\sum_0^d u_j^2)^{\alpha/2}$  is subharmonic in  $R_+^{d+1}$  for some  $0 < \alpha < 1$ . Hence, by Doob's inequality,

$$E_z \left[ \sup_{0 \leq t < \zeta} |u_j(z(t))| \right] \leq E_z \left[ \sup_{0 \leq t < \zeta} |F(z(t))|^{1/\alpha} \right] \leq C_{1/\alpha}^{1/\alpha} E_z [|F(z(\zeta))|^{1/\alpha}]$$

and therefore

$$\begin{aligned} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |u_j(z(t))| \right] dx &\leq C \int_{R^d} E_{x,y} \left[ \sum_0^d |R_j f(x(\zeta))| \right] dx \\ &= C \|f\|_{H^1(R^d)}. \end{aligned}$$

Since  $H^1(R^d) \cap L^2(R^d)$  is dense in  $H^1(R^d)$ , this completes the proof. Q.E.D.

**Corollary (3.3.1).** For each  $1 < p < \infty$ , the mapping  $f \rightarrow u_f$  is an isomorphism from  $L^p(R^d)$  onto  $\mathcal{H}^p(R_+^{d+1})$ . Also,  $f \rightarrow u_f$  is an isomorphism from  $H^1(R^d)$  onto  $\mathcal{H}^1(R_+^{d+1})$ . Finally,  $R_j$ ,  $1 \leq j \leq d$ , maps  $H^1(R^d)$  continuously into itself.

**Corollary (3.3.2).** For each  $1 < p < \infty$ ,  $H^1(R^d) \cap L^\infty(R^d)$  is dense in  $L^p(R^d)$ . Moreover, if  $f \in L^p(R^d) \cap L^\infty(R^d)$ , then there exists a sequence  $\{f_n\} \subseteq H^1(R^d) \cap L^\infty(R^d)$  such that  $f_n \rightarrow f$  in  $L^p(R^d)$  and  $\sup_n \|f_n\|_{L^\infty(R^d)} \leq 2 \|f\|_{L^\infty(R^d)}$ .

**Proof.** Let  $f \in L^p(R^d)$  for some  $1 < p < \infty$ . Define  $\eta(t) = u_f(z(t))$ ,  $0 \leq t < \zeta$ , and  $\eta_\alpha(t) = \eta(t \wedge \tau_\alpha)$ ,  $0 \leq t < \zeta$ , where  $\tau_\alpha = (\inf\{t \geq 0: |\eta(t) - \eta(0)| \vee |x(t)| \geq \alpha\}) \wedge \zeta$ . Clearly  $\Pi_y \eta_\alpha \in H^1(R^d) \cap L^\infty(R^d)$  for all  $\alpha > 0$  and  $y > 0$ . Moreover,  $f - \Pi_y \eta_\alpha = u_f(\cdot, 2y) + \Pi_y(\eta - \eta_\alpha)$ . Since  $u_f(\cdot, 2y) \rightarrow 0$  in  $L^p(R^d)$  (\*) as  $y \rightarrow \infty$ , it remains only to show that  $\|\Pi_y(\eta - \eta_\alpha)\|_p \rightarrow 0$  as  $\alpha \rightarrow \infty$  for each  $y > 0$ . But

$$\|\Pi_y(\eta - \eta_\alpha)\|_p^p \leq 2^p \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)|^p, \tau_\alpha < \zeta \right] dx \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty$$

since  $\tau_\alpha \nearrow \zeta$  (a.s.,  $P_z$ ) for all  $z \in R_+^{d+1}$ . Q.E.D.

**Remark (3.1).** A slight variation of the reasoning just given yields the following multi-dimensional analogue of the F. and M. Riesz theorem. Let  $u_0$  be a harmonic function in  $R_+^{d+1}$ . Then  $u_0 = u_f$  for some  $f \in H^1(R^d)$  if and only if there exist functions  $u_1, \dots, u_d$  which are harmonic in  $R_+^{d+1}$  such that the  $(d + 1)$ -triple  $(u_0, \dots, u_d)$  satisfies the generalized Cauchy-Riemann equation of M. Riesz in  $R_+^{d+1}$  and

(\*) This is obvious, since  $p_y \rightarrow 0$  in  $L^q$  and  $\|p_y\|_{L^1(R^d)} = 1$ .

$$\sup_{y>0} \int_{R^d} \left( \sum_0^d u_j^2(x, y) \right)^{1/2} dx < \infty.$$

The “only if” statement is obvious; simply take  $u_j = u_{R_j f}$ . To prove the “if” assertion it is enough to show that  $u_0 \in \mathfrak{H}^1(R_+^{d+1})$ . But by the theorem of Stein and Weiss at the end of §2, we know that  $(\sum_0^d u_j^2(x, y))^{1/2}$  is subharmonic in  $R_+^{d+1}$  for some  $0 < \alpha < 1$ . Hence, reasoning just as we did in the proof of Theorem (3.3),

$$\begin{aligned} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta_h} |u_0(z(t))| \right] dx &< C_{1/\alpha} \int_{R^d} E_{x,y} \left[ \left( \sum_0^d u_j^2(x(\zeta_h), h) \right)^{1/2} \right] dx \\ &= C_{1/\alpha} \int_{R^d} \left( \sum_0^d u_j^2(x, h) \right)^{1/2} dx \end{aligned}$$

for all  $0 < h < y$ , where  $\zeta_h = \inf\{t \geq 0: y(t) \leq h\}$ .

**Remark (3.2).** We have shown that if  $f \in L^1(R^d)$  satisfies

$$(*) \quad \sup_{y>0} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |u_f(z(t))| \right] dx < \infty,$$

then  $f \in H^1(R^d)$ . In particular,  $\int_{R^d} f(x) dx = 0$ . In fact, using the lemma on p. 225 of Stein [10], it is easy to show that  $u_f(\cdot, y) \rightarrow 0$  in  $L^1(R^d)$  as  $y \rightarrow \infty$ , and therefore, since the  $R_j$  are continuous on  $H^1(R^d)$  into itself and commute with convolution by  $p_y$ ,  $u_f(\cdot, y) \rightarrow 0$  in  $H^1(R^d)$ . However, we have found no direct probabilistic proof of any of these properties of  $f$  directly from (\*). It would be interesting to find such a proof.

**4. An interpolation theorem.** Let  $\mathfrak{H}^\infty(\Omega)$  stand for the class of  $\eta: [0, \infty) \times \Omega \rightarrow R$  such that  $\langle \eta(t) - \eta(0), \mathfrak{M}_t, P_z \rangle$  is a continuous local martingale on  $[0, \zeta]$  for each  $z \in R_+^{d+1}$  and for which there exists a number  $B < \infty$  with the property that

$$\sup_{z \in R_+^{d+1}} P_z \left( \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)| > B \right) = 0.$$

**Lemma (4.1).** Let  $\langle X, \mathfrak{B}, \lambda \rangle$  be a measure space and suppose  $\mathfrak{T}$  is a subadditive mapping of  $\mathfrak{H}^1(\Omega) \cap \mathfrak{H}^\infty(\Omega)$  into a measurable function on  $\langle X, \mathfrak{B} \rangle$ . Further, assume that there exist numbers  $A, B$ , and  $1 < p < \infty$  such that

$$\lambda(|\mathfrak{T}\eta| \geq \gamma) \leq \frac{A}{\gamma} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)| \right] dx$$

and

$$\lambda(|\mathfrak{T}\eta| \geq \gamma) \leq \frac{B}{\gamma^p} \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)|^p \right] dx,$$

for all  $\gamma > 0$  and some  $y > 0$ . Then for each  $1 < r < p$ ,  $\mathfrak{F}$  has a unique extension to  $\mathfrak{K}^r(\Omega)$  such that

$$(4.1) \quad \|\mathfrak{F}\eta\|_{L^r(\lambda)} \leq C \left( \int_{\mathbb{R}^d} E_{x,y} \left[ \sup_{0 \leq t < \zeta} |\eta(t) - \eta(0)|^r \right] dx \right)^{1/r}.$$

The constant  $C$  in (4.1) depends only on  $p, r, A$ , and  $B$ .

**Proof.** Since  $\mathfrak{K}^1(\Omega) \cap \mathfrak{K}^\infty(\Omega)$  is dense in  $\mathfrak{K}^r(\Omega)$  with respect to the norm on the right-hand side of (4.1) (cf. the proof of Corollary (3.3.2)), it suffices to prove (4.1) for  $\eta \in \mathfrak{K}^1(\Omega) \cap \mathfrak{K}^\infty(\Omega)$ . Let such an  $\eta$  be given. For convenience we will use  $\|\eta\|$  to denote  $\sup |\eta(t) - \eta(0)|$ . By subadditivity

$$\lambda(|\mathfrak{F}\eta| \geq \gamma) \leq \lambda(|\mathfrak{F}\eta^{(\gamma)}| \geq \gamma/2) + \lambda(|\mathfrak{F}\eta_{(\gamma)}| \geq \gamma/2),$$

where  $\eta^{(\gamma)}(t) = \eta(t \wedge \tau_\gamma)$ ,  $0 \leq t < \zeta$ ,  $\eta_{(\gamma)} = \eta - \eta^{(\gamma)}$ , and  $\tau_\gamma = \inf\{t \geq 0: |\eta(t)$

$$\begin{aligned} \lambda(|\mathfrak{F}\eta^{(\gamma)}| \geq \gamma/2) &\leq \frac{2^p B}{\gamma^p} \int_{\mathbb{R}^d} E_{x,y} [ \|\eta\|^p \wedge \gamma^p ] dx \\ &\leq \frac{2^p B}{\gamma^p} \int_{\mathbb{R}^d} E_{x,y} [ \|\eta\|^p, \|\eta\| \leq \gamma ] dx + 2^p B \int_{\mathbb{R}^d} P_{x,y} (\|\eta\| \geq \gamma) dx. \end{aligned}$$

Hence

$$\begin{aligned} r \int_0^\infty \gamma^{r-1} \lambda(|\mathfrak{F}\eta^{(\gamma)}| \geq \gamma/2) d\gamma &\leq 2^p r B \int_{\mathbb{R}^d} dx \int_0^\infty \gamma^{r-p-1} E_{x,y} [ \|\eta\|^p, \|\eta\| \leq \gamma ] d\gamma \\ &\quad + 2^p r B \int_{\mathbb{R}^d} dx \int_0^\infty \gamma^{r-1} P_{x,y} (\|\eta\| \geq \gamma) d\gamma \\ &\leq 2^p r p B \int_{\mathbb{R}^d} dx \int_0^\infty \gamma^{r-p-1} d\gamma \int_0^\gamma \alpha^{p-1} P_{x,y} (\|\eta\| \geq \alpha) d\alpha \\ &\quad + 2^p B \int_{\mathbb{R}^d} E_{x,y} [ \|\eta\|^r ] dx \\ &= \frac{2^p r p B}{p-r} \int_{\mathbb{R}^d} dx \int_0^\infty \alpha^{r-1} P_{x,y} (\|\eta\| \geq \alpha) d\alpha \\ &\quad + 2^p B \int_{\mathbb{R}^d} E_{x,y} [ \|\eta\|^r ] dx \\ &= 2^p B \frac{2p-r}{p-r} \int_{\mathbb{R}^d} E_{x,y} [ \|\eta\|^r ] dx. \end{aligned}$$

Also,

$$\lambda(|\mathfrak{F}\eta_{(\gamma)}| \geq \gamma/2) \leq \frac{4A}{\gamma} \int_{\mathbb{R}^d} E_{x,y} [ \|\eta\|, \|\eta\| \geq \gamma ] dx,$$

and so

$$\begin{aligned}
 & r \int_0^\infty \gamma^{r-1} \lambda(|\nabla \eta_{(\gamma)}| \geq \gamma/2) d\gamma \\
 & \leq 4Ar \int_{R^d} dx \int_0^\infty \gamma^{r-2} E_{x,y} [|\eta|, \|\eta\| \geq \gamma] d\gamma \\
 & = 4Ar \int_{R^d} dx \int_0^\infty d\gamma \gamma^{r-2} \left[ \int_\gamma^\infty P_{x,y}(\|\eta\| \geq \alpha) d\alpha + \gamma P_{x,y}(\|\eta\| \geq \gamma) \right] \\
 & \leq \frac{8Ar}{r-1} \int_{R^d} dx \int_0^\infty \alpha^{r-1} P_{x,y}(\|\eta\| \geq \alpha) d\alpha \\
 & = \frac{8A}{r-1} \int_{R^d} E_{x,y} [|\eta|^r] dx.
 \end{aligned}$$

Combining these, we arrive at (4.1). Q.E.D.

**Theorem (4.1).** *Let  $\langle X, \mathfrak{B}, \lambda \rangle$  be a  $\sigma$ -finite measure space and suppose  $T$  is a subadditive mapping of  $H^1(R^d) \cap L^\infty(R^d)$  into measurable functions on  $\langle X, \mathfrak{B} \rangle$ . If there exist numbers  $A, B$ , and  $1 < p < \infty$  such that*

$$\lambda(|Tf| \geq \gamma) \leq (A/\gamma) \|f\|_{H^1(R^d)}$$

and

$$\lambda(|Tf| \geq \gamma) \leq (B/\gamma^p) \|f\|_{L^p(R^d)}^p$$

for all  $\gamma > 0$ , then for  $1 < r < p$ ,  $T$  has a unique extension to  $L^r(R^d)$  such that

$$(4.2) \quad \|Tf\|_{L^r(\lambda)} \leq C \|f\|_{L^r(R^d)}.$$

The constant  $C$  depends only on  $A, B, p, r$ , and  $d$ .

**Proof.** By Corollary (3.3.2),  $H^1(R^d) \cap L^\infty(R^d)$  is dense in  $L^r(R^d)$ . Hence it suffices to prove (4.2) for  $f \in H^1(R^d) \cap L^\infty(R^d)$ . Also, we may assume  $\lambda$  is finite.

Given  $y > 0$ , define  $\mathfrak{F}_y = T \circ \Pi_y$  on  $\mathfrak{C}^1(\Omega) \cap \mathfrak{C}^\infty(\Omega)$ . Then it is easily checked that  $\mathfrak{F}_y$  satisfies the hypotheses of Lemma (4.1). Hence  $\mathfrak{F}_y$  has a unique extension to  $\mathfrak{H}^r(\Omega)$  such that

$$(4.3) \quad \|\mathfrak{F}_y \eta\|_{L^r(\lambda)} \leq C \left( \int_{R^d} E_{x,y} \left[ \sup_{0 \leq t < \tau} |\eta(t) - \eta(0)|^r \right] dx \right)^{1/r}.$$

The constant  $C$  in (4.3) is independent of  $y > 0$ .

Now let  $f \in H^1(R^d) \cap L^\infty(R^d)$  be given. Then  $f = u_f(\cdot, 2y) + \Pi_y \eta_f$  and so  $|Tf| \leq |Tu_f(\cdot, 2y)| + |\mathfrak{F}_y \eta_f|$ . Since  $u_f(\cdot, 2y) \rightarrow 0$  in  $L^p(R^d)$ , the hypotheses of the theorem guarantee that  $Tu_f(\cdot, 2y) \rightarrow 0$  in  $L^r(\lambda)$  as  $y \rightarrow \infty$ . Hence, by Fatou's lemma,

$$\begin{aligned}
 \|Tf\|_{L^r(\lambda)} & \leq \liminf_{y \rightarrow \infty} \|\mathfrak{F}_y \eta_f\|_{L^r(R^d)} \leq C \|\eta_f\|_{\mathfrak{B}C^r(\Omega)} \\
 & \leq 2C \|u_f\|_{\mathfrak{B}C^r(R^d)} \leq C' \|f\|_{L^r(R^d)}. \quad \text{Q.E.D.}
 \end{aligned}$$

**Corollary (4.1.1).** *For each  $1 \leq j \leq d$  and  $1 < p < \infty$ ,  $R_j$  has a unique continuous extension to  $L^p(\mathbb{R}^d)$  into itself. In particular,  $H^p(\mathbb{R}^d)$  is isomorphic to  $L^p(\mathbb{R}^d)$  for  $1 < p < \infty$ .<sup>(8)</sup>*

**Corollary (4.1.2).** *Suppose  $T$  is a linear map on  $H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  into measurable functions on  $\mathbb{R}^d$ . Assume that*

$$\|Tf\|_{L^2(\mathbb{R}^d)} \leq A\|f\|_{L^2(\mathbb{R}^d)}$$

and

$$\|T^*f\|_{\text{B.M.O.}(\mathbb{R}^d)} \leq B\|f\|_{L^\infty(\mathbb{R}^d)}$$

where  $T^*$  is the adjoint of  $T$  on  $L^2(\mathbb{R}^d)$ . Then  $T$  has a unique bounded extension as a map from  $H^r(\mathbb{R}^d)$  into  $L^r(\mathbb{R}^d)$  for  $1 \leq r \leq 2$ . Moreover, if  $T$  commutes with all the  $R_j$ ,  $1 \leq j \leq d$ , then  $T$  maps  $H^1(\mathbb{R}^d)$  continuously into itself.

**Proof.** Suppose  $f \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  is given. Given  $\phi \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , we have

$$\left| \int_{\mathbb{R}^d} Tf(x)\phi(x) dx \right| \leq C\|f\|_{H^1(\mathbb{R}^d)}\|\phi\|_{L^\infty(\mathbb{R}^d)},$$

and therefore  $T$  extends uniquely as a bounded map of  $H^1(\mathbb{R}^d)$  into  $L^1(\mathbb{R}^d)$ . We now apply Theorem (4.1) to conclude that  $T$  has a unique bounded extension as a map from  $H^r(\mathbb{R}^d)$  into  $L^r(\mathbb{R}^d)$ ,  $1 < r \leq 2$ . Finally, if  $T$  commutes with all the  $R_j$ , then

$$\|R_j Tf\|_{L^1(\mathbb{R}^d)} = \|TR_j f\|_{L^1(\mathbb{R}^d)} \leq C\|R_j f\|_{H^1(\mathbb{R}^d)} \leq C'\|f\|_{H^1(\mathbb{R}^d)}$$

for all  $f \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  and  $1 \leq j \leq d$ . Q.E.D.

**Corollary (4.1.3).** *If  $K \in L^1(\mathbb{R}^d)$  satisfies the conditions of Lemma (2.2), then  $\|K * f\|_{H^r(\mathbb{R}^d)} \leq C_r(B)\|f\|_{H^r(\mathbb{R}^d)}$ ,  $1 \leq r < \infty$ , where  $C_r(B)$  depends only on  $r$  and  $B$ .*

**Remark (4.1).** Corollary (4.1.3) contains the hard part of some of Hörmander's results in [5]. In particular, his result about "almost  $L^1$ " operators follows from Corollary (4.1.3) and the type of reasoning used in showing that  $R_j$  is bounded on  $L^\infty(\mathbb{R}^d)$  to  $\text{B.M.O.}(\mathbb{R}^d)$ .

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK, NEW YORK 10012 (Current address of S. R. S. Varadhan)

*Current address* (D. Stroock): Department of Mathematics, University of Colorado, Boulder, Colorado 80302