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$H^{r,\infty}(R)$- AND $W^{r,\infty}(R)$-SPLINES

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ABSTRACT. Let E be a subset of R the real line and $f: E \to R$. Necessary and sufficient conditions are derived for $\inf(\|D^r x\|_{L^\infty}: x|_E = f)$ to have a solution. When restricted to quasi-uniform partitions E, necessary and sufficient conditions are derived for the solution to be in L^∞. For finite partitions E it is shown that a solution to the L^∞ infimum problem can be obtained by solving $\inf(\|D^r x\|_{L^p}: x|_E = f)$ and letting p go to infinity. In this way it was discovered that solutions to the L^∞ problem could be chosen to be piecewise polynomial (of degree r or less). The solutions to the L^p problem are called $H^{r,p}$-splines and were studied extensively by Golomb in [3].

1. **Introduction.** Let $H^{r,p}$, $r = 1, 2, \ldots$, and $1 \leq p \leq \infty$, denote the space of real-valued functions x defined on the real line R which have an absolutely continuous $(r-1)$th derivative and in addition $D^r x \in L^p(R) \equiv L^p$ (here $D^r \equiv d^r/dt^r$). In [3] Golomb considered the following problem: Given $E \subset R$, $f: E \to R$, and $1 < p < \infty$ solve

$$\inf_{x \in H^{r,p}, x|_E = f} \|D^r x\|_{L^p}. \tag{1.1}$$

A set $E = \{t_i\}_{i=-\infty}^{\infty} \subset R$ is called quasi-uniform if there is a $\delta > 0$ so that $\delta < t_{i+1} - t_i < 1/\delta$ for all integers i. For quasi-uniform partitions E, Golomb proved [3]:

Theorem 1.1. *Let $E = \{t_i\}_{i=-\infty}^{\infty}$ be a quasi-uniform partition and suppose $f: E \to R$. Then* (1.1) *has a* (*unique*) *solution if and only if the* (r)*th divided differences of f are in l^p. That is,*

$$\sum_{i=-\infty}^{\infty} |f(t_i, \ldots, t_{i+r})|^p < \infty. \tag{1.2}$$

The author proved in [7] the following.

Theorem 1.2. *Let $E = \{t_i\}_{i=-\infty}^{\infty}$ be a quasi-uniform partition and $f: E \to R$. Then* (1.1) *has a* (*unique*) *solution x which is in L^p if and only if $\{f(t_i)\}_{i=-\infty}^{\infty} \in l^p$.*

This paper extends the two above results to the case $p = \infty$ (except of course we do not get uniqueness).

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For any $y \in H^{r,p}$ $(1 < p < \infty)$ and $f \equiv y\,|_E$ we let $S_E^{r,p}y$ be a solution to (1.1) (note the solution is unique if and only if cardinality $E \geq r$). For finite E we let p go to infinity and extract a subsequence which converges in H^{r,p_o}, $1 < p_o < \infty$, to a solution of (1.1) with $p = \infty$.

We will call $S_E^{r,p}x$ the $H^{r,p}$-spline interpolating $x\,|_E$, and if $S_E^{r,p}x$ is in $W^{r,p} \equiv \{H^{r,p} \cap L^p\}$ then we call $S_E^{r,p}x$ the $W^{r,p}$-spline interpolating $x\,|_E$. The idea of letting $p \to \infty$ was suggested by Mangasarian and Schumaker in [5]. Problems similar to (1.1) with $p = \infty$ and interpolation conditions on the derivatives have been studied by Favard [1] and more recently by Glaeser [2]. In particular, Glaeser obtains solutions which are piecewise polynomial. Jean Merrien in [6] makes use of divided differences to decide whether a function defined on a closed subset of R has a C^m extension. Finally, Jerome and Schumaker in [4] present some closely related ideas by considering the problem of determining whether a function is in $H^{r,p}$, $1 \leq p \leq \infty$.

Existence of $H^{r,\infty}$-splines. Suppose $E = \{t_i\}_{i=0}^k$, $t_i < t_{i+1}$, and $k > r$. For $k \leq r$ the existence of an $H^{r,\infty}$-spline is obvious. Suppose $x \in H^{r,1} \cap H^{r,\infty}$. We will prove

Theorem 2.1. *There is a sequence $\{p_n\}$, $p_n \to \infty$, so that for any $1 \leq p < \infty$*

$$S_E^{r,p_n}x \to x_* \quad \text{in } H^{r,p} \text{ as } p_n \to \infty, \tag{2.1}$$

where x_* is an $H^{r,\infty}$-spline interpolating $x\,|_E$.

This theorem will follow from several lemmas which will now be presented. For any p satisfying $1 < p < \infty$ we know [3] that $S_E^{r,p}x$ is a polynomial of degree $r-1$ or less on the infinite intervals $(-\infty, t_o)$, (t_k, ∞). On the interval $[t_o, t_k]$ we have the representation [3]

$$S_E^{r,p}x = \sum_{i=0}^{k-1} \chi_{[t_i,t_{i+1})}\{P_p^i + Q_p^i\} \tag{2.2}$$

where

$$\begin{aligned} &\text{(i)} && P_p^i(t) = \sum_{j=0}^{r-1} \frac{D^j S_E^{r,p}x(t_i)}{j!}(t - t_i)^j, \\ (2.3)\quad &\text{(ii)} && Q_p^i(t) = \int_{t_1}^{t} \int_{t_1}^{\tau_{-1}} \cdots \int_{t_1}^{\tau_{r-1}} |g_p^i(\tau_r)|^{1/(p-1)} \cdot \operatorname{sgn}(g_p^i(\tau_r))\, d\tau_r \cdots d\tau_1, \\ &\text{(iii)} && g_p^i \in \mathcal{P}^r \equiv \{\text{the set of polynomials of degree } r-1 \text{ or less}\}. \end{aligned}$$

Lemma 2.1. *The set $\{\|D^r S_E^{r,p}x\|_{L^\infty} : 1 < p_o \leq p < \infty\}$ is bounded.*

Proof. Since $S_E^{r,p}x$ depends only on the values of x on E we can assume that

$$(\operatorname{supp} D^r x) \subset [t_o, t_k]. \tag{2.4}$$

Using the minimality of the spline, we get

$$(2.5) \quad \|D^r S_E^{r,p} x\|_{L^p} \le \|D^r x\|_{L^p} \le \|D^r x\|_{L^\infty}(t_k - t_o)^{1/p}, \qquad 1 < p < \infty.$$

Setting $M_1 = \|D^r x\|_{L^\infty} \cdot \max\{(t_k - t_o)^{1/p_o}, 1\}$ we have

$$(2.6) \qquad \|D^r S_E^{r,p} x\|_{\underline{L^p}} \le M_1, \qquad p_o \le p < \infty.$$

Now clearly for $i = 0, \ldots, k-1$

$$(2.7) \qquad \|D^r S_E^{r,p} x\|_{L^p}^{p-1} \ge \left(\int_{t_i}^{t_{i+1}} |g_p^i|^{p/(p-1)} \right)^{(p-1)/p} \ge M_2 \|g_p^i\|_{L^\infty(t_i, t_{i+1})}.$$

We may assume that $M_2 > 0$ is independent of p between p_o and ∞ (e.g. consider the continuous map $(g,q) \to \|g\|_{L^q(t_i,t_{i+1})}$ where $q \in [1, p_o/(p_o - 1)]$ and $g \in \mathcal{P}^r$ which satisfy $\|g\|_{L^\infty(t_i,t_{i+1})} = 1$). Thus we have, taking $(p-1)$th roots,

$$(2.8) \qquad \begin{aligned} \|D^r S_E^{r,p} x\|_{L^p} &\ge M_2^{1/(p-1)} \max_{0 \le i \le k-1} \| \, |g_p^i|^{1/(p-1)} \|_{L^\infty(t_i,t_{i+1})} \\ &= M_2^{1/(p-1)} \|D^r S_E^{r,p} x\|_{L^\infty}. \end{aligned}$$

Setting $M_3 = \sup_{p \ge p_o} M_2^{-1/(p-1)} M_1$ we get

$$(2.9) \qquad M_3 \ge \|D^r S_E^{r,p} x\|_{L^\infty},$$

thereby completing the proof of the lemma.

The following technical result will be of use to us.

Lemma 2.2. *Let $I \subset R$ be a compact interval and m denote Lebesgue measure. Suppose, as before, that $\mathcal{P}^r = \mathrm{sp}\{1, t, \ldots, t^{r-1}\}$ and set*

$$(2.10) \qquad \delta_r^\alpha = \inf_{g \in \mathcal{P}^r, \|g\|_{L^\infty(I)} = 1} m\{t \in I \colon |g(t)| \ge \alpha\}.$$

Then

$$(2.11) \qquad \delta_r^\alpha \to m(I) \quad \text{as } \alpha \to 0.$$

This lemma follows directly from the local compactness of $\mathcal{P}^r$ and the analytic nature of the polynomials. As a simple corollary, we obtain

Corollary 2.1. *With I as in Lemma 2.2, we have*

$$(2.12) \qquad \inf_{g \in \mathcal{P}^r} m\{t \in I \colon |g(t)| \ge \alpha \|g\|_{L^\infty(I)}\} \to m(I) \quad \text{as } \alpha \to 0.$$

The interest in the preceding lemma and corollary lies in the fact that by applying them to a subsequence of $\{D^r S_E^{r,p} x\}_{p \ge p_o}$ we can prove that this subsequence converges in measure.

Lemma 2.3. *There is a sequence* $\{p_n\}$, $p_n \to \infty$, *so that*

$$(2.13) \qquad |D^r S_E^{r,p_n} x(t)| \xrightarrow{m} \sum_{i=0}^{k-1} K_i \chi_{[t_i,t_{i+1})}(t).$$

The symbol $\to^m$ means converges in (Lebesgue) measure, K_i are nonnegative constants, and $\chi_{[t_i,t_{i+1})}$ is the characteristic function of the interval $[t_i, t_{i+1})$.

Proof. From Lemma 2.1 we know that there is a constant M_3 so that

$$(2.14) \qquad \|D^r S_E^{r,p} x\|_{L^\infty} \le M_3 < \infty, \qquad p \ge p_o.$$

Thus, clearly,

$$(2.15) \qquad \|D^r S_E^{r,p} x\|_{L^\infty[t_i,t_{i+1})} \le M_3, \qquad i = 0, \dots, k-1.$$

We choose a sequence of numbers $\{p_n\}_{n=1}^\infty$ going to infinity so that for $i = 0, 1, \dots, k-1$

$$(2.16) \qquad \|D^r S_E^{r,p_n} x\|_{L^\infty[t_i,t_{i+1})} \to K_i \quad \text{as } p_n \to \infty.$$

Furthermore by (2.3) it follows that

$$(2.17) \qquad \|\, |g_{p_n}^i|^{1/(p_n-1)} \|_{L^\infty[t_i,t_{i+1})} \to K_i \quad \text{as } p_n \to \infty.$$

For $i = 0, 1, \dots, k-1$ and $1 > \varepsilon > 0$ given, set

$$(2.18) \qquad A_{\varepsilon,n}^i = \{t \in [t_i, t_{i+1}) \colon |g_{p_n}^i(t)| \ge \varepsilon \|g_{p_n}^i\|_{L^\infty[t_i,t_{i+1})}\}.$$

Then for $t \in A_{\varepsilon,n}^i$ we have

$$(2.19) \qquad \begin{aligned} \|D^r S_E^{r,p_n} x\|_{L^\infty[t_i,t_{i+1})} &\ge |g_{p_n}^i(t)|^{1/(p_n-1)} \\ &\ge \varepsilon^{1/(p_n-1)} \|D^r S_E^{r,p_n} x\|_{L^\infty[t_i,t_{i+1})}. \end{aligned}$$

Now as $p_n \to \infty$ both end terms of (2.19) converge to K_i. Since Corollary 2.1 tells us that as $\varepsilon \to 0$, $m\{[t_i, t_{i+1}] \setminus A_{\varepsilon,n}^i\} \to 0$, it follows that

$$(2.20) \qquad |D^r S_E^{r,p_n} x| \xrightarrow{m} \sum_{i=0}^{k-1} K_i \chi_{[t_i,t_{i+1})}(t).$$

As a corollary we obtain

Corollary 2.2. *There is a sequence* $\{p_n\}$, $p_n \to \infty$, *so that*

$$(2.21) \qquad (D^r S_E^{r,p_n} x) \xrightarrow{m} \sum_{i=0}^{k-1} K_i (\operatorname{sgn} g_*^i) \chi_{[t_i,t_{i+1})},$$

for some $g_*^i \in \mathcal{P}^r$ (*here* sgn *is the usual signum function*).

Proof. We first choose a sequence $\{q_n\}$ with $q_n \to \infty$ so that (2.13) holds. We note that

$$D^r S_E^{r,q_n} x = \sum_{i=0}^{k-1} \chi_{[t_i,t_{i+1})} |g_{q_n}^i|^{1/(q_n-1)} \operatorname{sgn} g_{q_n}^i \tag{2.22}$$

where $g_{q_n}^i \in \mathcal{P}^r$. We choose a subsequence $\{p_n\}$ of $\{q_n\}$ so that for $i = 0, 1, \ldots, k-1$

$$\chi_{[t_i,t_{i+1})} \operatorname{sgn} g_{p_n}^i \xrightarrow{m} \chi_{[t_i,t_{i+1})} \operatorname{sgn} g_*^i \tag{2.23}$$

for some $g_*^i \in \mathcal{P}^r$. Clearly then for $i = 0, 1, \ldots, k-1$

$$\chi_{[t_i,t_{i+1})} |g_{p_n}^i|^{1/(p_n-1)} \operatorname{sgn} g_{p_n}^i \xrightarrow{m} K_i(\operatorname{sgn} g_*^i)\chi_{[t_i,t_{i+1})} \tag{2.24}$$

and adding the individual results of (2.24) yields (2.21).

The limit in (2.21) is now our candidate for the (r)th derivative of an $H^{r,\infty}$-spline interpolating $x|_E$. Thus we must prove the following:

Lemma 2.4. *With the same notation as in Corollary 2.2 it is the case that*

$$\left\| \sum_{i=0}^{k-1} K_i(\operatorname{sgn} g_*^i)\chi_{[t_i,t_{i+1})} \right\|_{L^\infty} \le \|D^r x\|_{L^\infty}. \tag{2.25}$$

Proof. Let $\{p_n\}$ be as in Corollary 2.2; then by the minimality of the $H^{r,p}$-spline we have

$$\|D^r S_E^{r,p_n} x\|_{L^{p_n}} \le \|D^r x\|_{L^\infty}(t_k - t_o)^{1/p_n}. \tag{2.26}$$

Since $D^r S_E^{r,p_n} x \to^m \sum_{i=0}^{k-1} K_i \operatorname{sgn} g_*^i \chi_{[t_i,t_{i+1})}$, given $\varepsilon > 0$, the sets

$$B_{\varepsilon,n}^i = \left\{ t \in [t_i, t_{i+1}) \colon \left| D^r S_E^{r,p_n} x(t) - \sum_{i=0}^{k-1} K_i \operatorname{sgn} g_*^i(t)\chi_{[t_i,t_{i+1})}(t) \right| < \varepsilon \right\} \tag{2.27}$$

have measure larger than $\frac{1}{2}(t_{i+1} - t_i)$, for $i = 0, \ldots, k-1$ and p_n large enough. Let us calculate for large p_n

$$\begin{aligned}\left(\int_{t_i}^{t_{i+1}} |D^r S_E^{r,p_n} x|^{p_n} \right)^{1/p_n} &\ge \left(\int_{B_{\varepsilon,n}^i} |D^r S_E^{r,p_n} x|^{p_n} \right)^{1/p_n} \\ &\ge [\tfrac{1}{2}(t_{i+1} - t_i)]^{1/p_n}(K_i - \varepsilon).\end{aligned} \tag{2.28}$$

As $p_n \to \infty$, $\beta^{1/p_n} \to 1$ if $\beta > 0$. Thus by (2.26) and (2.28) we have

$$K_i - \varepsilon \le \|D^r x\|_{L^\infty}, \qquad \varepsilon > 0. \tag{2.29}$$

This of course implies (2.25).

Since the only restriction on x was that $x \in H^{r,1} \cap H^{r,\infty}$, we may conclude

Corollary 2.3. *For all* $z \in H^{r,\infty}$ *which satisfy* $z|_E = x|_E$ *we have* (*with notation as above*)

$$\left\| \sum_{i=0}^{k-1} K_i \operatorname{sgn} g_*^i \chi_{[t_i,t_{i+1})} \right\|_{L^\infty} \leq \|D^r z\|_{L^\infty}. \tag{2.30}$$

We are now in a position to prove Theorem 2.1. Given $x \in H^{r,1} \cap H^{r,\infty}$ there is a sequence $p_n \to \infty$ as in Corollary 2.2 so that (2.21) holds. For finite measure spaces convergence in measure and boundedness imply convergence in the L^p-norm, $1 \leq p < \infty$. Thus

$$D^r S_E^{r,p_n} x \to \sum_{i=0}^{k-1} K_i(\operatorname{sgn} g_*^i) X_{[t_i,t_{i+1})}, \quad \text{in } L^p,\ 1 \leq p < \infty. \tag{2.31}$$

Since $S_E^{r,p_n} x|_E = x|_E$ the usual arguments imply that $S_E^{r,p_n} x$ converges in $H^{r,p}$ $(1 \leq p < \infty)$, say to x_*. Clearly $x_* \in H^{r,\infty}$ and $x_*|_E = x|_E$. Corollary 2.3 implies that x_* is an $H^{r,\infty}$-spline interpolating $x|_E$. This completes the proof.

From (2.32) it is easy to see that the limit x_* of the splines $S_E^{r,p_n} x$ is piecewise polynomial of degree r with no more than $r - 1$ changes of sign in the (r)th derivative between interpolation values. Furthermore, between interpolation values the (r)th derivative is of constant absolute magnitude.

Although it appears to be necessary to go to a subsequence to prove Theorem 2.1, we do have the following continuity result.

Lemma 2.5. *With* E, x, *and* x_* *as above we have*

$$\lim_{p\to\infty} \|D^r S_E^{r,p} x\|_{L^p} = \|D^r x_*\|_{L^\infty}. \tag{2.32}$$

Proof. For $1 < p < \infty$

$$\begin{aligned} &\text{(i)} && \|D^r S_E^{r,p} x\|_{L^p} \leq \|D^r x_*\|_{L^p}, \\ &\text{(ii)} && \|D^r x_*\|_{L^p} \to \|D^r x_*\|_{L^\infty} \quad \text{as } p \to \infty. \end{aligned} \tag{2.33}$$

Therefore

$$\overline{\lim_{p\to\infty}} \|D^r S_E^{r,p} x\|_{L^p} \leq \|D^r x_*\|_{L^\infty}. \tag{2.34}$$

On the other hand if

$$\underline{\lim_{p\to\infty}} \|D^r S_E^{r,p} x\|_{L^p} = M < \|D^r x_*\|_{L^\infty}, \tag{2.35}$$

then it is easy to see that there is a subsequence $p_n \to \infty$ as in Corollary 2.2 so that

$$\begin{aligned} &\text{(i)} && S_E^{r,p_n} x \xrightarrow{m} x_{**} && \text{as } p_n \to \infty, \\ &\text{(ii)} && \|D^r S_E^{r,p_n} x\|_{L^{p_n}} \to M && \text{as } p_n \to \infty, \end{aligned} \tag{2.36}$$

where x_{**} is an $H^{r,\infty}$-spline interpolating $x\,|_E$. But then, it can be shown that

$$\|D^r x_{**}\|_{L^\infty} \leq M < \|D^r x_*\|_{L^\infty}, \tag{2.37}$$

which is a contradiction. Thus the lim sup and lim inf are both equal to $\|D^r x_*\|_{L^\infty}$ and the lemma is proved.

We collect these results in

Theorem 2.2. *Let $E \subset R$ be a finite set and $f: E \to R$. Then there is an $H^{r,\infty}$-spline x_* interpolating f which is the limit of a subsequence of $\{S_E^{r,p} x_*: 1 < p < \infty\}$ in the H^{r,p_0} norm $(1 \leq p_0 < \infty)$. Moreover, $|D^r x_*|$ is constant on the connected components of $R\backslash E$, and $D^r x_*$ changes sign at most $r-1$ times on the connected components of $R\backslash E$. $D^r x_*$ vanishes on the unbounded components of $R\backslash E$, and thus x_* is a polynomial of degree $r-1$ or less on these sets. Finally* $\lim_{p\to\infty} \|D^r S_E^{r,p} x_*\|_{L^p} = \|D^r x_*\|_{L^\infty}$.

3. **$H^{r,\infty}$- and $W^{r,\infty}$- splines on infinite sets.** Suppose $E = \{t_i\}_{i=-\infty}^{\infty}$, $t_{i+1} > t_i$, and $f: E \to R$. Set

$$\begin{aligned} &\text{(i)} \quad E_N = \{t_i\}_{i=-N}^{N}, \qquad N = 1, 2, \ldots, \\ &\text{(ii)} \quad f_N = f\,|_{E_N}. \end{aligned} \tag{3.1}$$

We let x_*^N be an extremal $H^{r,\infty}$-extension of f_N as described in Theorem 2.2. It is easy to see that

$$\|D^r x_*^N\|_{L^\infty} \leq \|D^r x_*^{N+1}\|_{L^\infty}, \tag{3.2}$$

and if y is any $H^{r,\infty}$-extension of f then

$$\|D^r y\|_{L^\infty} \geq \|D^r x_*^N\|_{L^\infty}. \tag{3.3}$$

Lemma 3.1. *If there is an $H^{r,\infty}$-extension y of f, then there is an $H^{r,\infty}$-spline interpolating f on E.*

Proof. Using Theorem 2.2 we write

$$D^r x_*^N = \sum_{i=-N}^{N} K_{i,N} \operatorname{sgn} g_{i,N} \chi_{[t_i,t_{i+1})} \tag{3.4}$$

where $g_{i,N} \in \mathcal{P}^r$. Since the set $\{K_{i,N}\}$ is bounded (see 3.3), we may select $\{N_k\}$ so that for all i:

$$K_{i,N_k} \operatorname{sgn} g_{i,N_k} \chi_{[t_i,t_{i+1})} \xrightarrow{m} K_i \operatorname{sgn}(g_i) \chi_{[t_i,t_{i+1})} \tag{3.5}$$

where $g_i \in \mathcal{P}^r$. Thus $\{x_*^{N_k}\}$ converges locally in $H^{r,p}$, $1 \leq p < \infty$, to say x_*. The function x_* is in $H^{r,\infty}$ because the set $\{K_i\}$ is bounded. Furthermore,

$$\|D^r x_*\|_{L^\infty} = \lim_{N\to\infty} \|D^r x_*^N\|_{L^\infty}; \tag{3.6}$$

this, in conjunction with (3.3), tells us that x_* is an extremal $H^{r,\infty}$-extension of f.

We can now prove a theorem which is similar to Theorem 3.2 of [3].

Theorem 3.1. *Let* $E = \{t_i\}_{i=-\infty}^{\infty}$, $\delta \leq t_{i+1} - t_i \leq 1/\delta$ *for some* $\delta > 0$. *Then there exists an* $H^{r,\infty}$ *-spline interpolating* f *on* E *if and only if*

$$\sup_{-\infty<i<\infty} |f(t_i, \ldots, t_{i+r})| < \infty, \tag{3.7}$$

where $f(t_i, \ldots, t_{i+r})$ *is the* (r)*th divided difference of* f *on the set* $\{t_i, \ldots, t_{i+r}\}$.

Proof. In the course of the proof of Theorem 3.1 of [3] Golomb defines the functions $G_i \in H^{r,\infty}$ which satisfy, for $i \geq 0$,

$$\text{(3.8)}\quad \begin{array}{llll} \text{(i)} & G_i(t_j) = 0, & & j = i+1, \ldots, i+r-1, \\ \text{(ii)} & G_i(t_j, \ldots, t_{j+r}) = \delta_{i,j}, & & j = 0, 1, 2, \ldots, \\ \text{(iii)} & G_i(t) = 0, & & t < t_i, \\ \text{(iv)} & D^r G_i(t) = 0, & t_{i+r} < t, & i = 0, 1, 2, \ldots, \\ \text{(v)} & \|D^r G_i(t)\|_{L^\infty} \leq C < \infty, & & i = 0, 1, 2, \ldots, \end{array}$$

and for $i < 0$

$$\text{(3.9)}\quad \begin{array}{lll} \text{(i)} & G_i(t_j) = 0, & j = i-1, \ldots, i-r+1, \\ \text{(ii)} & G_i(t_j, \ldots, t_{j-r}) = \delta_{ij}, & j = 0, -1, -2, \ldots, \\ \text{(iii)} & G_i(t) = 0, & t > t_j, \\ \text{(iv)} & D^r G_i(t) = 0, & t_{i-r} > t, \\ \text{(v)} & \|D^r G_i(t)\|_{L^\infty} \leq C < \infty, & i = -1, -2, \ldots. \end{array}$$

Let $g \in \mathcal{P}^{r-1}$ so that $g(t_i) = f(t_i)$ for $i = 0, 1, \ldots, r-1$, and suppose

$$H_n = g + \sum_{|i| \leq n} f(t_i, \ldots, t_{i+r}) G_i, \qquad n = 1, 2, \ldots. \tag{3.10}$$

Due to (3.8) and (3.9) the H_n have the properties

$$\text{(3.11)}\quad \begin{array}{ll} \text{(i)} & H_{n-1}(t) = H_n(t), \qquad t \in [t_{-n}, t_n],\ n = 1, 2, \ldots, \\ \text{(ii)} & |D^r H_n(t)| \leq C \sum_{i=j-r-1}^{j+r+1} |f(t_i, \ldots t_{i+r})| \quad \text{for } t \in [t_j, t_{j+1}]. \end{array}$$

If we denote by H the pointwise limit of the H_n as $n \to \infty$, it is easy to see that $H \in H^{r,\infty}$ and H interpolates f. Thus f has an $H^{r,\infty}$-extension and hence Lemma 3.1 implies that f has an $H^{r,\infty}$-spline interpolating f on E.

Conversely suppose that $\sup_{-\infty<i<\infty} |f(t_i, \ldots, t_{i+r})| = \infty$. For any function $x \in H^{r,\infty}$ we have

$$
(3.12)\qquad \begin{aligned} x(t_i,\ldots,t_{i+r}) &= (t_{i+r}-t_i)^{-1}\{x(t_{i+1},\ldots,t_{i+r}) - x(t_i,\ldots,t_{i+r-1})\} \\ &= (t_{i+r}-t_i)^{-1}\{D^{r-1}x(\eta_i) - D^{r-1}x(\xi_i)\}/(m-1)! \end{aligned}
$$

where $t_i \le \xi_i \le t_{i+r-1}$ and $t_{i+1} \le \eta_i \le t_{i+r}$. Thus we have

$$
(3.13)\qquad |x(t_i,\ldots,t_{i+r})| \le |\eta_i - \xi_i|^{-1}|D^{r-1}x(\eta_i) - D^{r-1}x(\xi_i)| \le \|D^r x\|_{L^\infty}.
$$

Therefore if $x|_E = f$ the divided differences would be unbounded and, by (3.13), x could not be in $H^{r,\infty}$. This completes the proof.

Another consequence of Lemma 3.1 is

Theorem 3.2. *Let $E \subset R$ and f map E into R. Then there is an $H^{r,\infty}$-spline interpolating f on E if and only if there is an $H^{r,\infty}$-extension of f.*

Proof. If there is an $H^{r,\infty}$-spline interpolating f, then it is an $H^{r,\infty}$-extension of f. On the other hand if there is an $H^{r,\infty}$-extension of f, then the extremal problem breaks up into at most a countable number of uncoupled problems as in [3]. Applying Lemma 3.1 to each problem separately yields the theorem.

Finally, when considering the $W^{r,\infty}$ case, we have

Theorem 3.3. *If $E = \{t_i\}_{i=-\infty}^{\infty}$, $\delta < t_{i+1} - t_i < 1/\delta$ for some $\delta > 0$, then $f\colon E \to R$ has a $W^{r,\infty}$-spline interpolant on E if and only if*

$$
(3.14)\qquad \sup_{-\infty<i<\infty} |f(t_i)| < \infty.
$$

Proof. Obviously if f is unbounded on E then its interpolant would be unbounded and hence not in $W^{r,\infty}$. If (3.14) holds then we must have

$$
(3.15)\qquad \sup_{-\infty<i<\infty} |f(t_i,\ldots,t_{i+r})| < \infty,
$$

and hence Theorem 3.1 guarantees that f has an $H^{r,\infty}$-spline interpolating it on E. We will now show that x_* must be bounded. If x_* were not bounded, there is a sequence $\{\tau_j\}_{j=0}^{\infty}$ so that $|x_*(\tau_j)| \to \infty$ as $j \to \infty$. Consider the divided differences

$$
(3.16)\qquad \{x_*(\tau_j, t_{l_j},\ldots,t_{l_j+r-1})\}, \qquad j = 0, 1, \ldots,
$$

where the index l_j satisfies $t_{l_j-1} \le \tau_j < t_{l_j}$. Since $\delta < t_{i+1} - t_i < 1/\delta$, (3.16) must be unbounded. Now (3.13) implies that x_* is not in $H^{r,\infty}$. This contradiction completes the proof.

4. **Remarks.** It would be interesting to know whether the limit in Theorem 2.1 is unique independent of the sequence $\{p_n\}$. If the limit were unique, then it would single out a specific solution to the extremal problem. This work together with [3] and [7] constitutes a complete theory of the $H^{r,p}$- and $W^{r,p}$-splines on quasi-uniform partitions.

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