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A GENERALIZATION OF THE $\cos \pi\rho$ THEOREM

BY

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ABSTRACT. Let f be an entire function, and let β and λ be positive numbers with $\beta \leq \pi$ and $\beta\lambda < \pi$. Let $E(r) = \{\theta: \log|f(re^{i\theta})| > \cos \beta\lambda \log M(r)\}$. It is proved that either there exist arbitrarily large values of r for which $E(r)$ contains an interval of length at least 2β, or else $\lim_{r\to\infty} r^{-\lambda} \log M(r, f)$ exists and is positive or infinite. For $\beta = \pi$ this is Kjellberg's refinement of the $\cos \pi\rho$ theorem.

1. Introduction. Let f be an entire function. The classical $\cos \pi\rho$ theorem (see [4, Chapter 3] for its history) asserts that if f has order ρ, with $0 < \rho < 1$, then

(1) $$\limsup_{r\to\infty} \frac{\log m(r)}{\log M(r)} \geq \cos \pi\rho,$$

where $M(r)$ and $m(r)$ denote $\sup|f(z)|$ and $\inf|f(z)|$ on $|z| = r$, respectively.

Kjellberg [11] proved a striking improvement of this theorem. He showed that, for any number $\lambda \in (0, 1)$, either $\log m(r) > \cos \pi\lambda \log M(r)$ holds for certain arbitrarily large values of r or else $\lim_{r\to\infty} r^{-\lambda} \log M(r)$ exists and is positive or infinite. (The case $\lambda = \frac{1}{2}$ had been proved earlier by Heins [7].) A consequence of Kjellberg's theorem is that if f has lower order $\rho^* \in (0, 1)$ then the lim sup in (1) is $\geq \cos \pi\rho^*$. We remark that in this theorem it is not necessary to make any assumption about the order of f.

In this note I shall prove the following result:

Theorem 1. *Let f be a nonconstant entire function. Let β and λ be numbers with $0 < \lambda < \infty$, $0 < \beta \leq \pi$, $\beta\lambda < \pi$. Then either*

(a) *there exist arbitrarily large values of r for which the set of θ such that $\log|f(re^{i\theta})| > \cos \beta\lambda \log M(r)$ contains an interval of length at least 2β, or else*

(b) $\lim_{r\to\infty} r^{-\lambda} \log M(r)$ *exists, and is positive or infinite.*

For $\beta = \pi$ this is Kjellberg's theorem. For $\beta = \pi/2\lambda$ the theorem provides a sharpening of results of Arima [1] and Heins [8, p. 121].

The possibility that there might be a result like the one in Theorem 1 was suggested to me by A. Weitsman. I would also like to acknowledge some very

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Drasin and Shea [5], [6] have characterized functions extremal for the cos $\pi\rho$ theorem, that is, those entire functions f of order $\rho \in (0, 1)$ for which equality holds in (1). It would be interesting to determine what sort of functions are extremal, in a similar sense, for Theorem 1.

Theorem 1 suggests an analogous problem for meromorphic functions. For a given number α, what can we say about the size of the set

$$E_\alpha(r) = \{\theta: \log|f(re^{i\theta})| > \alpha T(r, f)\},$$

where $T(r, f)$ denotes the Nevanlinna characteristic of f? For $\alpha = 0$ the author has proved [2], [3] the "spread relation":

$$\limsup_{r\to\infty} \operatorname{meas} E_0(r) \geq \min\{2\pi, 4\rho^{-1}\sin^{-1}(\delta/2)^{1/2}\},$$

where ρ is the lower order of f and $\delta = \delta(\infty, f)$ is the Nevanlinna deficiency of f at ∞. It is also known ([12], [13], [14], [15]) that certain hypotheses on α, δ, and ρ insure that $\limsup_{r\to\infty} E_\alpha(r) = 2\pi$.

The proof of Theorem 1 depends on two key inequalities involving an auxiliary function $\mu(r)$. In §2 we state the inequalities and then show how the conclusion of the theorem follows from them. In §§3, 4 we obtain some results about harmonic functions which are needed to prove the inequalities, and in §§5, 6 we prove the inequalities themselves.

2. The auxiliary functions and key inequalities. Let f be entire and nonconstant. Consider the function u defined by

$$u(r, \theta, \phi) = \int_{-\theta}^{\theta} \log|f(re^{i(\omega+\phi)})|\, d\omega$$

where $0 \leq r < \infty$, $0 \leq \theta \leq \pi$, and ϕ is any real number. This function was introduced by the author in [2], where it was shown (Statements (3.9) and (3.10)) that

(2) *for each fixed ϕ, $u(r, \theta, \phi)$ is a subharmonic function of $re^{i\theta}$ in $0 < \theta < \pi$ and, for each fixed $\theta \in [0, \pi]$, $u(r, \theta, \phi)$ is a subharmonic function of $re^{i\phi}$ in the whole plane.*

We remark that the statements (3.9) and (3.10) of [2] do not cover the cases when θ is fixed and has the value zero or π. However, for $\theta = 0$ we have $u = 0$ and for $\theta = \pi$ we have

$$u(r, \pi, \phi) = 2\pi[N(r, 0, f) + k\log r + \log|c_k|]$$

where N has its usual meaning and c_k is the first nonvanishing coefficient in the Maclaurin series of f. The function in the brackets is a convex function of $\log r$, hence is a subharmonic function of $re^{i\phi}$. Thus (2) still holds when $\theta = 0$ and $\theta = \pi$.

Now consider the function $v(z)$ defined in the upper half plane by

$$v(re^{i\theta}) = \sup_{\phi} u(r, \theta, \phi) \qquad (0 \le \theta \le \pi). \tag{3}$$

Alternatively,

$$v(re^{i\theta}) = \sup_{I} \int_I \log |f(re^{i\omega})| \, d\omega \tag{4}$$

where the sup is taken over all ω-intervals I of length exactly 2θ. This $v(z)$ is the same, except for a factor of 2π, as the functions $m_1(z)$ and $T_1(z)$ considered by the author in [2].

Proposition 1. (a) *For each fixed* $re^{i\theta}$ *there exists an interval* I *of length* 2θ *for which the* sup *in* (3) *is attained.*

(b) $v(z)$ *is subharmonic in* $\operatorname{Im} z > 0$ *and continuous on* $\operatorname{Im} z \ge 0$, *except perhaps at* $z = 0$.

(c) *For each fixed* $\beta \in (0, \pi]$, $v(re^{i\beta})$ *is a nondecreasing convex function of* $\log r$, $0 < r < \infty$.

(d) *Define*

$$v_\theta(r) = \lim_{\theta \to 0+} \theta^{-1}[v(re^{i\theta}) - v(r)] = \lim_{\theta \to 0+} \theta^{-1} v(re^{i\theta}).$$

Then $v_\theta(r) = 2 \log M(r, f)$ $(0 < r < \infty)$.

Proof. (a) For $re^{i\theta}$ fixed, $u(r, \theta, \phi)$ is a continuous periodic function of ϕ. Take a ϕ for which $u(r, \theta, \phi)$ is maximal, and let I be the interval of length 2θ centered at ϕ.

(b) The continuity statement follows from a routine argument. The definition (3), together with (2), shows that $v(re^{i\theta})$ is the supremum of a family of subharmonic functions of $re^{i\theta}$. Such a function is always subharmonic, provided it is upper semicontinuous, and this is certainly the case here.

(c) The definition (3), together with (2), allows us to interpret $v(re^{i\beta})$ as the maximum modulus of a function of $re^{i\phi}$ which is subharmonic in the whole plane. This implies the conclusion (c).

(d) For any interval I of length 2θ we have $\int_I \log|f(re^{i\omega})| \, d\omega \le 2\theta \log M(r)$. This implies

$$\limsup_{\theta \to 0+} \theta^{-1} v(re^{i\theta}) \le 2 \log M(r). \tag{5}$$

On the other hand, let $re^{i\phi_0}$ be a point such that $\log|f(re^{i\phi_0})| = \log M(r)$. Then $v(re^{i\theta}) \ge \int_{-\theta}^{\theta} \log |f(r \exp\{i(\phi_0 + \omega)\})| \, d\omega$. Dividing by θ and letting $\theta \to 0$ we obtain

$$\liminf_{\theta \to 0+} \theta^{-1} v(re^{i\theta}) \ge 2 \log M(r),$$

which with (5), proves (d). This completes the proof of Proposition 1.

Fix $\beta \in (0, \pi]$. Let $I(r)$ be an interval of length 2β such that $v(re^{i\beta}) = \int_{I(r)} \log|f(re^{i\omega})|\, d\omega$. Define

$$\mu(r) = \inf\{\log|f(re^{i\omega})| : \omega \in I(r)\}.$$

Then conclusion (a) of Theorem 1 will hold if

$$\mu(t) > (\cos \beta\lambda) \log M(t) \tag{6}$$

for arbitrarily large values of t.

In the inequalities below it is assumed that β and λ satisfy the hypotheses of Theorem 1.

Key inequality I. There exist positive constants C_1, C_2, depending only on β and λ, such that whenever $f(0) = 1$, we have

$$\int_r^s \frac{\mu(t) - (\cos \beta\lambda) \log M(t)}{t^{1+\lambda}}\, dt > C_1 \frac{\log M(r)}{r^\lambda} - C_2 \frac{\log M(2s)}{s^\lambda} \qquad (0 < r < s < \infty). \tag{7}$$

Key inequality II. Let

$$Q(r, t) = 2r\pi^{-2}(r^2 - t^2)^{-1} \log rt^{-1}, \qquad \gamma = \beta/\pi.$$

Then, if $\limsup_{r \to \infty} r^{-\lambda} \log M(r) < \infty$, we have

$$\log M(r^\gamma) \le \int_0^\infty [\mu(t^\gamma) + \log M(t^\gamma)] Q(r, t)\, dt \qquad (0 < r < \infty). \tag{8}$$

Once we have these inequalities the proof of Theorem 1 is completed by exactly the same reasoning as that used by Kjellberg in [10] and [11]. Let

$$A = \liminf_{r \to \infty} r^{-\lambda} \log M(r), \qquad B = \limsup_{r \to \infty} r^{-\lambda} \log M(r).$$

If $A = B = \infty$ then conclusion (b) of Theorem 1 holds. If $B = \infty$ and $A < \infty$ we can find arbitrarily large values of r and s, with $r < s$, such that the right-hand side of (7) is positive. So, if $f(0) = 1$, then (6) holds for some $t > r$ and we are done. If $B = 0$ and $f(0) = 1$ then $r^{-\lambda} \log M(r) > 0$ for $r > 0$. For each fixed r the right-hand side of (7) is positive for all sufficiently large s, and again we are done.

The restriction $f(0) = 1$ can be removed in the usual way. Let g be the entire function with $g(0) = 1$ and $f(z) = cz^k g(z)$ $(c \neq 0)$. Then

$$\log M(r, g) = \log M(r, f) - \log|c| - k \log r, \tag{9}$$

and $\mu(r, g)$ can be chosen so that (9) holds with μ in place of $\log M$. Using (7) with g in place of f we easily deduce

$$\int_r^s \frac{\mu(t, f) - \cos\beta\lambda \log M(t, f)}{t^{1+\lambda}}\,dt > c_1 r^{-\lambda}(\log M(r, f) - \log|c| - k\log r)$$
$$- c_2 s^{-\lambda}(\log M(2s, f) - \log|c| - k\log 2s)$$
$$- (\lambda r^{\lambda})^{-1}(1 - \cos\beta\lambda)\log|c|^{-1} \qquad (1 \le r < s < \infty).$$

Arguing as above, with obvious modifications, we find that if $B = \infty$ and $A < \infty$, or if $B = 0$ and f is not a polynomial, then (6) holds for arbitrarily large values of t. (For f a polynomial (6) holds for all sufficiently large values of t.)

Now consider the case $0 < B < \infty$, so that (8) holds. If (6) is false for all sufficiently large t then

$$\mu(t) \le (\cos\beta\lambda)\log M(t) \qquad (t \ge t_0). \tag{10}$$

Dividing f by a large positive constant, if necessary, we can assume that (10) holds for all $t > 0$. (See the argument on p. 6 of [11].) Putting (10) in (8) we obtain

$$\log M(r^{\gamma}) \le \int_0^{\infty} (1 + \cos\beta\lambda)\log M(t^{\gamma})Q(r, t)\,dt.$$

Proceeding as in §4 of [11], with $\gamma\lambda$ in place of λ, we arrive at

$$\lim_{r\to\infty} \frac{\log M(r^{\gamma})}{r^{\gamma\lambda}} = B > 0$$

so that (b) of Theorem 1 holds.

3. **A class of harmonic functions.** In this section $B(t)$ will always stand for a nondecreasing convex function of $\log t$ on $(0, \infty)$ satisfying

$$B(0) = B(0+) = 0, \qquad B(t) = O(t^{\rho}) \qquad (t \to \infty)$$

for some $\rho \in (0, 1)$.

The function $B(t)$ is absolutely continuous. Let $B_1(t)$ denote its logarithmic derivative, $B_1(t) = tB'(t)$. Then B_1 exists a.e., and is a nonnegative nondecreasing function of t.

Since $B(2t) \ge \int_t^{2t} B_1(s)s^{-1}\,ds \ge B_1(t)\log 2$ it follows that

$$B_1(t) = O(t^{\rho}) \qquad (t \to \infty). \tag{11}$$

Similarly, $B(t^{1/2}) - B(t) \ge B_1(t)(\log t^{1/2} - \log t)$, so

$$\lim_{t\to 0}\left(\log\frac{1}{t}\right)B_1(t) = 0. \tag{12}$$

The Poisson integral

$$h(re^{i\theta}) = \frac{1}{\pi}\int_0^\infty B(t)\frac{r\sin\theta}{t^2 + r^2 + 2tr\cos\theta}\,dt$$

is harmonic in the slit plane $|\arg z| < \pi$, is zero on the positive axis and tends to $B(r)$ as $\theta \to \pi -$, the convergence being uniform on bounded subsets of $(0, \infty)$.

The purpose of this section is to obtain some results about $h_\theta = \partial h/\partial\theta$. These results generalize known properties of entire functions. Let

$$F(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{a_n}\right)$$

where $0 < a_n < a_{n+1}$ and $n^{1/\rho} = O(a_n)$. Then $B(t) = N(t, 0, f) = \Sigma \log^+(t/a_n)$ satisfies our hypotheses, and $B_1(t) = n(t, 0, f)$. In this case the Poisson integral h has a representation $h(re^{i\theta}) = \pi^{-1}\int_0^\theta \log|F(re^{i\phi})|\,d\phi$, since the right-hand side is a function harmonic in the upper half plane which has the same boundary values as h. Thus $h_\theta(re^{i\theta}) = \pi^{-1}\log F(re^{i\theta})$. In particular,

$$h_\theta(r) = \pi^{-1}\log M(r, F), \qquad h_\theta(re^{i\pi}) = \pi^{-1}\log m(r, F).$$

The reader might find it helpful to keep this special case in mind in what follows.

Proposition 2.

$$h_\theta(re^{i\theta}) = \frac{1}{\pi}\int_0^\infty \log\left|1 + \frac{r}{t}e^{i\theta}\right| dB_1(t) \qquad (|\theta| < \pi). \tag{13}$$

This generalizes the well-known formula $\log|F(re^{i\theta})| = \int_0^\infty \log|1 + rt^{-1}e^{i\theta}|\,dn(t)$.

Proof. We differentiate the Poisson integral with respect to θ; use

$$\frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r^2 + t^2 + 2tr\cos\theta}\right) = -\frac{\partial}{\partial t}\operatorname{Re}\frac{re^{i\theta}}{(t + re^{i\theta})}$$

and integrate by parts. The result is

$$h_\theta(re^{i\theta}) = \frac{1}{\pi}\int_0^\infty B_1(t)\operatorname{Re}\left(\frac{re^{i\theta}}{t(t + re^{i\theta})}\right)dt.$$

Using

$$\operatorname{Re}\frac{re^{i\theta}}{t(t + re^{i\theta})} = -\frac{\partial}{\partial t}\log\left|1 + \frac{re^{i\theta}}{t}\right|,$$

doing another integration by parts, and observing (11), (12), we obtain (13).

Proposition 3.

$$\lim_{\theta\to\pi-}\frac{B(r) - h(re^{i\theta})}{\pi - \theta} = \lim_{\theta\to\pi-} h_\theta(re^{i\theta}) = \frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{r}{t}\right| dB_1(t). \tag{14}$$

The above integral is always well defined, but it may be $-\infty$ *for some values of* r.

Proof. Fix $r \in (0, \infty)$. Since $\log|1 + re^{i\theta}/t|$ is a decreasing function of θ on $(0, \pi)$, and since $\log|1 + re^{i\theta}/t| \le \log(1 + r/t)$ $(0 < \theta \le \pi)$ with $\int_0^\infty \log(1 + r/t)\, dB_1(t) = h_\theta(r) < \infty$, the monotone convergence theorem shows that

$$\lim_{\theta \to \pi-} \frac{1}{\pi} \int_0^\infty \log\left|1 + \frac{re^{i\theta}}{t}\right| dB_1(t) = \frac{1}{\pi} \int_0^\infty \log\left|1 - \frac{r}{t}\right| dB_1(t).$$

Because of (13), this proves the second equality in (14).

The proof of the other equality seems to require a slightly more elaborate argument. Take $\theta \in (0, \pi)$. From Proposition 2 we deduce

$$h(re^{i\theta}) = \int_0^\theta h_\theta(re^{i\phi})\, d\phi = \frac{1}{\pi} \int_0^\infty dB_1(t) \int_0^\theta \log\left|1 + \frac{re^{i\phi}}{t}\right| d\phi.$$

Now

$$B(r) = \int_0^\infty \log^+ \frac{r}{t}\, dB_1(t) = \frac{1}{\pi} \int_0^\infty dB_1(t) \int_0^\pi \log\left|1 + \frac{r}{t} e^{i\phi}\right| d\phi.$$

So, setting $A(r, \theta) = \pi^{-1} \int_\theta^\pi \log|1 + re^{i\phi}|\, d\phi$ we see that

$$B(r) - h(re^{i\theta}) = \int_0^\infty A(rt^{-1}, \theta)\, dB_1(t). \tag{15}$$

A calculation shows

$$\pi(\pi - \theta) \frac{\partial}{\partial\theta} \frac{A(r, \theta)}{\pi - \theta} = -\log|1 + re^{i\theta}| + \frac{1}{\pi - \theta} \int_\theta^\pi \log|1 + re^{i\phi}|\, d\phi.$$

The monotonicity of $\log|1 + re^{i\phi}|$ thus implies

$$\frac{\partial}{\partial\theta} \frac{A(r, \theta)}{\pi - \theta} < 0 \qquad (0 < \theta < \pi).$$

Hence, for r and t fixed, $(\pi - \theta)^{-1} A(r/t, \theta) \searrow$ as $\theta \nearrow \pi$. In particular,

$$\frac{1}{\pi - \theta} A\left(\frac{r}{t}, \theta\right) < \frac{1}{\pi} A\left(\frac{r}{t}, 0\right) = \log^+ \frac{r}{t}.$$

Now $\int_0^\infty \log^+(r/t)\, dB_1(t) = B(r) < \infty$, so, when we divide (15) by $\pi - \theta$ and let $\theta \nearrow \pi$ the monotone convergence theorem is again applicable. Since $\lim_{\theta \to \pi-} (\pi - \theta)^{-1} A(r/t, \theta) = \pi^{-1} \log|1 - r/t|$, the other equality in (14) is thus established.

From now on we will denote the quantity appearing in (14) by $h_\theta(-r)$.

Proposition 4. *Let* $\sigma \in (0, 1)$. *There exist positive constants* k_1, k_2, *depending only on* σ, *such that*

$$\int_r^s \frac{h_\theta(-t) - (\cos \pi\sigma) h_\theta(t)}{t^{1+\sigma}}\, dt > k_1 \frac{h_\theta(r)}{r^\sigma} - k_2 \frac{h(s)}{s^\sigma} \qquad (0 < r < s < \infty). \tag{16}$$

This generalizes the inequality (23) in Kjellberg's paper [10].

Proof. Let J be the integral in (16). Using Propositions 3 and 2, with $\theta = 0$, we deduce

$$J = \int_0^\infty dB_1(u) \int_r^s t^{-(1+\sigma)}[\log|1 - t/u| - \cos \pi\sigma \log(1 + t/u)]\, dt. \tag{17}$$

Kjellberg has shown [9], [10, p. 192] that

$$\int_r^s t^{-(1+\sigma)}[\log|1 - t/u| - \cos \pi\sigma \log(1 + t/u)]\, dt$$
$$> k_1 r^{-\sigma} \log(1 + r/u) - k_2 s^{-\sigma} \log(1 + s/u) \qquad (0 < u < \infty),$$

where k_1, k_2 are positive and depend only on σ. Putting this in (17), and using Proposition 2, with $\theta = 0$, we obtain (16).

Proposition 5.

$$h_\theta(r) = \int_0^\infty [h_\theta(t) + h_\theta(-t)] Q(r, t)\, dt. \tag{18}$$

Here Q is as in the statement of key inequality II. This generalizes the identity (15) in Kjellberg's paper [11].

Proof. The function h_θ is harmonic in the half plane $|\theta| < \pi/2$ and is continuous on the closure (we define $h_\theta(0) = 0$). From (13) it follows easily that $0 \le h_\theta(re^{i\theta}) \le h_\theta(r) = O(r^\rho)$ ($|\theta| \le \pi/2$, $r \to \infty$). Thus h_θ can be represented in the half plane by the Poisson integral of its boundary values. Since $h_\theta(iy) = h_\theta(-iy)$ for real y, we have

$$h_\theta(r) = \frac{2r}{\pi} \int_0^\infty h_\theta(iy) \frac{dy}{r^2 + y^2} \qquad (0 < r < \infty). \tag{19}$$

Now we show that h_θ is also the Poisson integral of its boundary values on the real axis. Take $\delta \in (0, \frac{1}{2}\pi)$ and consider $h_\theta(re^{i(\theta - \delta)})$ as a function of $re^{i\theta}$ in the upper half plane. It follows easily from (13) that $\sup_{0 \le \theta \le \pi} |h_\theta(re^{i(\theta-\delta)})| = O(r^\rho)$ $(r \to \infty)$ for each fixed δ. Since $h_\theta(re^{i(\theta-\delta)})$ is continuous in the closed half plane, we have

$$h_\theta(iye^{-i\delta}) = \frac{y}{\pi} \int_0^\infty [h_\theta(te^{-i\delta}) + h_\theta(te^{i(\pi-\delta)})] \frac{dt}{t^2 + y^2} \qquad (0 < y < \infty).$$

Let $\delta \downarrow 0$. Then $h_\theta(te^{-i\delta}) \uparrow h_\theta(t)$ and $h_\theta(te^{i(\pi-\delta)}) \downarrow h_\theta(-t)$, with $h_\theta(te^{i(\pi-\delta)}) \le h_\theta(t)$. Since $h_\theta(t)$ is integrable with respect to $(t^2 + y^2)^{-1} dt$, we can apply the monotone convergence theorem and conclude

$$h_\theta(iy) = \frac{y}{\pi} \int_0^\infty [h_\theta(t) + h_\theta(-t)] \frac{dt}{t^2 + y^2} \qquad (0 < y < \infty). \tag{20}$$

Putting (20) in (19) and changing the order of integration, we get (18). (To see that Fubini's theorem is applicable here, consider, for fixed r,

$$F(t, y) = [h_\theta(t) + h_\theta(-t)]\frac{y}{(r^2 + y^2)(t^2 + y^2)}$$

$$= 2h_\theta(t)\frac{y}{(r^2 + y^2)(t^2 + y^2)} - [h_\theta(t) - h_\theta(-t)]\frac{y}{(r^2 + y^2)(t^2 + y^2)}$$

$$= F_1(t, y) - F_2(t, y).$$

Then $F_1 \geq 0$, $F_2 \geq 0$, and it is easy to verify that $\int_0^\infty dt \int_0^\infty F_1(t, y)\,dy < \infty$.)

4. More results on harmonic functions.

Proposition 6. *Let h be harmonic and bounded in $|z| < R$. Let $\alpha \in (0, 1)$. Then*

$$|h_\theta(z)| \leq k(\alpha)\frac{|z|}{R}\sup_\phi |h(Re^{i\phi})| \qquad (|z| \leq \alpha R),$$

where $k(\alpha)$ depends only on α.

Proof. We have

$$h(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\phi})\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \phi)}\,d\phi.$$

Differentiation with respect to θ and simple estimates show

$$|h_\theta(re^{i\theta})| \leq \sup_\phi |h(Re^{i\phi})| \cdot \frac{(R^2 - r^2)2rR}{(R - r)^4} \qquad (0 < r < R).$$

If $r \leq \alpha R$ then $(R - r)^3 \geq (1 - \alpha)^3 R^3$, so

$$\frac{(R^2 - r^2)2rR}{(R - r)^4} = \frac{(R + r)2rR}{(R - r)^3} \leq \frac{4rR^2}{(1 - \alpha)^3 R^3} = k(\alpha)\frac{r}{R}$$

and we are done.

The next result is a local version of Proposition 4 in which no assumption is made about the growth of the boundary function $B(t)$.

Proposition 7. *Let $B(t)$ be a nondecreasing convex function of* $\log t$ *on $(0, \infty)$ with $B(0) = B(0+) = 0$. Let $g(\phi)$ be bounded and measurable on $(0, \pi)$. Let h be the function which is bounded and harmonic in the half disk $\{z\colon |z| < R,$ $\operatorname{Im} z > 0\}$ and has the following boundary values:*

$$h(Re^{i\phi}) = g(\phi), \quad h(r) = 0, \quad h(-r) = B(r) \qquad (0 < r < R).$$

Let $\sigma \in (0, 1)$, $\alpha \in (0, 1)$. Suppose $0 < r < s = \alpha R$. Then

$$\int_r^s \frac{h_\theta(-t) - (\cos \pi\sigma)h_\theta(t)}{t^{1+\sigma}}\,dt > k_1\frac{h_\theta(r)}{r^\sigma} - k(\alpha, \sigma)\frac{B(\alpha^{-1}R) + M_1}{s^\sigma} \tag{21}$$

where k_1 is as in Proposition 4, $k(\alpha, \sigma)$ is a positive constant depending only on α and σ, and $M_1 = \sup_{0<\phi<\pi}|g(\phi)|$.

Proof. Define $B^*(t)$ by

$$B^*(t) = B(t) \qquad (0 < t \le R)$$
$$= B_1(R)\log(t/R) + B(R) \qquad (R < t < \infty)$$

where $B_1(t) = tB'(t)$. Then B^* satisfies the hypotheses of the B in §3. Define b_1 in the slit plane $|\arg z| < \pi$ by

$$b_1(re^{i\theta}) = \frac{1}{\pi}\int_0^\infty B^*(t)\frac{r\sin\theta}{t^2 + r^2 + 2tr\cos\theta}\,dt.$$

Define b_2 in the half disk by $b_2 = b - b_1$. Then $b_2(x) = 0$ for $-R < x < R$, so b_2 has a harmonic extension to the full disk $|z| < R$.

Let J be the integral in (21), and let J_1, J_2 be the corresponding integrals with b replaced by b_1 and b_2. Proposition 4 can be used to estimate J_1, so we have

(22)
$$J = J_1 + J_2 > k_1\frac{(b_1)_\theta(r)}{r^\sigma} - k_2\frac{(b_1)_\theta(s)}{s^\sigma} + J_2$$
$$= k_1\frac{b_\theta(r)}{r^\sigma} - k_1\frac{(b_2)_\theta(r)}{r^\sigma} - k_2\frac{(b_1)_\theta(s)}{s^\lambda} + J_2.$$

Let $M_2 = \sup_{0<\phi<\pi}|b_2(\mathrm{Re}^{i\phi})|$. By Proposition 6 we have

(23)
$$r^{-\sigma}|(b_2)_\theta(r)| \le k(\alpha)rR^{-1}r^{-\sigma}M_2 < k(\alpha)M_2R^{-\sigma}.$$

Another application of Proposition 6 gives

(24)
$$|J_2| \le \int_r^s \frac{|(b_2)_\theta(-t)| + |(b_2)_\theta(t)|}{t^{1+\sigma}}\,dt$$
$$\le 2k(\alpha)M_2R^{-1}\int_r^s t^{-\sigma}\,dt < 2k(\alpha)(1-\sigma)^{-1}M_2R^{-1}s^{1-\sigma}$$
$$< 2k(\alpha)(1-\sigma)^{-1}M_2R^{-\sigma}.$$

Using Proposition 2 with $\theta = 0$, and integrating by parts twice, we find

$$(b_1)_\theta(s) = \frac{1}{\pi}\int_0^\infty \frac{s}{(t+s)^2}B^*(t)\,dt$$
$$= \frac{1}{\pi}\int_0^R \frac{s}{(t+s)^2}B(t)\,dt + \frac{1}{\pi}\int_R^\infty [B(R) + B_1(R)\log(t/R)]\frac{s}{(t+s)^2}\,dt.$$

Since $B(t) \le B(R)$ for $0 < t < R$ and $s = \alpha R$, we have

$$(h_1)_\theta(s) \le \frac{1}{\pi} B(R) \int_0^\infty \frac{s}{(t+s)^2}\,dt + \frac{1}{\pi} B_1(R) \int_R^\infty (\log(t/R)) \frac{s}{(t+s)^2}\,dt$$

$$= \frac{1}{\pi} B(R) + \frac{1}{\pi} B_1(R) \int_1^\infty \log t \frac{\alpha}{(t+\alpha)^2}\,dt.$$

Now $(\log t)\,\alpha/(t+\alpha)^2 < t^{1/2}/(1+t)^2$ $(1 < t < \infty,\ 0 < \alpha < 1)$, so the last integral is < 2. Hence

(25) $$(h_1)_\theta(s) < \pi^{-1}(B(R) + 2B_1(R)).$$

Using (23), (24), (25) in (22) and remembering $s = \alpha R$, we obtain

(26) $$J > k_1 h_\theta(r)/r^\sigma - k_1(\alpha, \sigma) s^{-\sigma}(B(R) + 2B_1(R) + M_2)$$

where $k_1(\alpha, \sigma)$ is a positive constant depending on α and σ.

Now $h_2 = h - h_1$, so $M_2 \le M_1 + \sup_{\theta\in(0,\pi)} |h_1(Re^{i\theta})|$.

Using the Poisson integral representation of h_1 and the definition of B^*, one can easily deduce $|h_1(Re^{i\theta})| \le B_1(R) + B(R)$ $(0 < \theta < \pi)$. Hence

(27) $$M_2 \le M_1 + B_1(R) + B(R).$$

Putting (27) in (26), and using

$$B(R) \le B(\alpha^{-1}R), \qquad B_1(R) \le \frac{1}{(\log \alpha^{-1})} \int_R^{\alpha^{-1}R} B_1(t) \frac{dt}{t} \le \frac{B(\alpha^{-1}R)}{(\log \alpha^{-1})},$$

we obtain (21).

5. **Proof of key inequality I.** We are assuming $f(0) = 1$. This implies that $v(z)$, defined by (3), is continuous in the closed upper half plane, with $v(0) = 0$.

Fix $R > 0$. Define D by $D = \{z: 0 < |z| < R,\ 0 < \arg z < \beta\}$. Let H be the bounded harmonic function in D which has the following boundary values:

$$H(r) = 0, \qquad H(re^{i\beta}) = v(re^{i\beta}) \qquad (0 \le r < R),$$

$$H(Re^{i\theta}) = \begin{cases} 2\pi \log M(R, f) & (0 < \theta < \tfrac{1}{2}\beta), \\ 4\pi \log M(R, f) & (\tfrac{1}{2}\beta < \theta < \beta). \end{cases}$$

Let $\gamma = \beta/\pi$, and define $h(z)$ in the upper half disk of radius $R^{1/\gamma}$ by $h(z) = H(z^\gamma)$ $(0 < |z| < R^{1/\gamma},\ 0 < \arg z < \pi)$. Then h is the function considered in Proposition 7, with $B(t) = v(t^\gamma e^{i\beta})$, the R there replaced by $R^{1/\gamma}$, and

$$g(\phi) = 2\pi \log M(R, f) \qquad (0 < \phi < \pi/2),$$

$$= 4\pi \log M(R, f) \qquad (\pi/2 < \phi < \pi).$$

The function $B(t)$ satisfies the hypothesis of Proposition 7, by virtue of Proposition 1.

Let $s = 2^{-1/2}R$ and let $0 < r < s$. Using (21), with $\sigma = \gamma\lambda$ ($= \beta\lambda/\pi < 1$) and $\alpha = 2^{-1/2\gamma}$, we obtain

$$\int_{r^{1/\gamma}}^{s^{1/\gamma}} \frac{h_\theta(-t) - \cos \pi\lambda h_\theta(t)}{t^{1+\sigma}}\,dt \tag{28}$$
$$> k_1 \frac{h_\theta(r^{1/\gamma})}{r^\lambda} - k_2 \frac{B(2^{\frac{1}{2}\gamma} R^{1/\gamma}) + 4\pi \log M(R)}{s^\lambda},$$

where k_1, k_2 depend on β and λ. Now $h_\theta(t) = \gamma H_\theta(t^\gamma)$, $h_\theta(-t) = \gamma H_\theta(t^\gamma e^{i\beta})$. Changing variables in (28), and using $B(2^{\frac{1}{2}\gamma} R^{1/\gamma}) = v(\sqrt{2}\, Re^{i\beta}) = v(2se^{i\beta})$, we obtain

$$\int_r^s \frac{H_\theta(te^{i\beta}) - \cos \pi\lambda H_\theta(t)}{t^{1+\lambda}}\,dt > k_1 \frac{\gamma H_\theta(r)}{r^\lambda} - k_2 \frac{v(2se^{i\beta}) + 4\pi \log M(\sqrt{2}\, s)}{s^\lambda}.$$

Since $v(2se^{i\beta}) \le 2\beta \log M(2s)$, $\log M(\sqrt{2}\, s) \le \log M(2s)$, we have

$$\int_r^s \frac{H_\theta(te^{i\beta}) - \cos \pi\lambda H_\theta(t)}{t^{1+\lambda}}\,dt > C_1 \frac{H_\theta(r)}{r^\lambda} - C_2 \frac{\log M(2s)}{s^\lambda}, \tag{29}$$

where C_1, C_2 are positive constants depending on β and λ.

By Proposition 1, v is subharmonic in D. The harmonic function H majorizes v on the boundary of D (since $v(r) = 0$ for $r \ge 0$ and $v(Re^{i\theta}) \le 2\pi \log M(R)$). It follows that

$$v(z) \le H(z) \quad \text{for all } z \in D. \tag{30}$$

Since $v(r) = H(r) = 0$ for $r > 0$, it follows from (30) and Proposition 1 that

$$H_\theta(r) \ge v_\theta(r) = 2 \log M(r) \qquad (0 < r < R). \tag{31}$$

Here, and in what follows, $H_\theta(r)$ and $H_\theta(re^{i\beta})$ are understood to be one-sided derivatives computed from inside D.

I claim that the following inequality also holds:

$$H_\theta(te^{i\beta}) + H_\theta(t) \le 2(\mu(t) + \log M(t)) \qquad (0 < t < R). \tag{32}$$

Let us assume (32). Using it together with (31), we find

$$\begin{aligned} H_\theta(te^{i\beta}) - \cos \beta\lambda H_\theta(t) &= [H_\theta(te^{i\beta}) + H_\theta(t)] - (1 + \cos \beta\lambda) H_\theta(t) \\ &\le 2(\mu(t) + \log M(t)) - 2(1 + \cos \beta\lambda) \log M(t) \\ &= 2(\mu(t) - \cos \beta\lambda \log M(t)). \end{aligned} \tag{33}$$

Using (33) and (31) in (29), we obtain key inequality I.

To prove (32) we introduce another auxiliary function $w(z)$. It is defined in the angle $0 \le \theta \le \frac{1}{2}\beta$ by

$$w(re^{i\theta}) = \sup_E \int_E \log|f(re^{i\omega})|\,d\omega \qquad (0 < r < \infty,\ 0 \le \theta \le \tfrac{1}{2}\beta) \tag{34}$$

where the sup is taken over all sets E of the following form:

$$E = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3], \tag{35}$$

with

$$a_1 \le b_1 \le a_2 \le b_2 \le a_3 \le b_3,$$

$$b_2 - a_2 = 2\theta, \qquad (b_1 - a_1) + (b_3 - a_3) = 2\theta,$$

$$a_2 - b_1 = a_3 - b_2 = \beta - 2\theta.$$

Note that for $\theta = \beta/2$ the sets E are simply intervals of length 2β. Thus $w(re^{i\beta/2}) = v(re^{i\beta})$.

Lemma. (a) *w is subharmonic in $0 < \arg z < \frac{1}{2}\beta$ and continuous on $0 \le \arg z \le \frac{1}{2}\beta$.*

(b) $$\limsup_{\theta \to \frac{1}{2}\beta -} \frac{w(re^{i\beta/2}) - w(re^{i\theta})}{\beta/2 - \theta} \le 2(\mu(r) + \log M(r)) \qquad (0 < r < \infty).$$

Once the Lemma is proved, we obtain (32) as follows. Let $D_1 = \{z: 0 < |z| < R,\ 0 < \arg z < \beta/2\}$ and define $H_1(z)$ on D_1 by

$$H_1(re^{i\theta}) = H(r\exp\{i(\beta/2 + \theta)\}) - H(r\exp\{i(\beta/2 - \theta)\}).$$

Then H_1 is harmonic in D_1 and has the following boundary values:

$$H_1(r) = 0, \qquad H_1(re^{i\beta/2}) = H(re^{i\beta}) \qquad (0 \le r < R),$$

$$H_1(Re^{i\theta}) = 2\pi \log M(r, f) \qquad (0 < \theta < \tfrac{1}{2}\beta).$$

A look at the definition of w shows

$$\begin{aligned} &w(re^{i\beta/2}) = v(re^{i\beta}) = H(re^{i\beta}) = H_1(re^{i\beta/2}) \qquad (0 < r < R), \\ &w(r) = 0 \quad (0 < r < R), \qquad w(Re^{i\theta}) \le 2\pi \log M(r, f). \end{aligned} \tag{36}$$

Thus H_1 majorizes w on the boundary of D_1, hence it also majorizes w inside D_1. Using this, together with (36), we obtain

$$\limsup_{\theta \to \beta/2} \frac{w(re^{i\beta/2}) - w(re^{i\theta})}{\beta/2 - \theta} \ge (H_1)_\theta(re^{i\beta/2}) = H_\theta(re^{i\beta}) + H_\theta(r) \qquad (0 < r < R),$$

which, together with part (b) of the Lemma, proves (32).

Proof of the Lemma. The continuity statement follows from a routine argument which we leave to the reader.

For $r > 0$, $0 < \rho < r$, $-\pi \le \psi \le \pi$, define $r(\psi) > 0$ and $\alpha(\psi) \in (-\pi/2, \pi/2)$ by $r + \rho e^{i\psi} = r(\psi)e^{i\alpha(\psi)}$. With this notation, a function s defined on an open set D is subharmonic in D if and only if it is upper semicontinuous and, if for each $re^{i\theta} \in D$, there exists $\rho_0 > 0$ such that $s(re^{i\theta}) \le \frac{1}{2}\pi^{-1}\int_{-\pi}^{\pi} s(r(\psi)\exp\{i(\theta + \alpha(\psi))\})\,d\psi$ holds whenever $0 < \rho < \rho_0$.

For fixed $re^{i\theta}$ with $0 < r < \infty$, $0 < \theta < \frac{1}{2}\beta$ it is easily shown that there exists a set E of the form (35) for which the supremum in (34) is attained. Let a_j, b_j be the endpoints of the intervals defining one such extremal E and set

$$\sigma_1 = \tfrac{1}{2}(b_1 - a_1), \qquad \sigma_2 = \tfrac{1}{2}(b_3 - a_3),$$

$$\phi_1 = \tfrac{1}{2}(a_1 + b_1), \qquad \phi_2 = \tfrac{1}{2}(a_2 + b_2), \qquad \phi_3 = \tfrac{1}{2}(a_3 + b_3).$$

In terms of the function u introduced in §2 we have

$$u(re^{i\theta}) = u(r, \sigma_1, \phi_1) + u(r, \theta, \phi_2) + u(r, \sigma_2, \phi_3). \tag{37}$$

Assume $\sigma_1 > 0$. Choose $\rho_0 \in (0, r)$ such that whenever $0 < \rho < \rho_0$ we have $0 < \theta + \alpha(\psi) < \frac{1}{2}\beta$, $0 < \sigma_1 + \alpha(\psi) < \frac{1}{2}\beta$ $(-\pi \le \psi \le \pi)$. Define

$$E(\psi) = [a_1 - \alpha(\psi), b_1 + \alpha(\psi)]$$
$$\cup [a_2 - \alpha(\psi), b_2 + \alpha(\psi)] \cup [a_3 - \alpha(\psi), b_3 - \alpha(\psi)].$$

Then $E(\psi)$ satisfies (35), with θ replaced by $\theta + \alpha(\psi)$. Hence

$$u(r(\psi)\exp\{i(\theta + \alpha(\psi))\}) \ge \int_{E(\psi)} \log|f(r(\psi)\exp\{i\omega\})|\, d\omega. \tag{38}$$

Now

$$\begin{aligned}\int_{E(\psi)} \log|f(r(\psi)e^{i\omega})|\, d\omega &= u(r(\psi), \sigma_1 + \alpha(\Psi), \phi_1) \\ &\quad + u(r(\psi), \theta + \alpha(\psi), \phi_2) + u(r(\psi), \sigma_2, \phi_3 - \alpha(\psi)).\end{aligned} \tag{39}$$

Substitute (39) in (38), divide by 2π, and integrate from $\psi = -\pi$ to $\psi = \pi$. The subharmonicity properties of u mentioned in (2) yield

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(r(\psi)\exp\{i(\theta + \alpha(\psi))\})\, d\psi \ge u(r, \sigma_1, \phi_1) + u(r, \theta, \phi_2) + u(r, \sigma_2, \phi_3). \tag{40}$$

(We used here the fact that $\int_0^{2\pi} u(r(\psi), \sigma_2, \phi_3 - \alpha(\psi))\, d\psi = \int_0^{2\pi} u(r(\psi), \sigma_2, \phi_3 + \alpha(\psi))\, d\psi$.)

Comparing (40) with (37) we see that w satisfies the criterion for subharmonicity at $re^{i\theta}$. (We were assuming $\sigma_1 > 0$. If $\sigma_1 = 0$ then $\sigma_2 = \theta - \sigma_1 > 0$, and we repeat the above argument with the roles of $[a_1, b_1]$, $[a_3, b_3]$ interchanged.) Thus part (a) of the Lemma is proved.

Recall that $I(r)$ was chosen to be an interval of length 2β such that $v(re^{i\beta}) = \int_{I(r)} \log|f(re^{i\omega})|\, d\omega$, and $\mu(r)$ is the inf of $\log|f(re^{i\omega})|$ over $I(r)$. Fix r, and let ω_0 be a point of $I(r)$ such that $\mu(r) = \log|f(re^{i\omega_0})|$. (It may happen that $\mu(r) = -\infty$, but this does not affect the argument.)

Write $I(r) = [a, b]$. Note $b - a = 2\beta$. We have

$$u(re^{i\beta/2}) = \int_a^b \log|f(re^{i\omega})|\, d\omega. \tag{41}$$

Let $c = \frac{1}{2}(a + b)$. For the proof of (b) we consider five cases.

Case I. $\omega_0 = a$.

Case II. $\omega_0 \in (a, c)$.

Case III. $\omega_0 = c$.

Case IV. $\omega_0 \in (c, b)$.

Case V. $\omega_0 = b$.

Assume Case I. For $0 < \theta < \frac{1}{2}\beta$ define $E(\theta) = [a, a] \cup [a + \beta - 2\theta, a + \beta] \cup [a + 2\beta - 2\theta, b]$. Then $E(\theta)$ has the form (35). Thus

$$u(re^{i\theta}) \geq \int_{E(\theta)} \log|f(re^{i\omega})|\, d\omega. \tag{42}$$

Using this and (41) we see that

$$u(re^{i\beta/2}) - u(re^{i\theta}) \leq \int_a^{a+\beta-2\theta} + \int_{a+\beta}^{a+2\beta-2\theta} \log|f(re^{i\omega})|\, d\omega.$$

Divide by $\beta - 2\theta$ and let $\theta \to \frac{1}{2}\beta$. The result is

$$\limsup_{\theta \to \frac{1}{2}\beta-} \frac{u(re^{i\beta/2}) - u(re^{i\theta})}{\beta - 2\theta} \leq \log|f(re^{ia})| + \log M(r).$$

Since $\log|f(re^{ia})| = \mu(r)$, the inequality above is equivalent to (b).

Now assume Case II. Let $I_1(\theta)$ be the interval with center ω_0 and length $\beta - 2\theta$, and let $I_2(\theta)$ be the interval of length $\beta - 2\theta$ whose left endpoint lies 2θ units to the right of the right endpoint of I_1. For θ sufficiently close to $\frac{1}{2}\beta$ we have $I_1 \cup I_2 \subset [a, b]$. Let $E(\theta)$ be the complement of $I_1 \cup I_2$ in $[a, b]$. Then $E(\theta)$ has the form (35). Thus (42) holds, and we have, for θ sufficiently close to $\frac{1}{2}\beta$,

$$u(re^{i\beta/2}) - u(re^{i\theta}) \leq \int_{I_1(\theta)} + \int_{I_2(\theta)} \log|f(re^{i\omega})|\, d\omega.$$

Divide by $\beta - 2\theta$ and let $\theta \to \frac{1}{2}\beta$. The first term on the right tends to $\mu(r)$ and the second one is dominated by $\log M(r)$. This proves (b) for Case II.

For Case III we let $E(\theta)$ consist of two intervals of length 2θ and one degenerate interval. The right endpoint of the first interval is $\frac{1}{2}\beta - \theta$ units to the left of c, and the left endpoint of the second interval is $\frac{1}{2}\beta - \theta$ units of the right of c. Then $E(\theta)$ has the form (35), and we deduce this time

$$u(re^{i\beta/2}) - u(re^{i\theta}) \leq \int_{c-(\frac{1}{2}\beta-\theta)}^{c+(\frac{1}{2}\beta-\theta)} + \int_a^{a+\frac{1}{2}(\beta-\theta)} + \int_{b-\frac{1}{2}(\beta-\theta)}^{b} \log|f(re^{i\omega})|\, d\omega.$$

Divide by $\beta - 2\theta$ and let $\theta \to \frac{1}{2}\beta$. The first term on the right tends to $\mu(r)$ and the sum of the other two is dominated by $\log M(r)$. This proves (b) for Case III.

Cases IV and V are handled in a fashion similar to II and I, respectively. This completes the proof of the Lemma.

6. **Proof of key inequality II.** We established (18) under the hypothesis that the function $B(t)$ whose Poisson integral is b satisfies $B(0) = 0$. However, the formula is still valid if we only assume

$$B(t) = B^*(t) + A_1 \log t + A_2, \tag{43}$$

where B^* satisfies all the hypotheses of the B in §3, and A_1, A_2 are constants. To see this, simply observe that in this case, with obvious notation, $b(re^{i\theta}) = b^*(re^{i\theta}) + \pi^{-1}A_1\theta \log r + \pi^{-1}A_2\theta$, and (18) can be established for each of the harmonic functions on the right.

As in §5 we set $B(t) = v(t^\gamma e^{i\beta})$, where $\gamma = \beta/\pi$. We are assuming $\log M(r) = O(r^\lambda)$. Since $v(te^{i\beta}) \le 2\pi \log M(t)$, it follows that $B(t) = O(t^{\beta\lambda/\pi})$ $(t \to \infty)$. Since $\beta\lambda/\pi < 1$, $B(t)$ satisfies the growth condition of §3. We can write

$$\log|f(z)| = \log|f_1(z)| + A_1 \log|z| + A_2 \tag{44}$$

with f_1 entire and $f_1(0) = 1$. It follows from this that $B(t)$ can be written in the form (43). Let b be the Poisson integral of $B(t)$, as in §2, and define $H(z)$ by $H(z) = b(z^{1/\gamma})$ $(0 < \arg z < \beta)$. Using (18), we obtain

$$H_\theta(r^\gamma) = \int_0^\infty [H_\theta(t^\gamma) + H_\theta(t^\gamma e^{i\beta})]Q(r, t)\, dt.$$

To prove our key inequality (8) all we need to do is show that (31) and (32) are true for the H being considered in this section. (In this case these inequalities are to hold for $0 < r < \infty$.)

Consider first (31). The function H and v are harmonic and subharmonic, respectively, in the angle $0 < \arg z < \beta$, and they are equal on the boundary, with the possible exception of $z = 0$, where well-defined boundary values need not exist. However, by considering the decomposition (43) one can easily deduce that in fact $H(re^{i\theta}) - v(re^{i\theta})$ tends to zero uniformly in θ as $r \to 0$. Since v and H are both $O(r^\lambda)$ in the angle as $r \to \infty$, and since $\beta\lambda < \pi$, we once again can conclude that $v(z) \le H(z)$ inside the angle. This is exactly what we needed to prove (31).

To prove (32) we define H_1 just as before, except now its domain is the full angle $0 < \arg z < \frac{1}{2}\beta$. Arguing as above, we conclude that H_1 majorizes the function w inside this angle, and the deduction of (32) proceeds as in §5.

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