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SOMEWHERE LOCALLY FLAT CODIMENSION ONE MANIFOLDS WITH 1-ULC COMPLEMENTS ARE LOCALLY FLAT

BY

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ABSTRACT. The purpose of this paper is to prove a taming theorem for a codimension one manifold that is locally flat at some point and has 1-ULC complement. We also prove that any two sufficiently close locally flat embeddings of a codimension one manifold are ambient isotopic. Since this paper was first submitted, R. Daverman has shown that, given any point on a codimension one manifold with 1-ULC complement, some neighborhood of that point lies on a codimension one sphere that is locally flat at some points and has 1-ULC complement. Hence the two papers combined prove that a codimension one manifold is locally flat if and only if its complement is 1-ULC.

Suppose M is a topological $(n-1)$-manifold, Q is a topological n-manifold, and $h\colon M \to \operatorname{Int} Q$ is an embedding. We say that h can be locally approximated by locally flat embeddings if for each $x \in M$ there is a neighborhood U of x in M such that for each $\epsilon > 0$ there is a locally flat embedding $f\colon U \to Q$ such that $d(f, h \mid U) < \epsilon$. We shall show first that h can be locally approximated by locally flat embeddings if $Q - h(M)$ is 1-ULC, $n \geq 5$, and there is some open set $U \subset M$ such that $h \mid U$ can be approximated by locally flat embeddings. Then we show that $h(M)$ is locally flat. Daverman has used this fact to show that $h(M)$ is locally flat if its complement is 1-ULC.(3) We also obtain that two close locally flat approximations of a fixed embedding are ambient isotopic by an ϵ-push.

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Key words and phrases. Codimension one, homeomorphic approximation, 1-ULC complement, radial engulfing.

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(2) Mainly because of suggestions from the referee, the present level of readability of this paper is far superior to that of the original manuscript. Since the readers will probably never know how indebted they are to his diligence and perseverance, we take it upon ourselves to express their thanks to him.

(3) A. V. Černavskiĭ has announced a very nice proof of this same result that is independent of this paper.

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Throughout this paper we will assume that M and Q are topological $(n-1)$- and n-manifolds which are separable metric with metric d. We allow our manifolds to be possibly noncompact and with or without boundary. Throughout we assume that $n \geq 5$. Let $A \cup B \subset Q$ and $\epsilon: Q \to (0, \infty)$ be a map. Denote by $N(A, \epsilon)$ the set $\{y \in Q \mid \text{for some } x \in A,\ d(x, y) < \epsilon(x)\}$. A homeomorphism H of Q is an ϵ-push of (Q, A) fixed on B if there is an isotopy H_t of Q such that $H_0 = \text{Identity}$, for each $t \in [0, 1]$, $H_t = \text{Identity}$ on B and outside $N(A, \epsilon)$, $d(H_t(x), x) < \epsilon(x)$ for each $t \in [0, 1]$ and each $x \in Q$, and $H_1 = H$. Denote by D^k the k-cell $[-1, 1]^k$ and by aD^k the k-cell $[-a, a]^k$ for any $a > 0$. Let $U \subset Q$ be an open set and $x \in \mathrm{Cl}\, U$. U is 1-LC at x if for each $\epsilon > 0$ there is a $\delta > 0$ such that each map $f: \partial D^2 \to N(x, \delta) \cap U$ extends to a map $F: D^2 \to N(x, \epsilon) \cap U$.

For Theorem 1 we use essentially the same notation as in Bryant [1]. Let $h: D^k \times [-1, 1] \to E^n$ be an embedding. We use the following notation:

$D(a, b) = h(D^k \times [a, b])$,

$D(a) = h(D^k \times \{a\})$,

$D(x, a, b) = h(x \times [a, b])$, and

$N(a, b; \epsilon) = N(h(D^k \times [a, b]), \epsilon) = N(D(a, b), \epsilon)$.

Theorem 1. *Let $f: D^{n-2} \times [-1, 1] \to E^n$ $(n \geq 5)$ be a topological embedding with $E^n - D(-1, 1)$ 1-ULC at each point of $D(-1, 1)$. Let $\epsilon > 0$. Then there is a PL homeomorphism $h: E^n \to E^n$ such that $h \circ f \circ \pi: D^{n-2} \times [-1, 1] \to E^n$ is an ϵ-approximation of f (where $\pi: D^{n-2} \times [-1, 1] \to D^{n-2} \times [-1, 0]$ is defined by $\pi(x, t) = (x, (t-1)/2)$).*

Proof. Choose real numbers $a_0, a_1, \cdots, a_q$ such that $-1 = a_0 < a_1 < a_2 < \cdots < a_q = 1$, such that diameter $D(x, a_{i-1}, a_i) < \epsilon/3$ for each $x \in D^{n-2}$ and each i $(0 < i \leq q)$ and such that $a_{i-1} > (a_i - 1)/2$. Let $b_i = (a_i - 1)/2$. Then we have $b_{i-1} < b_i < a_{i-1} < a_i$ for each i $(0 < i \leq q)$.

Choose ϵ' $(0 < \epsilon' < \epsilon/3)$ so that for any x and $x' \in D^{n-2}$ we have distance$(D(x, -1, 1), D(x', -1, 1)) < 3\epsilon'$ implies that distance$(D(x, t), D(x', t)) < \epsilon/3$ for all $t \in [-1, 1]$. Choose η_0 $(0 < \eta_0 < \epsilon')$ so that for any $x, x' \in D^{n-2}$ we have distance$(D(x, -1, 1), D(x', -1, 1)) < \eta_0$ implies that $D(x', -1, 1) \subseteq N(D(x, -1, 1), \epsilon')$.

Choose η_1 $(0 < \eta_1 < \frac{1}{2}\eta_0)$ so that $N(b_1, a_1; 2\eta_1) \cap [D(-1) \cup D(a_2, 1)] = \emptyset$ and so that for any $x, x' \in D^{n-2}$ distance$(D(x, -1, 1), D(x', -1, 1)) < 2\eta_1$ implies that $D(x', -1, 1) \subseteq N(D(x, -1, 1), \eta_0)$. Choose $\tilde{\eta}_1$ $(0 < \tilde{\eta}_1 < \eta_1)$ so that $N(b_1, a_1; \tilde{\eta}_1) \cap N(a_1, 1; \tilde{\eta}_1) \subseteq N(a_1; \eta)$. By Bryant's Theorem 4.4, there exists a neighborhood W_1 of $D(a_1, 1)$ and a PL homeomorphism $h_1: E^n \to E^n$ satisfying

(a) $h_1 = \text{identity}$ except on $N(b_1, a_1; \tilde{\eta}_1)$,

(b) $h_1 = \text{identity}$ on W_1,

(c) $D(b_1, 1) \subseteq h_1(N(a_1, 1; \tilde{\eta}_1)$, and

(d) for each $y \in E^n$ either $h_1(y) = y$ or there exists an $x \in D^{n-2}$ such that both y and $h_1(y)$ lie in $N(D(x, b_1, a_1), \tilde{\eta}_1)$.

Furthermore it follows that $D(b_1, a_2) \subseteq h_1(N(a_1, a_2; \eta_1))$. To prove this, see Figure 1 or note that $D(b_1, a_1) \subseteq D(b_1, 1) \subseteq h_1(N(a_1, 1; \tilde{\eta}_1))$ and that $D(b_1, a_1) \cap [N(a_1, 1; \tilde{\eta}_1) - N(b_1, a_1; \tilde{\eta}_1)] = \emptyset$. Since h_1 = identity except on

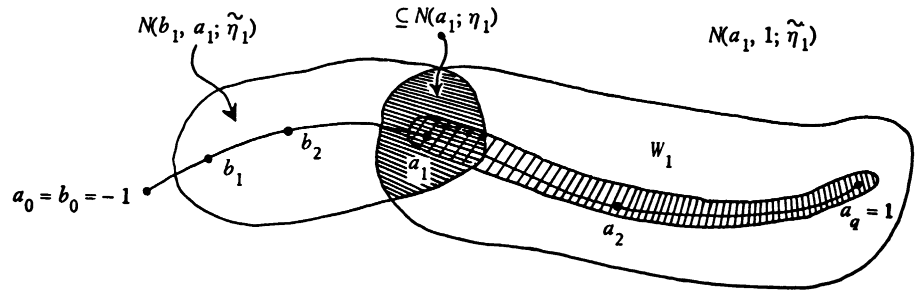

Figure 1

$N(b_1, a_1; \tilde{\eta}_1)$ it follows that $D(b_1, a_1) \subseteq h_1(N(a_1, 1; \tilde{\eta}) \cap N(b_1, a_1; \tilde{\eta}_1)) \subseteq h_1(N(a_1; \eta_1)) \subseteq h_1(N(a_1, a_2; \eta_1))$. Also $D(a_1, a_2) \subseteq W_1 \cap N(a_1, a_2; \eta_1) = h_1(W_1 \cap N(a_1, a_2; \eta_1)) \subseteq h_1(N(a_1, a_2; \eta_1))$. Together we have $D(b_1, a_2) = D(b_1, a_1) \cup D(a_1, a_2) \subseteq h_1(N(a_1, a_2; \eta_1))$ as claimed.

Now choose η_2 $(0 < \eta_2 < \frac{1}{2}\eta_1)$ so that $N(b_2, a_2; 2\eta_2) \cap [D(-1, b_1) \cup D(a_3, 1)] = \emptyset$, so that $N(b_2, a_2; \eta_2) \subseteq h_1(N(a_1, a_2; \eta_1))$ and so that for any $x, x' \in D^{n-2}$ distance $(D(x, -1, 1), D(x', -1, 1)) < 2\eta_2$ implies that $D(x', -1, 1) \subseteq N(D(x, -1, 1), \frac{1}{2}\eta_1)$. Then choose $\tilde{\eta}_2$ $(0 < \tilde{\eta}_2 < \eta_2)$ so that $N(b_2, a_2; \tilde{\eta}_2) \cap N(a_2, 1; \tilde{\eta}_2) \subseteq N(a_2; \eta_2)$. Then by Bryant's Theorem 4.4 there exists a neighborhood W_2 of $D(a_2, 1)$ and a PL homeomorphism $h_2: E^n \to E^n$ satisfying

(a) h_2 = identity except on $N(b_2, a_2; \tilde{\eta}_2)$,

(b) h_2 = identity on W_2,

(c) $D(b_2, 1) \subseteq h_2(N(a_2, 1; \tilde{\eta}_2)$, and

(d) for each $y \in E^n$ either $h_2(y) = y$ or there exists an $x \in D^{n-2}$ such that both y and $h_2(y)$ lie in $N(D(x, b_2, a_2), \tilde{\eta}_2)$.

Furthermore, just as for h_1, we can prove that $D(b_2, a_3) \subseteq h_2(N(a_2, a_3; \eta_2))$.

In general, for $2 \le k \le q$ we choose η_k analogous to η_2 above and get W_k and h_k from Theorem 4.4. More specifically, choose η_k $(0 < \eta_k < \frac{1}{2}\eta_{k-1})$ so that $N(b_k, a_k; 2\eta_k) \cap [D(-1, b_{k-1}) \cup D(a_{k+1}; 1)] = \emptyset$, so that $N(b_k, a_k; \eta_k) \subseteq h_{k-1}(N(a_{k-1}, a_k; \eta_{k-1}))$ and so that for any $x, x' \in D^{n-2}$

$$\text{distance}(D(x, -1, 1), D(x' - 1, 1)) < 2\eta_k$$

implies that $D(x', -1, 1) \subseteq N(D(x, -1, 1), \frac{1}{2}\eta_{k-1})$. Finally we choose $\tilde{\eta}_k$ $(0 < \tilde{\eta}_k < \eta_k)$ so that $N(b_k, a_k; \tilde{\eta}_k) \cap N(a_k, 1; \tilde{\eta}_k) \subseteq N(a_k; \eta_k)$. Then by Bryant's Theorem 4.4 there exists a neighborhood W_k of $D(a_k, 1)$ and a PL homeomorphism $h_k: E^n \to E^n$ satisfying

(a) h_k = identity except on $N(b_k, a_k; \tilde{\eta}_k)$,

(b) h_k = identity on W_k,

(c) $D(b_k, 1) \subseteq h_k(N(a_k, 1; \tilde{\eta}_k))$, and

(d) for each $y \in E^n$ either $h_k(y) = y$ or there exists $x \in D^{n-2}$ such that both y and $h_k(y)$ lie in $N(D(x, b_k, a_k), \tilde{\eta}_k)$.

Furthermore the same reasoning as for h_1 proves that $D(b_k, a_{k+1}) \subseteq h_k(N(a_k, a_{k+1}; \eta_k))$. Hence η_{k+1} can be chosen, as claimed, to satisfy $N(b_{k+1}, a_{k+1}; \eta_{k+1}) \subseteq h_k(a_k, a_{k+1}; \eta_k)$. This property is also used in the next paragraph.

The homeomorphism $h: E^n \to E^n$ required by the theorem is just $h = h_1^{-1} \circ h_2^{-1} \circ \cdots \circ h_q^{-1}$. We now show that h has the required properties. First note that $h_q^{-1} | D(-1, b_{q-1})$ = identity, that $D(b_{q-1}, b_q) \subseteq D(b_{q-1}, a_q) \subseteq h_{q-1}(N(a_{q-1}, a_q; \eta_{q-1})) = h_q \circ h_{q-1}(N(a_{q-1}, a_q; \eta_{q-1}))$, so $h_q^{-1}(D(b_{q-1}, b_q)) \subseteq h_{q-1}(N(a_{q-1}, a_q; \eta_{q-1}))$ and finally $h_q^{-1}(D(b_q)) \subseteq N(a_q, \eta_q)$. Continuing we get $h_{q-1}^{-1} \circ h_q^{-1} | D(-1, b_{q-2})$ = identity. From the furthermore property of h_{q-1} we get

$$D(b_{q-2}, b_{q-1}) \subseteq D(b_{q-2}, a_{q-1}) \subseteq h_{q-2}(N(a_{q-2}, a_{q-1}; \eta_{q-2}))$$
$$= h_{q-1} \circ h_{q-2}(N(a_{q-2}, a_{q-1}; \eta_{q-2}))$$

so

$$h_{q-1}^{-1} \circ h_q^{-1}(D(b_{q-2}, b_{q-1})) = h_{q-1}^{-1}(D(b_{q-2}, b_{q-1})) \subseteq h_{q-2}(N(a_{q-2}, a_{q-1}; \eta_{q-2})).$$

From the above statement about h_q^{-1} it follows immediately that $h_{q-1}^{-1} \circ h_q^{-1}(D(b_{q-1}, b_q)) \subseteq N(a_{q-1}, a_q; \eta_{q-1})$. Finally $h_{q-1}^{-1} \circ h_q^{-1}(D(b_q)) = h_q^{-1}(D(b_q)) \subseteq N(a_q, \eta_q)$. In general, for each k $(1 \le k \le q)$ we have $h_k^{-1} \circ \cdots \circ h_q^{-1} | D(-1, b_{k-1})$ = identity, $h_k^{-1} \circ \cdots \circ h_q^{-1}(D(b_{k-1}, b_k)) = h_k^{-1}(D(b_{k-1}, b_k)) \subseteq h_{k-1}(N(a_{k-1}, a_k; \eta_{k-1}))$, $h_k^{-1} \circ \cdots \circ h_q^{-1}(D(b_k, b_{k+1})) \subseteq N(a_k, a_{k+1}; \eta_k)$ and for each i $(k+1 \le i < q)$ we have $h_k^{-1} \circ \cdots \circ h_q^{-1}(D(b_i, b_{i+1})) = h_{k+1}^{-1} \circ \cdots \circ h_q^{-1}(D(b_i, b_{i+1})) \subseteq N(a_i, a_{i+1}; \eta_i)$. Thus h has the property that for each k $(0 \le k < q)$ $h(D(b_k, b_{k+1})) \subseteq N(a_k, a_{k+1}; \eta_k) \subseteq N(a_k, a_{k+1}; \epsilon)$. In other words, h has moved points about the correct distance lengthwise along D.

We now establish that h has not moved points too far sideways. Let $y \in E^n$. Then either $h_q^{-1}(y) = y$ or else there exists an $x_q \in D^{n-2}$ such that both $h_q^{-1}(y)$ and y lie in $N(D(x_q, -1, 1), \eta_q)$. Similarly either $h_{q-1}^{-1} \circ h_q^{-1}(y) = h_q^{-1}(y)$ or else there exists an $x_{q-1} \in D^{n-2}$ such that both lie in $N(D(x_{q-1}, -1, 1), \eta_{q-1})$. By our choice of η_{q-1} it follows that either $h_{q-1}^{-1} \circ h_q^{-1}(y) = y$ or else both lie in $N(D(x_{q-1}, -1, 1), \eta_{q-2})$ for some $x_{q-1} \in D^{n-2}$. Continuing we get that for each k $(1 \leq k \leq q)$ either $h_k^{-1} \circ \cdots \circ h_q^{-1}(y) = y$ or else both lie in $N(D(x_k, -1, 1), \eta_{k-1})$ for some $x_k \in D^{n-2}$. Hence h has the property that for each $y \in E^n$ either $h(y) = y$ or else both lie in $N(D(x', -1, 1), \eta_0)$ for some $x' \in D^{n-2}$. In particular, if $y = f \circ \pi(x, t)$ (with $a_k \leq t \leq a_{k+1}$), then $h(y) \in N(D(x, -1, 1), 2\epsilon')$. From the last paragraph we know that there exists an (x'', t'') such that $a_k \leq t'' \leq a_{k+1}$ the distance from $h(y)$ to $f(x'', t'') < \epsilon'$. Hence from our choice of ϵ' we have distance from $f(x'', t'')$ to $f(x, t'')$ is less than $\epsilon/3$ and since distance from $f(x, t'')$ to $f(x, t)$ is less than $\epsilon/3$ we have that distance from $h(y) = h \circ f \circ \pi(x, t)$ to $f(x, t)$ is less than ϵ. □

The following is an immediate corollary, but states the result in the form we wish to use later.

Corollary. *Let M be a connected topological $(n-1)$-manifold and let Q be a topological n-manifold $(n \geq 5)$. Let $g: M \to \operatorname{Int} Q$ be a topological embedding with $Q - g(M)$ 1-LC at each point of $g(M)$. Suppose that g can be locally approximated by locally flat embeddings at some point of M (for example, if g is locally flat at some point of M), then g can be locally approximated by locally flat embeddings at each point of M.*

Proof. For each $x \in M$ there exists an open n-cell $\mathcal{O} \subseteq Q$ and an embedding $f: D^{n-2} \times [-1, 1] \to M$ such that $x \in \operatorname{Int} f(D^{n-2} \times [-1, 1])$, $g \circ f(D^{n-2} \times [-1, 1]) \subseteq \mathcal{O}$ and $g \circ f | D^{n-2} \times [-1, 0]$ can be ϵ-approximated by locally flat embeddings for every $\epsilon > 0$. This provides us with a close locally flat approximation G of $g \circ f \circ \pi: D^{n-2} \times [-1, 1] \to \mathcal{O}$ and hence Theorem 1 gives us a homeomorphism $h: \mathcal{O} \to \mathcal{O}$ so that $h \circ G$ is a close locally flat approximation of $h \circ g \circ f \circ \pi$ which is in turn close to $g \circ f$.

Lemma 1. *Suppose $h: D^{n-1} \to E^n$ is an embedding, $0 < a < b < 1$, N is a neighborhood of $h(bD^{n-1})$, and $\epsilon > 0$. Then there is a $\delta > 0$ and neighborhoods $U' \subset V'$ of $h(aD^{n-1})$ such that if $g: (D^k - \operatorname{Int} aD^k) \times D^{n-k-1} \to E^n$ is an embedding such that $d(g, h | (D^k - \operatorname{Int} aD^k) \times D^{n-k-1}) < \delta$ then there is a neighborhood W of $g(\operatorname{Int} bD^k - aD^k \times \operatorname{Int} D^{n-k-1})$ such that if $f: D^{n-1} \to E^n$ is an embedding such that $d(f, h) < \delta$ and $f | (D^k - \operatorname{Int} aD^k) \times D^{n-k-1} = g$ then $U = U' \cup W$ is a neighborhood of $f(\operatorname{Int} bD^k \times aD^{n-k-1})$ in N that is separated by $f(D^{n-1})$ into two sides U_1 and U_2 and $V = V' \cup W$ is a neighborhood of $f(\operatorname{Int} bD^k \times aD^{n-k-1})$ that is separated by $f(D^{n-1})$ into two sides V_1 and V_2, $U_1 \subset V_1$, $U_2 \subset V_2$, $V \cap N(h(D^k - bD^{n-k-1}), \epsilon) \subset W$, $V \cap f(D^k - bD^k) \times bD^{n-k-1} = \emptyset$, and there are homotopies $f_t: \operatorname{Cl} U_1 \cup \operatorname{Cl} V_2 \to \operatorname{Cl} V$ and $g_t: \operatorname{Cl} U_2 \cup \operatorname{Cl} V_1$*

$\to \mathrm{Cl}\, V$ *satisfying the following conditions*:

1. $f_0 = Identity$ *and* $g_0 = Identity$,
2. $f_t | \mathrm{Cl}\, V_2 = Identity$ *and* $g_t | \mathrm{Cl}\, V_1 = Identity$ *for all* $t \in [0, 1]$.
3. $d(f_t, \mathrm{Id}) < \epsilon$ *and* $d(g_t, \mathrm{Id}) < \epsilon$ *for all* $t \in [0, 1]$,
4. $f_1(U_1) \subset \mathrm{Cl}\, V_2$ *and* $g_1(U_2) \subset \mathrm{Cl}\, V_1$, *and*
5. $f_t(U_1) \cap V_2 = \emptyset = g_t(U_2) \cap V_1$ *for all* $t \in [0, 1]$.

Proof. First choose $a < m_1 < m_2 < m_3 < m_4 < b$. Let N_1 be a neighborhood of $h(bD^{n-1})$ in N and δ be so small that N_1 $\epsilon/2$-retracts onto $f(bD^{n-1})$ for any δ-approximation f of h. Let $N_2 \subset N_1$ be a neighborhood of $h(M_4 D^{n-1})$ so small that $N_2 \cap f(D^{n-1}) \subset f(bD^{n-1})$ for any δ-approximation f of h. Next we claim that there is a neighborhood $V' \subset N_2$ of $h(m_3 D^{n-1})$ and $\delta > 0$ so small that $f(D^{n-1})$ separates V' for any embedding f such that $d(f, h) < \delta$. Let $V' \subset N_2$ and δ be chosen so that the inclusion $i: V' \to E^n - f(\partial D^{n-1})$ is homotopic to a retraction onto $f(m_3 D^{n-1})$ for any embedding $f: D^{n-1} \to E^n$ such that $d(f, h) < \delta$. Now pick points x and y in $V' - h(D^{n-1})$ which can be joined by an arc α from x to y that pierces $h(m_3 D^{n-1})$ once. Let β be an arc in $E^n - h(D^{n-1})$ such that $\alpha \cup \beta$ is a simple closed curve and suppose δ is less than

$$\min\{d(h(\partial D^{n-1}), \alpha[0, 1]),\ d(h(D^{n-1}), \beta[0, 1])\}.$$

Now suppose that $f: D^{n-1} \to E^n$ is an embedding such that $d(f, h) < \delta$ and such that there is an arc, say γ, from x to y in $V' - f(D^{n-1})$. Then $\gamma \cup \beta$, we can assume, is a simple closed curve in $E^n - f(D^{n-1})$ so $\gamma \cup \beta$ does not link $f(\partial D^{n-1})$. But $\gamma \cup \beta$ is homotopic to $\alpha \cup \beta$ in $E^n - f(\partial D^{n-1})$ so $\alpha \cup \beta$ does not link $f(\partial D^{n-1})$. Also $f|\partial D^{n-1}$ is homotopic to $h|\partial D^{n-1}$ in $E^n - \alpha \cup \beta$ so $\alpha \cup \beta$ does not link $h(\partial D^{n-1})$. But $\alpha \cup \beta$ pierces $h(D^{n-1})$ once so $\alpha \cup \beta$ does link $h(\partial D^{n-1})$. Therefore we have a contradiction and so V' is separated by $f(D^{n-1})$. Furthermore no simple closed curve in $V' - f(D^{n-1})$ links $f(\partial D^{n-1})$ so no simple closed curve in $V' - f(D^{n-1})$ can pierce $f(D^{n-1})$ once. Thus $V' - f(D^{n-1})$ can be written as the union of V_1' and V_2' and each set approaches $f(D^{n-1})$ from one side, i.e. there is a neighborhood V'' of $f(\mathrm{Int}\, D^{n-1})$ that is separated by $f(\mathrm{Int}\, D^{n-1})$ onto components V_1'' and V_2'' and $V_1' \cap V_2'' = \emptyset = V_2' \cap V_1''$. Now let $N_3 \subset V'$ be a neighborhood of $h(m_2 D^{n-1})$ and let δ be so small that N_3 deforms by an $\epsilon/2$-homotopy in V' to a retraction onto $f(m_2 D^{n-1})$ for any embedding f such that $d(f, h) < \delta$. Suppose further that N_4, a neighborhood of $h(m_1 D^{n-1})$, $N_4 \subset N_3$, and δ are such that $N_4 \cap f(D^{n-1}) \subset f(m_2 D^{n-1})$ for any δ-approximation f of h. Last we choose $U' \subset N_4$, U' a neighborhood of $h(aD^{n-1})$, and $\delta > 0$ so that U' is separated by $f(D^{n-1})$ for any embedding f such that $d(f, h) < \delta$.

Now let $g: (D^k - aD^k) \times D^{n-k-1} \to E^n$ be an embedding such that $d(g, h \mid (D^k - aD^k) \times D^{n-k-1}) < \delta$ and we pick a small neighborhood W of $g((\text{Int } bD^k - aD^k) \times \text{Int } bD^{n-k-1})$ that is separated by the image of g into two components W_1 and W_2 and suppose that

$$W \cap \text{Image of } g = g((\text{Int } bD^k - aD^k) \times \text{Int } bD^{n-k-1}).$$

Suppose that $f: D^{n-1} \to E^n$ is an embedding such that f extends g and $d(f, h) < \delta$. Then V' and U' are separated by $f(D^{n-1})$ into sides $U'_1 \subset V'_1$ and $U'_2 \subset V'_2$. Suppose that W_1 and W_2 are indexed so that $W_1 \cap V'_2 = \emptyset = W_2 \cap V'_1$. Then $U = U' \cup W$ and $V = V' \cap W$ are separated by $h(D^{n-1})$ into components $U_1 = U'_1 \cup W_1$, $U_2 = U'_2 \cup W_2$, $V_1 = V'_1 \cup W_1$, and $V_2 = V'_2 \cup W_2$ and $U_1 \subset V_1$ and $U_2 \subset V_2$ and $\text{Cl } V_1 \cap \text{Cl } V_2 \subset f(D^{n-1})$. Therefore it follows from the fact that $V' \subset N_2 \subset N_1$ that there are $\epsilon/2$-retractions $r'_1, r'_2: \text{Cl } V' \to \text{Cl } V'_1, \text{Cl } V'_2$, respectively. If W is chosen sufficiently close to the image of g then r'_1 and r'_2 can be extended to $\epsilon/2$-retractions $r_1, r_2: \text{Cl } V \to \text{Cl } V_1, \text{Cl } V_2$, respectively. Similarly it is possible to define homotopies $f'_t: \text{Cl } U_1 \to \text{Cl } V$ and $g'_t: \text{Cl } U_2 \to \text{Cl } V$ satisfying the conditions 1′. $f'_0 = \text{Id}$, $g'_0 = \text{Id}$; 2′. $f'_t \mid \text{Cl } U_1 \cap \text{Cl } V_2 = \text{Id} = g'_t \mid \text{Cl } U_2 \cap V_1$; 3′. $d(f'_t, \text{Id}) < \epsilon/2$ and $d(g'_t, \text{Id}) < \epsilon/2$; 4′. $f'_1(\text{Cl } U_1) \subset f(m_2 D^{n-1})$ and $g'_1(\text{Cl } U_1) \subset f(m_2 D^{n-1})$. Now define $f_t: \text{Cl } U_1 \cup \text{Cl } V_2 \to \text{Cl } V_1$ by $f_t = r_1 \circ f'_t$ on $\text{Cl } U_1$ and $f_t = \text{Id}$ on $\text{Cl } V_2$. Analogously set $g_t = r_2 \circ g'_t$ on $\text{Cl } U_2$ and $g_t = \text{Id}$ on $\text{Cl } V_1$. Then conditions 1–4 follow in turn from conditions 1′–4′ and condition 5 follows from the definitions of r_1 and r_2.

Lemma 2. *Suppose $h: D^{n-1} \to E^n$ is a topological embedding, $0 < a < 1$, and $\epsilon > 0$. Then there is a $\delta > 0$ such that if $h_0, h_1: D^{n-1} \to E^n$ are locally flat embeddings such that $d(h_i, h) < \delta$ and $h_0 \mid (D^k - aD^k) \times D^{n-k-1} = h_1 \mid (D^k - aD^k) \times D^{n-k-1}$ then for any neighborhood N of $h_1(D^k \times aD^{n-k-1})$ there is an ϵ-push H of $(E^n, h(aD^{n-1}))$ such that $Hh_0(D^k \times aD^{n-k-1}) \subset N$.*

Proof. Pick a' and b so that $a < a' < b$ and $h(bD^{n-1}) \subset N(h(aD^{n-1}), \epsilon)$. Then apply Lemma 1 with ϵ replaced by $\epsilon/3$ and a replaced by a'. Thus we obtain neighborhoods $U \subset V \subset N(h(aD^{n-1}), \epsilon)$ such that the conclusions of Lemma 1 are satisfied. Let $U_1(h_0)$ and $U_2(h_0)$ be the two sides of $h(D^{n-1})$ in U. Similarly define $U_i(h_j)$ and $V_i(h_j)$ so that $U_1(h_0)$ and $U_1(h_1)$ are on the "same side". Since $V_2(h_0)$ is 1-LC at each point of $h_0(D^{n-1}) \cap V_2(h_0)$ and $V_1(h_1)$ is 1-LC at each point of $h_1(D^{n-1}) \cap V_1(h_1)$ it is possible to modify the homotopies f_t and g_t given in Lemma 1 which retract $\text{Cl } U_1(h_0) \cup \text{Cl } V_2(h_0)$ onto $\text{Cl } V_2(h_0)$ and $\text{Cl } U_2(h_1) \cup \text{Cl } V_1(h_1)$ onto $\text{Cl } V_1(h_1)$, respectively, and to use radial engulfing (see Proposition 7 and Lemma 2 of [6]). The result is that for any closed set C_1 of $\text{Cl } V_2(h_0)$ and C_2 of $\text{Cl } V_1(h_1)$ such that $C_1 \cap V \cap h_0(D^{n-1}) = \emptyset =$

$C_2 \cap V \cap h_1(D^{n-1})$ there are $\epsilon/2$-pushes F and G of $(E^n, h(aD^{n-1}))$ fixed on C_1 and C_2, respectively such that $F(V_2(h_0)) \cup G(V_1(h_1)) \supset U$. However, we can choose C_1 and C_2 so that $V \subset C_1 \cup C_2 \cup U \cup N(h(D^k \times (D^{n-k-1} - a'D^{n-k-1})), \epsilon) \subset F(V_2(h_0)) \cup G(V_1(h_1)) \cup N(h(D^k \times (D^{n-k-1} - a'D^{n-k-1})), \epsilon)$. Since F and G can be chosen to be fixed outside V, ϵ can be chosen so small that $G^{-1}Fh_0(\text{Int } bD^k \times aD^{n-k-1}) \subset V_1(h_1)$. Thus without loss of generality we may assume that $h_0 | (D^k - aD^k) \times D^{n-k-1} = h_1 | (D^k - aD^k) \times D^{n-k-1}$, $h_0(\text{Int } bD^k \times bD^{n-k-1}) \subset V_1(h_1)$, and $d(h_i, h) < \delta$. It is possible now to modify the homotopies constructed in Lemma 1 and use radial engulfing to construct $\epsilon/2$-pushes F' and G' of $(E^n, h(aD^{n-1}))$ fixed on $h_0((D^k - bD^k) \times aD^{n-k-1})$ and on $V_2(h_1)$ such that $F'(V_2(h_1) \cup N) \cup G'(V_1(h_0)) \cup N(h(D^k \times (D^{n-k-1} - a'D^{n-k-1}), \epsilon)) \supset V$. Thus $F'^{-1}G'h_0(bD^k \times aD^{n-k-1}) \subset N$ if ϵ is sufficiently small. But $F'^{-1}G'$ is fixed on $h_1((D^k - bD^k) \times aD^{n-k-1})$ so $F'^{-1}G'h_0(D^k \times aD^{n-k-1}) \subset N$.

Addendum to Lemma 2. In the case that $h_0(D^k \times bD^{n-k-1}) \cap V_2(h_1) = \emptyset$ the ϵ-push H can be constructed to be fixed on $\text{Cl } V_2(h_1)$.

Theorem 2. *Suppose $h: D^{n-1} \to E^n$ is an embedding and $\epsilon > 0$. Then there is a $\delta > 0$ such that if $h_0, h_1: D^{n-1} \to E^n$ are locally flat embeddings such that $d(h_i, h) < \delta$ and h_0 agrees with h_1 on $(D^k - \tfrac{1}{2}D^k) \times D^{n-k-1}$ then there is an ϵ-push H of $(E^n, h(\tfrac{1}{2}D^{n-1}))$ such that $Hh_0 | D^k \times \tfrac{1}{2}D^{n-k-1} = h_1 | D^k \times \tfrac{1}{2}D^{n-k-1}$*

Proof. The proof is a slight modification of [7] or Lemma 5 of [6]. Let h_0, $h_1: D^{n-1} \to E^n$ be close locally flat approximations of h and $\tfrac{1}{2} < a < 1$. Then h_0 and h_1 can be extended to embeddings of $D^n = D^{n-1} \times [-1, 1]$ into E^n such that h_0 and h_1 agree on $(D^k - aD^k) \times D^{n-k}$. By first applying Lemma 2 and then modifying h_0 by squeezing toward the set $h_0(D^k \times D^{n-k-1})$ it follows that we can assume that $h_0(D^k \times \tfrac{1}{2}D^{n-k}) \subset h_1(D^n)$. Let $-1 = t_0 < t_1 < \cdots < t_{2m} = 1$. We can use Lemma 2 and the Addendum to Lemma 2 to modify h_0 by an ϵ'-push so that for $0 \leq i \leq (m-1)$, $h_1^{-1}h_0(D^k \times aD^{n-k-1} \times t_{2i+1}) \subset D^{n-1} \times (2i, 2i+2)$. Thus for ϵ' small enough and the t_i chosen sufficiently dense

$$d(h_1^{-1}h_0 | D^k \times aD^{n-k-1} \times [-1, 1], \text{Id})$$

is small. Thus it follows from the Main Lemma of [4] that there is an ϵ'-push H' of (D^n, aD^n) fixed on ∂D^n such that $H'h_1^{-1}h_0 | D^k \times \tfrac{1}{2}D^{n-k} = \text{Id} | D^k \times \tfrac{1}{2}D^{n-k}$. Thus we can define H to be fixed outside $h_1(D^n)$ and equal $h_1H'h_1^{-1}$ on $h_1(D^n)$. Then $Hh_0 | D^k \times \tfrac{1}{2}D^{n-k-1} = h_1H'h_1^{-1}h_0 | D^k \times \tfrac{1}{2}D^{n-k-1} = h_1 | D^k \times \tfrac{1}{2}D^{n-k-1}$. For ϵ' sufficiently small and a chosen close enough to $\tfrac{1}{2}$, H will be an ϵ-push of $h(\tfrac{1}{2}D^{n-1})$.

Theorem 3. *Suppose M is a topological $(n-1)$-manifold, Q is a topological n-manifold, $h\colon M \to \operatorname{Int} Q$ is an embedding and $\epsilon\colon Q \to R^+$ is a map. Then there is a positive function δ on M such that if $h_0, h_1\colon M \to Q$ are locally flat embeddings such that $d(h_i(x), h(x)) < \delta(x)$ for all $x \in M$ then there is an ϵ-push H of $(Q, h(M))$ such that $Hh_0 = h_1$.*

Proof. The proof is virtually identical to the proof of Theorem 5.1 of [4] using our Theorem 2 instead of their Main Lemma to get h_0 and h_1 to agree on a closed subset in Int M. Then a little push-pull near the boundary does the rest.

Theorem 4. *Suppose M and Q are topological $(n-1)$- and n-manifolds, $h\colon M \to \operatorname{Int} Q$ is an embedding that can be locally approximated (see introduction) by locally flat embeddings, and ϵ is a nonnegative map on M. Then there is an embedding $f\colon M \to Q$ such that $d(f(x), h(x)) < \epsilon(x)$ for all $x \in M$ and f is locally flat at any point x for which $\epsilon(x) > 0$.*

Proof. It is sufficient to prove the following statement. Suppose A_1 is closed in M, A_2 is compact, U_1 and U_2 are open sets such that $U_1 \supset A_1$, $U_2 \supset A_2$ and $h_i = h|U_i$ can be approximated by locally flat embeddings for $i = 1, 2$. Then there is an open set $U \subset M$ such that $A_1 \cup A_2 \subset U$ and $h|U$ can be approximated by locally flat embeddings. Let P_1, P_2 and P_3 be polyhedral neighborhoods of A_1, A_2, and $A_1 \cap A_2$ such that $P_1 \cap P_2 \subset P_3 \subset U_1 \cap U_2$, $P_1 \subset U_1$, and $P_2 \subset U_2$. Choose $\epsilon < \frac{1}{2} d(Q(P_2 - P_3), P_1)$. Let δ be given by Theorem 3 depending on ϵ for $h|P_3$. Then there is an ϵ-push H of (Q, P_3) such that $Hh_2 = h_1$ on P_3. Now define f to be h_1 on P_1 and Hh_2 on P_2. It is clear from $\epsilon < d(Q(P_2 - P_3), P_1)$ that f is an embedding.

Theorem 5. *Suppose $h\colon D^{n-1} \to \operatorname{Int} Q$ is a topological embedding, h can be locally approximated by locally flat embeddings, U is a connected open subset of Q such that $h(\operatorname{Int} D^{n-1}) \subset U$ and U is separated by $h(D^{n-1})$ into components U_1 and U_2, and U_1 is 1-LC at each point of $h_1(\operatorname{Int} D^{n-1})$. Then $h(\operatorname{Int} D^{n-1})$ has a collar in $\operatorname{Cl} U_1$.*

Proof. Suppose $\epsilon\colon \operatorname{Int} aD^{n-1} \to R^+$ is a map. It follows from Theorem 4 that there is a locally flat $\epsilon(x)$-approximation f of $h|\operatorname{Int} aD^{n-1}$. But from the proof of Lemma 2 we can push $f(\operatorname{Int} aD^{n-1})$ into U_1. Thus it suffices to show that there is a $\delta\colon \operatorname{Int} aD^{n-1} \to R^+$ such that if $f_0, f_1\colon \operatorname{Int} aD^{n-1} \to U_1$ are disjoint locally flat $\delta(x)$-approximations of $h|\operatorname{Int} aD^{n-1}$, then there is an embedding $F\colon \operatorname{Int} aD^{n-1} \times [0, 1] \to U_1$ such that $F(x, i) = f_i(x)$ for $i = 0, 1$ and $\operatorname{diam}(F(x \times [0, 1])) < \epsilon(x)$ for all $x \in \operatorname{Int} aD^{n-1}$. Choose $\epsilon'\colon \operatorname{Int} aD^{n-1} \to R^+$ so small that if $x, y \in \operatorname{Int} aD^{n-1}$, $d(x, y) < \epsilon'(x)$, and $d(f_0, h) < \epsilon'$, then $d(f_0(x), f_0(y)) < \epsilon(x)$. Now we can extend such an f_0 to $\overline{f}_0\colon \operatorname{Int} aD^{n-1} \times [0, 1]$ so that $d(\overline{f}_0(x), \overline{f}_0(y)) < \epsilon(x)$

wherever $x, y \in M \times [0, 1]$ and $d(x, y) < \epsilon'(x)$. From Lemma 2 it follows that we can engulf $f_1(\text{Int } aD^{n-1})$ and so we assume that $f_1(\text{Int } aD^{n-1}) \subset \overline{f}_0(M \times [t_0, 1])$ for some $t_0 > 0$ and that $d(\overline{f}_0^{-1} f_1, \text{incl.} \mid M \times 0) < \delta(x)$ where $\delta(x)$ is chosen from Theorem 5 depending on $\min\{1/2, \epsilon'\}$. Thus there is an ϵ'-push H of $(M \times [-1, 1], M \times 0)$ such that $H(x, t_1) = f_0^{-1} f_1(x)$ for all $x \in M$ and for some fixed $t_1 > t_0$. Let $S_2: [-1, 1] \to [-1, 1]$ be defined by setting

$$S_2(x) = \begin{cases} -1, & x = -1, \\ 0, & x = -1/2, \\ t_0, & x = t_0, \\ 1, & x = 1 \end{cases}$$

and extending linearly. Set $S = \text{Id} \times S_2: \text{Int } aD^{n-1} \times [-1, 1] \to \text{Int } aD^{n-1} \times [-1, 1]$. Then set $F = \overline{f}_0 SHS^{-1}: M \times [0, t_1] \to U_1$. $F(x, 0) = \overline{f}_0(x, 0) = f(x)$, $F(x, t_1) = \overline{f}_0 S \overline{f}_0^{-1} f_1(x) = f_1(x)$. But t_1 can be chosen so small that $\text{diam } SHS^{-1}(x \times [0, t_1]) < \epsilon'(x)$ for all $x \in \text{Int } aD^{n-1}$ so $\text{diam } F(x \times [0, t_1]) < \epsilon(x)$ for all x. Using a sequence of such F's it can be shown that $h(aD^{n-1}) \cup f_0(\text{Int } aD^{n-1})$ is the boundary of an n-cell; so $h(\text{Int } D^{n-1})$ is locally flat in U_1 at each point.

Corollary 6. *Suppose M and Q are topological $(n-1)$- and n-manifolds, respectively, and $h: M \to \text{Int } Q$ is an embedding such that $Q - h(M)$ is 1-LC at each point of $h(M)$ and for some open set $U \subset M$ there are locally flat approximations of $h \mid U$. Then $h(M)$ is locally flat.*

Proof. It follows from Corollary to Theorem 1 that h can be locally approximated by locally flat embeddings and thus we can apply Theorem 5 on a neighborhood of any point of M. Thus $h(M)$ is locally flat at each point of $h(\text{Int } M)$. However, from [3] we then obtain that $h(M)$ is locally flat.

Theorem 7. *Suppose $h: D^{n-1} \to E^n$ is an embedding such that $h \mid D^{n-2} \times \{-1\}$ is locally flat and $E^n - h(D^{n-1})$ is 1-ULC. Then $h(D^{n-1})$ is locally flat.*

Proof. We adapt a covering argument due to Černavskiĭ [Theorems 1, 2]. Let $D_+ = D^{n-2} \times [0, 1]$ and $D_- = D^{n-2} \times [-1, 0]$. First we may assume that $h: D_+ \to E^n$ and $h \mid D^{n-2} \times \{0\} = \text{Identity}$. Extend h to $\partial D_- \cup D_+$, then restrict h to ∂D-union a small enough neighborhood of $\text{Int } D^{n-2} \times \{0\}$ in D_+ that the restriction is an embedding. Thus we can find an embedding $h': \partial D_- \cup D_+ \to E^n \subset S_n$ such that $h'(D_+) \subset h(D^{n-1})$ and $h' \mid \partial D_- = \text{Identity}$. Let $p: E^n \to S^n - \partial D_-$ be a covering projection such that $p(E^{n-1} \times \{0\}) = \text{Int } D_-^{n-1}$. There is a lifting $\widetilde{h}': D_+ - (D^{n-2} - \{0\}) \to E^n$ such that $\widetilde{h}'((0, \cdots, 0, 1)) \in E^n - p^{-1}(\text{Int } D_-)$. Let

$H: E^n \to E^n$ be an embedding that moves points vertically, is fixed near $\tilde{h}'((0, \cdots, 0, 1))$, and satisfies $H\tilde{h}'(D_+ - (D^{n-2} \times 0)) \subset p^{-1}(E^n - D_-)$. Define $f: D_+ \to E^n$ by $f = pH\tilde{h}'$. Then f can be extended by the identity to an embedding $\bar{f}: D^{n-1} \to E^n$. Also $E^n - \bar{f}(D^{n-1})$ is 1-LC at such point of $\bar{f}(D_+ - (D^{n-2} \times 0))$ and so $E^n - \bar{f}(D^{n-1})$ is 1-ULC. Thus $\bar{f}(D^{n-1})$ is locally flat by Corollary 6. But H was fixed near $\tilde{h}'((0, \cdots, 0, 1))$ thus $\bar{f}(D^{n-1})$ has an $(n-1)$-cell in common with $h(D^{n-1})$ and so again by Corollary 7 $h(D^{n-1})$ is locally flat.

Corollary 8. *Suppose M is a compact PL $(n-1)$-manifold, Q is a PL n-manifold, and $h: M \to \operatorname{Int} Q$ is an embedding that can be locally approximated by locally flat embeddings. Then there is an obstruction in $H^3(M, Z_2)$ which vanishes if and only if h can be approximated by PL locally flat embeddings.*

Proof. It follows from Theorem 5 that h can be approximated by locally flat embeddings and that, for some $\delta > 0$, and two locally flat approximations that are δ-close to h are ambient isotopic by a 1-push of $(Q, h(M))$. Let h_1 be a locally flat δ-approximation of h and let ν_1 be a topological tubular neighborhood of $h_1(M)$ (i.e., an open bicollar). Then ν_1 has two PL structures, namely the structure induced by the PL structure on Q and the product structure induced by $M \times E^1$. Let $\alpha_1 \in H^3(M, Z_2) \approx H^3(M \times E^1, Z_2)$ be the obstruction to isotoping the product structure on ν_1 to the induced structure on ν_1. If h_2 is another locally flat δ-approximation of h and $\alpha_2 \in H^3(M, Z_2)$ is defined analogous to α_1, then, using the fact that h_1 and h_2 are ambient isotopic, it is easy to prove that $\alpha_1 = \alpha_2$. Hence we have a well-defined element α of $H^3(M, Z_2)$. If $\alpha = 0$, then any such h_1 can be isotoped slightly to be made PL and hence h can be approximated by PL locally flat homeomorphisms. On the other hand, if h can be δ-approximated by a PL locally flat homeomorphism h', then we can use h' for determining α and since h' is PL locally flat the two structures on the tubular neighborhood of $h'(M)$ are the same and hence $\alpha = 0$.

REFERENCES

1. J. L. Bryant, *Euclidean n-space modulo an $(n-1)$-cell*, Trans. Amer. Math. Soc. 179 (1973), 181–192.

2. A. V. Černavskiĭ, *The k-stability of homeomorphisms and the union of cells*, Dokl. Akad. Nauk SSSR 180 (1968), 1045–1047 = Soviet Math. Dokl. 9 (1968), 729–732. MR 37 #6919.

3. ———, *Locally homotopic unknotted imbeddings of manifolds*, Dokl. Akad. Nauk SSSR 181 (1968), 290–293 = Soviet Math. Dokl. 9 (1968), 835–839. MR 38 #720.

4. R. D. Edwards and R. C. Kirby, *Deformations of spaces of imbeddings*, Ann. of Math. (2) 93 (1971), 63–88. MR 44 #1032.

5. R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the hauptvermutung*, Bull. Amer. Math. Soc. 75 (1969), 742–749. MR 39 #3500.

6. C. L. Seebeck III, *Collaring an* $(n-1)$*-manifold in an* n*-manifold*, Trans. Amer. Math. Soc. 148 (1970), 63–68. MR 41 #2692.

7. P. Wright, *A uniform generalized Schoenflies theorem*, Ann. of Math. (2) 89 (1969), 292–304. MR 40 #3556.

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