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ANALYTIC DOMINATION WITH QUADRATIC FORM TYPE ESTIMATES AND NONDEGENERACY OF GROUND STATES IN QUANTUM FIELD THEORY

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ABSTRACT. We present a theorem concerning the analytic domination by a semibounded selfadjoint operator H of another linear operator A which requires only the quadratic form type estimates

$$\|H^{-1/2}((\mathrm{ad}\,A)^n H)H^{-1/2}u\| \leq c_n\|u\|$$

instead of the norm estimates

$$\|(\mathrm{ad}\,A)^n Hu\| \leq c_n\|Hu\|$$

usually required for this type of theorem. We call the new estimates "quadratic form type", since they are sometimes equivalent to

$$|((\mathrm{ad}\,A)^n Hu, u)| \leq c_n|(Hu,u)|.$$

The theorem is then applied with H the Hamiltonian for the spatially cutoff boson field model with real, bounded below, even ordered polynomial self-interaction in one space dimension and $A = \pi(g)$, the conjugate momentum to the free field. When the underlying Hilbert space of this model is represented as $L^2(Q, dq)$ where dq is a probability measure on Q, the spectrum of the von Neumann algebra generated by bounded functions of certain field operators, then e^{-tH} *maximizes support* in the sense that $e^{-tH}f$ is nonzero almost everywhere whenever f is not identically zero.

I. **Introduction.** The Hamiltonian, H, of a physical system is a selfadjoint operator on a Hilbert space which is bounded below in that its spectrum is. If the infimum of the spectrum is an eigenvalue then any corresponding eigenvector is called a *ground state*. An important physical question is whether the ground state, (assuming it exists), is *nondegenerate*, that is, whether the infimum of the spectrum of H has multiplicity one as an eigenvalue. If the ground state is nondegenerate, it is also said to be unique, though technically this uniqueness is only up to complex multiples. This problem concerning the unbounded operator H can be turned into one involving only bounded operators by considering the generated semigroup, e^{-tH}, $t \geq 0$, since, by the spectral theorem, H has a nondegenerate ground state if and only if $\|e^{-tH}\|$ is an eigenvalue of multiplicity one for e^{-tH}.

In quantum field theory the underlying Hilbert space may be viewed as an L^2 space on which e^{-tH} is *positivity preserving* in the sense that, for $t \geq 0$, $e^{-tH}f$ is

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nonnegative almost everywhere whenever f in L^2 is nonnegative almost everywhere ([5], [2], [16] and [8]). We assume that H is a selfadjoint and bounded below operator, that e^{-tH} is positivity preserving and that H has a ground state, for the remainder of the introduction.

The semigroup $t \to e^{-tH}$ is said to be *ergodic* if for every pair, f and g, of nonnegative L^2 functions there is an $s > 0$ such that $(f, e^{-sH}g) > 0$. In an extension of the classical Perron-Frobenius theory, Glimm and Jaffe [5] showed that if e^{-tH} is ergodic then the ground state of H is nondegenerate. Simon [15] has characterized ergodic semigroups in the following way: $t \to e^{-tH}$ is ergodic if and only if $e^{-tH}f$ is positive almost everywhere whenever f is nonnegative almost everywhere. When e^{-tH} has this latter property it is said to be *positivity improving*.

Why should semigroups generated by quantum field theoretic Hamiltonians be positivity improving? In certain models the L^2 space in question may be thought of as $L^2(R^\infty, q)$ where R^∞ is the countable product of real lines, R, and q is a probability measure on R^∞ [14], while H may be thought of as a differential operator in infinitely many variables. In certain finite-dimensional approximations to these models, [11], [12], [3] H becomes an elliptic differential operator with analytic coefficients. By a theorem of Nelson every analytic vector for H is an analytic function [10]. (Recall that v is an analytic vector for H if there is an $s > 0$ such that $\sum_{n=0}^{\infty} (s^n/n!) \|H^n v\| < \infty$.) But $e^{-tH}f$ is an analytic vector for H with f arbitrary, so $e^{-tH}f$ is nonzero almost everywhere or identically zero. Since we are assuming e^{-tH} is positivity preserving, if f is nonnegative almost everywhere then $e^{-tH}f$ must be positive almost everywhere. Observe that the positivity improving property is really a consequence of the following property. e^{-tH} is said to be *support maximizing* if $e^{-tH}f$ is nonzero almost everywhere whenever f is not identically zero. An operator may be positivity improving but not support maximizing [17], though clearly if an operator is support maximizing and positivity preserving then it must also be positivity improving. For an example, observe that e^{-tH} maximizes support on $L^2(O,dx)$ whenever H is an elliptic differential operator with analytic coefficients on the connected open set O [10]. Thus, when e^{-tH} is positivity preserving the nondegeneracy problem is solved once we show e^{-tH} maximizes support.

The connection between Hamiltonians and differential operators is well known ([14], [3], [11], [12] and [7]),and one is led naturally to the question of whether e^{-tH} maximizes support when H is a Hamiltonian of quantum field theory. A first step in the solution of this problem is to consider models defined in Fock space (though in the cases we consider the nondegeneracy problem is solvable by other techniques). In a previous paper [17], it was shown that when H is the free boson Hamiltonian then e^{-tH} maximizes support on $L^2(R^\infty, q)$ and on $L^2(Q)$ where Q is the spectrum of the von Neumann algebra generated by certain bounded functions of the free field ([5], [16]). In fact e^{-tH} maximizes support when H is the quantization [1] of any selfadjoint operator with positive spectrum bounded away from zero ([7] or [17]). A similar result holds when H is the Hamiltonian for the linear external source model of a boson field. Also previously considered was the momentum cutoff polaron of fixed total momentum in n dimensions. Again the Hamiltonian generates a semigroup which maximizes support. Moreover this support maximizing property remains when cutoffs are removed in two space dimensions.

Notable by its absence from this list of Fock space models is the spatially cutoff $\lambda(\phi^4)_2$ model. It is this model we wish to consider.

To prove that e^{-tH} maximizes support it suffices to prove that every analytic vector for H is either identically zero or nonzero almost everywhere. This is because $\{e^{-tH}f: f \in L^2, t > 0\}$ is exactly the set of analytic vectors for H [10]. Note that every analytic vector for d/dx on $L^2(R^1, dx)$ is an analytic function, by the Paley-Wiener theorem, and is either identically zero or nonzero almost everywhere. Similarly, on $L^2(R^\infty, q)$ any analytic vector for $\partial/\partial x_i$, $i = 1, 2, \ldots$, is an analytic function of each coordinate function x_i, when the others are fixed. This implies (by the zero-one law) that any analytic vector for every $\partial/\partial x_i$ is nonzero almost everywhere or identically zero. The annihilation operator of wave function $f \in L^2(R^1, dx)$, $A(f)$ [1] corresponds to differentiation in the direction of f [14]. Consequently to prove that a function is either identically zero or nonzero almost everywhere one needs only to verify that the function is an analytic vector for the annihilation operators, $A(f)$, when f is in a dense subset of $L^2(R^1)$. (This discussion has been informal but can be made rigorous. However, we present a different approach in §III.)

Now the basic problem is one of analytic domination. Prove that every analytic vector for H is analytic for $A(f)$ when f is an infinitely differentiable function of compact support on R^1. The standard procedure for analytic domination is due to Nelson [10]. One is required to obtain the estimate $\|A(f)u\| \leq c_0\|Hu\|$ and the commutator estimates $\|(\mathrm{ad}\, A(f))^n Hu\| \leq c_n \|Hu\|$ for positive c_n's such that the power series $\sum_{n=0}^{\infty} (c_n S^n/n!)$ has positive radius of convergence and for all u in some linear subspace containing all analytic vectors for H and invariant under H and $A(f)$. In the $(\phi^4)_2$ model the first commutator estimate, $n = 1$, $\|[A(f), H]u\| \leq c_1\|Hu\|$ is unknown. Consequently, Nelson's theorem does not apply directly. However, quadratic form estimates

$$(\mathrm{ad}(A(f)))^n H \leq c_n H + r_n, \qquad r_n > 0,$$

are known [13]. We present an analytic domination theorem which requires only such quadratic form estimates (see Theorem 1). This theorem is then applied to the Hamiltonian, H, of the spatially cutoff one space dimensional boson field with real, bounded below, even ordered polynomial self-interaction to conclude that e^{-tH} maximizes support (see Corollary 7).

This approach is nonperturbative and makes no use of mass gaps.

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II. **Analytic domination.** In this section $\mathcal{H}$ is a complex Hilbert space. For any operator T, $D(T)$ denotes the domain of T while $D^\infty(T) = \bigcap_{n=0}^\infty D(T^n)$. The main result in this section is

Theorem 1. *Let H be a selfadjoint operator on $\mathcal{H}$ with infimum (spectrum $(H)) \geq 1$. Let A be another linear operator on $\mathcal{H}$ and denote $\bigcap_{n=0}^\infty \bigcap_{m=0}^\infty D(A^n H A^m)$ by E.*

Suppose the following conditions are met:

(2.1) (i) $D^\infty(H) \subset E$;
(ii) *for each nonnegative integer n there is a $c_n \geq 0$ such that*

$$\|H^{-1/2}((\operatorname{ad} A)^n(H))H^{-1/2}u\| \leq c_n\|u\| \tag{2.2}$$

for all u in $H^{1/2}E$; and

(iii) *the power series in s, $\sum_{n=1}^\infty (c_n s^n/n!)$ has positive radius of convergence.*

Then every analytic vector for H is an analytic vector for A.

Remarks. (a) We have put

$$(\operatorname{ad} A)^0(H) = A, \qquad (\operatorname{ad} A)^1(H) = (\operatorname{ad} A)(H) = [A,H] = AH - HA$$

and for $n > 1$,

$$(\operatorname{ad} A)^{n+1}(H) = (\operatorname{ad} A)((\operatorname{ad} A)^n(H)).$$

(b) If it is known that A leaves $D^\infty(H)$ invariant then one may replace E by $D^\infty(H)$.

In fact E may be replaced with any linear space containing all the analytic vectors for H and which is left invariant by H, A, $H^{1/2}$ and $H^{-1/2}$.

(c) We view (2.2) as a quadratic form type estimate for it implies

$$|(H^{-1/2}((\operatorname{ad} A)^n(H))H^{-1/2}u, u)| \leq c_n(u,u) \tag{2.3}$$

for $u \in H^{1/2}E$ and this may be rewritten as

$$|((\operatorname{ad} A)^n(H)u, u)| \leq c_n(Hu,u) \tag{2.4}$$

for $u \in E$.

If $(\operatorname{ad} A)^n H$ is symmetric, then (2.4) yields, through (2.3), [9, p. 310], the bilinear form estimate

$$|(H^{-1/2}((\operatorname{ad} A)^n H)H^{-1/2}u, v)| \leq c_n\|u\|\,\|v\|$$

for u, v in $H^{1/2}E$. This bilinear form estimate implies (2.2) so that (2.2) may be equivalent to the quadratic form estimate (2.4).

More generally, suppose $(\operatorname{ad} A)^n H = S + B$ where S is symmetric, $H^{-1/2}BH^{-1/2}$ is bounded on $H^{1/2}E$ with norm k and $|(Su,u)| \leq l(Hu,u)$, for u in E. Then

$$\|H^{-1/2}((\operatorname{ad} A)^n(H))H^{-1/2}u\| \leq (k + l)\|u\|$$

is again true. Similar considerations apply when skew-symmetry replaces symmetry.

(d) The estimates (2.3) and corresponding quadratic form estimates (2.4) may be compared to the usual norm estimates required for this type of theorem, $\|(\operatorname{ad} A)^n Hu\| \leq c_n \|Hu\|$ [10, p. 577] and $\|Au\| \leq c_0 \|H^{1/2}u\|$ [6, p. 247].

Now we proceed with the proof of Theorem 1. Renorm $\mathcal{H}$ with $\|u\|_1 = \|H^{-1/2}u\|$. Throughout this section we assume the hypothesis of Theorem 1.

Lemma 2. *Let u be any analytic vector for H. Then there is a $t > 0$ such that*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n u\|_1 < \infty. \tag{2.5}$$

Proof. Let X denote the normed space $\mathcal{H}$ with $\| \ \|_1$. Let B denote H restricted to E. Since $E \subset D(H)$, $D(B) = E$.

Let x be in $D(B)$. Then by definition of E, $x \in D(A)$ and

$$\begin{aligned}\|Ax\|_1 &= \|H^{-1/2}AH^{-1/2}H^{1/2}x\| \\ &\leq c_0 \|H^{1/2}x\| = c_0 \|Hx\|_1.\end{aligned}$$

Since, for x in $D(B)$ we have $Bx = Hx$ it follows that

$$\|Ax\|_1 \leq c_0 \|Bx\|_1. \tag{2.6}$$

Next observe that $A: E \to E$ so that for x in $D(B)$ we have $A^n BA^m x = A^n HA^m x$ and consequently $(\operatorname{ad} A)^n(B)x = (\operatorname{ad} A)^n(H)x$. Thus

$$\begin{aligned}\|(\operatorname{ad} A)^n Bx\|_1 &= \|H^{-1/2}((\operatorname{ad} A)^n(H))H^{-1/2}H^{1/2}x\| \\ &\leq c_n \|H^{1/2}x\| = c_n \|Hx\|_1\end{aligned}$$

and

$$\|(\operatorname{ad} A)^n Bx\|_1 \leq c_n \|BX\|_1. \tag{2.7}$$

Now let u be an analytic vector for H. Then u is in $D^\infty(H) \subset E$ so $Hu = Bu$. Inductively, assume $u \in D(B^n)$ and $B^n u = H^n u$. Since $H^n u$ is in $D^\infty(H) \subset E$ it follows that $H^n u$ is in $D(B)$ and $BH^n u = B^{n+1}u$. Thus u is also an analytic vector for B. Since $\|v\|_1 \leq \|v\|$ there is an $r > 0$ such that $\sum_{n=0}^{\infty} (r^n/n!) \|B^n u\|_1 < \infty$.

The desired conclusion now follows from (2.6), (2.7) and Nelson's theorem [10, p. 577]. Q.E.D.

The next lemma is combinatorial in nature and expresses $[A^n, H^{-1}]$ in terms of $[A^k, H]$. For related expressions see [6, p. 253] and [13, p. 431].

Let P_n be the set of ordered partitions of n. That is, every p in P_n is a k-tuple of positive integers, $(p(1), \ldots, p(k))$ with $p(1) + \cdots + p(k) = n$, for some integer k, $1 \leq k \leq n$. We put $k = |p|$ and $p! = p(1)! \cdots p(k)!$. Observe that there are exactly $\binom{n-1}{k-1}$ partitions p in P_n with $|p| = k$ for a total of

$$\sum_{k=1}^{n}\binom{n-1}{k-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$$

partitions p in P_n. (For any integers $0<a<b$, $\binom{b}{a}$ is the binomial coefficient $b!/a!(b-a)!$.)

Since $D^\infty(H)=H^{1/2}D^\infty(H)\subset H^{1/2}(E)$ is dense in $\mathcal{H}$, it follows from (2.2) that $R^{1/2}((\mathrm{ad}\,A)^n(H))R^{1/2}$ extends from $H^{1/2}(E)$ to a bounded linear operator on all of $\mathcal{H}$. We have put $R=H^{-1}$. Denote this extension by $-K_n$. Again, from (2.2)

$$\|K_n\|\le c_n. \tag{2.8}$$

Lemma 3. *On $D^\infty(H)$,*

$$A^nR=RA^n+\sum_{i=1}^{n}\sum_{p\in P_i}\frac{n!}{(n-i)!\,p!}R^{1/2}K_{p(1)}\cdots K_{p(|p|)}R^{1/2}A^{n-i}. \tag{2.9}$$

Proof. First observe that the operators A^j, A^jHA^l are defined on $D^\infty(H)$ since $D^\infty(H)\subset E$. Next observe that A^jR and A^jHA^lR are also defined on $D^\infty(H)$ because R: $D^\infty(H)\to D^\infty(H)$. Consequently, all the calculations to follow are valid on $D^\infty(H)$.

We will prove (2.9) using induction on n. Let $n=1$. Then

$$\begin{aligned}AR&=RA+(AR-RA)=RA+RHARHR-RHRAHR\\&=RA+R[H,A]R=RA+R^{1/2}K_1R^{1/2}.\end{aligned}$$

Before considering the general case observe that, on $D^\infty(H)$,

$$[A^n,H]=\sum_{j=1}^{n}\binom{n}{j}((\mathrm{ad}\,A)^j(H))A^{n-j}, \tag{2.10}$$

(see [6, p. 248] or [10, p. 576]).

Now suppose (2.9) is true for $n=1,2,\dots,m$. Then

$$\begin{aligned}A^{m+1}R&=RA^{m+1}+(A^{m+1}R-RA^{m+1})\\&=RA^{m+1}+RHA^{m+1}RHR-RHRA^{m+1}HR\\&=RA^{m+1}+R[H,A^{m+1}]R\\&=RA^{m+1}-R[A^{m+1},H]R\\&=RA^{m+1}-R\sum_{j=1}^{m+1}\binom{m+1}{j}((\mathrm{ad}\,A)^j(H))A^{m+1-j}R\\&=RA^{m+1}-\sum_{j=1}^{m+1}\binom{m+1}{j}R((\mathrm{ad}\,A)^j(H))R^{1/2}H^{1/2}A^{m+1-j}R\\&=RA^{m+1}+\sum_{j=1}^{m+1}\binom{m+1}{j}R^{1/2}K_jH^{1/2}A^{m+1-j}R\end{aligned}$$

$$= RA^{m+1} + \sum_{j=1}^{m+1} \binom{m+1}{j} R^{1/2} K_j R^{1/2} A^{m+1-j}$$

$$+ \sum_{j=1}^{m+1} \binom{m+1}{j} R^{1/2} K_j H^{1/2} [A^{m+1-j}, R]$$

$$= RA^{m+1} + \sum_{j=1}^{m+1} \binom{m+1}{j} R^{1/2} K_j R^{1/2} A^{m+1-j}$$

$$+ \sum_{j=1}^{m+1} \binom{m+1}{j} R^{1/2} K_j H^{1/2}$$

$$\cdot \sum_{i=1}^{m+1-j} \sum_{p\in P_i} \frac{(m+1-j)!}{(m+1-i-j)!\,p!} R^{1/2} K_{p(1)} \cdots K_{p(|p|)} R^{1/2} A^{m+1-i-j}$$

$$= RA^{m+1} + \sum_{j=1}^{m+1} \binom{m+1}{j} R^{1/2} K_j R^{1/2} A^{m+1-j}$$

$$+ \sum_{i=1}^{m+1} \sum_{j=1}^{i-1} \sum_{p\in P_{i-j}} \binom{m+1}{j} \frac{(m+1-j)!}{(m+1-i)!\,p!}$$

$$\cdot\, R^{1/2} K_j K_{p(1)} \cdots K_{p(|p|)} R^{1/2} A^{m+1-i}.$$

If $q = (j, p(1), \ldots, p(|p|))$ then

$$\binom{m+1}{j} \frac{(m+1-j)!}{(m+1-i)!\,p!} = \frac{(m+1)!}{(m+1-i)!\,q!}.$$

Since every partition q in P_i, $1 \le i \le m+1$, can be written uniquely as $\{i\}$ or as $\{j, p(1), \ldots, p(|p|)\}$ for some $1 \le j < i$ and p in P_{i-j} the lemma now follows. Q.E.D.

The final preliminary result needed is

Lemma 4. *Let u be any analytic vector for H. Then there is an $s > 0$ such that*

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|A^n R u\| < \infty. \tag{2.11}$$

Proof. From hypothesis (iii) in the statement of the theorem there follows the existence of a positive number M with the property that

$$0 \le c_n \le M^n n!. \tag{2.12}$$

By Lemma 2, there is a $t > 0$ such that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|R^{-1/2} A^n u\| < \infty. \tag{2.13}$$

Choose s so that

(2.14) $$0 < s < \text{minimum}\,(1/2M, t, t/2M).$$

Then by (2.8), (2.9), (2.12) and the fact that $\|R^{1/2}\| \leq 1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{s^n}{n!} \|[A^n, R]u\| \\
&\quad \leq \sum_{n=1}^{\infty} \frac{s^n}{n!} \sum_{j=1}^{n} \sum_{p \in P_j} \frac{n!}{(n-j)!\,p!} \|R^{1/2} K_{p(1)} \cdots K_{p(|p|)} R^{1/2} A^{n-j} u\| \\
&\quad \leq \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{p \in P_j} \frac{s^n}{(n-j)!\,p!} \|K_{p(1)}\| \cdots \|K_{p(|p|)}\| \cdot \|R^{1/2} A^{n-j} u\| \\
&\quad \leq \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{p \in P_j} \frac{s^n}{(n-j)!\,p!} M^{p(1)} p(1)! \cdots M^{p(|p|)} p(|p|)!\, \|R^{1/2} A^{n-j} u\| \\
&\quad = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{p \in P_j} \frac{M^j s^n}{(n-j)!} \|R^{1/2} A^{n-j} u\| \\
&\quad \leq \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{(2M)^j s^n}{(n-j)!} \|R^{1/2} A^{n-j} u\| \\
&\quad \leq \sum_{n=1}^{\infty} \sum_{j=1}^{n} (2Ms)^j \frac{t^{n-j}}{(n-j)!} \|R^{1/2} A^{n-j} u\| \\
&\quad = \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} (2Ms)^{n-j+1} \frac{t^{j-1}}{(j-1)!} \|R^{1/2} A^{j-1} u\| \\
&\quad = \sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} (2Ms)^n \right) \frac{t^{j-1}}{(j-1)!} \|R^{1/2} A^{j-1} u\| \\
&\quad = \frac{2Ms}{1-2Ms} \sum_{j=0}^{\infty} \frac{t^j}{j!} \|R^{1/2} A^j u\|.
\end{aligned}$$

(2.13) now yields

(2.15) $$\sum_{n=1}^{\infty} \frac{s^n}{n!} \|[A^n, R]u\| < \infty.$$

Since $\|v\| \leq \|H^{1/2} v\|$ and since $s < t$ we finally arrive at

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{s^n}{n!} \|A^n R u\| &\leq \sum_{n=0}^{\infty} \frac{s^n}{n!} \|R A^n u\| + \sum_{n=1}^{\infty} \frac{s^n}{n!} \|[A^n, R]u\| \\
&\leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \|R^{1/2} A^n u\| + \sum_{n=1}^{\infty} \frac{s^n}{n!} \|[A^n, R]u\| < \infty.
\end{aligned}$$

Q.E.D.

Proof of Theorem 1. Let u be an analytic vector for H. Then $u = e^{-tH} v$ for some v in $\mathcal{H}$. Since $Hu = e^{-tH/2} H e^{-tH/2} v$, Hu is also an analytic vector for H. Now apply Lemma 4 to Hu. Q.E.D.

III. **An application.** We adopt the notation of [3], [5] and [13]. Proofs and further references may be found in [1], [4], [5], [13] and [16]. Let $\hat{f}(k) = (2\pi)^{-1/2} \int e^{-ikx} f(x)\,dx$ be the Fourier transform of f and let $f^{\vee}$ denote the inverse Fourier transform of f. $C_0^\infty(R^n)$ = space of real valued infinitely differentiable functions of compact support.

Now g will denote a fixed $C_0^\infty(R^1)$ function with $0 \leq g(x) \leq 1$. P will denote a fixed polynomial, $P(x) = b_q x^q + \cdots + b_0$ with $b_q > 0$, and b_i real for all i and q an even integer. The corresponding spatially cutoff interaction Hamiltonian is

$$H_I(g) = \left(\sum_{j=0}^{q} b_j \int g(x)\colon \phi^j(x)\colon dx\right)^{**}.$$

$H_I(g)$ is selfadjoint with core D.

The operator $H_0 + H_I(g)$ defined on D is essentially selfadjoint and is bounded below. Let $c(g)$ denote a number such that infimum (spectrum (H)) $= 0$ where $H = (H_0 + H_I(g))^{**} + c(g)$.

We wish to apply the results of the preceding section to $\pi(f)$ and H for f in $C_0^\infty(R^1)$. What follows is the verification of the necessary hypothesis. We are assuming f is in $C_0^\infty(R^1)$.

First let

$$E = \bigcap_{n=0}^{\infty} \bigcap_{m=0}^{\infty} D(\pi(f)^n H \pi(f)^m). \tag{3.1}$$

$\pi(f)^m$ is relatively bounded with respect to $(N+1)^{m/2}$ and $\pi(f)^n H \pi(f)^m$ is relatively bounded with respect to $(H_0+1)(N+1)^{(n+m+q)/2}$. Consequently, from the higher order estimates [13, p. 443], it follows that

$$D^\infty(H) \subset E. \tag{3.2}$$

A formal calculation shows that on D

$$[\pi(f), \colon \phi^n(g)\colon] = -ni(4\pi)^{-1/2}\colon \phi^{n-1}(gf)\colon. \tag{3.3}$$

Repeated application of (3.3) along with a semiboundedness argument [13, p. 421] shows that for each positive integer n there are $c_n > 0$ and $r_n > 1$ such that for u in D

$$|((\operatorname{ad}(\pi(f)))^n H_I(g)u, u)| \leq c_n((H + r_n)u, u). \tag{3.4}$$

Since, for any h in $L^2(R^1)$,

$$[\pi(f), \phi(h)] = -(i/2) \int \hat{f}(-k)\hat{h}(k)\,dk, \tag{3.5}$$

valid on D we may, for $n > q$, choose

$$c_n = 0 \quad \text{and} \quad r_n = 1. \tag{3.6}$$

Consequently, $1 \leq r = \max\{r_n\} < \infty$ and from (3.4) there follows

$$|(R^{1/2}((\mathrm{ad}(\pi(f)))^n H_I(g))R^{1/2}u, u)| \leq c_n \|u\|^2 \tag{3.7}$$

for all u in $(H + r)^{1/2}D$. Here we have put $R = (H + r)^{-1}$.

Since $R^{1/2}((\mathrm{ad}(\pi(f)))^n H_i(g))R^{1/2}$ is either symmetric or skew-symmetric, for each n, the quadratic form estimate (3.7) gives a bilinear form estimate which finally yields an operator estimate [9, p. 310]

$$\|R^{1/2}((\mathrm{ad}\ \pi(f))^n H_I(g))R^{1/2}u\| \leq c_n \|u\| \tag{3.8}$$

for all u in $(H + r)^{1/2}D$.

Another formal calculation shows that on D

$$[\pi(f), H_0] = -i\phi((\hat{f}w^2)^\vee). \tag{3.9}$$

Since ϕ is relatively bounded by $(N + 1)^{1/2}$ and $(N + 1)^{1/2}$ is relatively bounded with respect to H [13, p. 436], r and c_1, c_2 may be redefined so that

$$\|R^{1/2}((\mathrm{ad}\ \pi(f))^n H)R^{1/2}u\| \leq c_n \|u\| \tag{3.10}$$

for all u in $(H + r)^{1/2}D$.

In order to extend (3.10) to all of $(H + r)^{1/2}E$ we need another lemma.

Lemma 5. *Let T be a nonnegative selfadjoint operator on a Hilbert space K with bounded inverse. Then any core for T is a core for $T^{1/2}$.*

Proof. Let D be a core for T, let v be an arbitrary element of K and let $\varepsilon > 0$ be given. $T^{1/2}$ is densely defined so there is a w in the domain of $T^{1/2}$ such that $\|v - w\| < \varepsilon/2$. Since T is invertible and D is a core for T, it follows that TD is dense in K. Consequently, there is a u in D such that $\|Tu - T^{1/2}w\| < (2\|T^{-1/2}\|)^{-1}\varepsilon$. Thus

$$\|T^{1/2}u - w\| = \|T^{-1/2}(Tu - T^{1/2}w)\| \leq \|T^{-1/2}\|\,\|Tu - T^{1/2}w\| < \varepsilon/2$$

and

$$\|T^{1/2}u - w\| \leq \|T^{1/2}u - w\| + \|w - v\| = \varepsilon.$$

This shows that $T^{1/2}D$ is dense. Since $T^{1/2}$ is invertible, D is a core for $T^{1/2}$ [9, p. 166]. Q.E.D.

Since D is a core for H [13, p. 438] and $H + r$ is invertible, we have that $(H + r)^{1/2}D$ is dense. Consequently (3.10) is valid for all u in $D((\mathrm{ad}\ \pi(f))^n H)R^{1/2}$. Since $(H + r)^{1/2}E$ is in this domain we have that

$$\|R^{1/2}((\mathrm{ad}\ \pi(f))^n H)R^{1/2}u\| \leq c_n \|u\| \tag{3.11}$$

for all u in $(H + r)^{1/2}E$.

$\pi(f)(N+1)^{-1/2}$ is a bounded operator [13, p. 424], $(N+1)^{+1/2}(H+r')^{-1/2}$ is a bounded operator for r' sufficiently large [13, p. 437] so by redefining r to be at least as large as r' one finds that there is a $c_0 > 0$ such that

$$\|R^{1/2}\pi(f)R^{1/2}u\| \leq c_0\|u\| \tag{3.12}$$

for all u in $H^{1/2}E$ (and in fact for all u).

From (3.6) it is clear that

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} c_n < \infty \tag{3.13}$$

for every $s > 0$.

Observe that (3.2), (3.11), (3.12) and (3.13) are exactly the hypothesis (i)–(iii) needed for the application of Theorem 1. Consequently we may conclude

Corollary 6. *Every analytic vector for H is an analytic vector for $\pi(f)$, providing f is in $C_0^\infty(R^1)$.*

Let M be the von Neumann algebra generated by $M = \{e^{i\phi(f)}: f \in C_0^\infty(R^1)\}$. Let Q be the spectrum of M. There is a probability measure λ on Q and a unitary operator W: $\mathfrak{F} \to L^2(Q,\lambda)$ such that $WMW^{-1} = L_\infty(Q)$ and $W1 = 1$ where the 1 on the right is the constant function while the 1 on the left is the complex number [5, p. 373].

Corollary 7. *$We^{-tH}W^{-1}$ maximizes support on $L^2(Q,\lambda)$.*

Proof. Let $u \in L^2(Q,\lambda)$ be arbitrary and let $v = e^{-tH}W^{-1}u$, so that v is an analytic vector for H.

Since $\{e^{i\pi(g)}, e^{i\phi(f)}: f \in C_0^\infty(R^1)\}$ is irreducible and since $e^{i\pi(g)}e^{i\phi(f)} = e^{-i(f,g)}e^{i\phi(f)}e^{i\pi(g)}$, it follows from [7] and the previous corollary that the largest projection in M which annihilates v is the zero projection. Consequently the largest projection in $L^\infty(Q)$ which annihilates v is the zero projection and so $Wv = We^{-tH}W^{-1}u$ is nonzero almost everywhere or zero almost everywhere. Q.E.D.

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