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GROUP ALGEBRAS WHOSE SIMPLE MODULES ARE INJECTIVE

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ABSTRACT. Let F be either a field of char 0 with all roots of unity or a field of char $p > 0$. Let G be a countable group. Then all simple $F[G]$-modules are injective if and only if G is locally finite with no elements of order char F and G has an abelian subgroup of finite index.

The condition that all simple modules over a ring be injective first appeared in a theorem due to Kaplansky: a commutative ring satisfies the condition if and only if it is von Neumann regular. Several people have studied the property for noncommutative rings, a recent example being [3]. The authors of that paper suggest the problem of characterizing group algebras with this condition. In this paper we make substantial progress by offering

Theorem 3. *Let F be either a field of* char 0 *with all roots of unity or a field of* char $p > 0$. *Let G be a countable group. Then all simple $F[G]$-modules are injective if and only if G is locally finite with no elements of order* char F *and G has an abelian subgroup of finite index.*

The proof is divided into three parts. In §1, we show that F is injective as an $F[G]$-module if and only if G is locally finite with no elements of order char F. In the second and crucial section, we show that for a certain class of rings ("locally Wedderburn algebras") the condition that all simple modules are injective is equivalent to the property that all simple modules are finite dimensional over their commuting rings. In §3, we prove the main theorem by showing that if all simple modules are finite dimensional over their commuting rings then G is abelian-by-finite.

We would like to thank D.S. Passman for suggestions that shortened and improved our work.

1. **Results of Villamayor.** Most of this section can be gleaned from several of Villamayor's papers. However, the statements and proofs that appear here are new.

The field F is a right $F[G]$-module under the trivial action (i.e. if $k \in F$, then $k \cdot a = k(a)\varepsilon$ where ε is the augmentation map). We characterize those G with F being $F[G]$-injective. By considering maps from ideals of $F[H]$ to F, the reader can casily prove

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Lemma 1. *If H is a subgroup of G and F is $F[G]$-injective then F is $F[H]$-injective.*

Lemma 2. *Let G be a finite group. F is $F[G]$-injective if and only if $|G| \neq 0$ in F.*

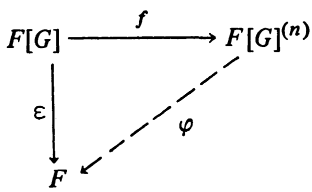
Proof. If $|G| \neq 0$ on F, then by Maschke's theorem $F[G]$ is semisimple and the result follows.

Suppose ch $F = p > 0$. By Lemma 1 the converse follows if we show that F is not $F[G]$-injective for $G = \langle x \rangle$ cyclic of order p. Let A be the augmentation ideal of $F[G]$ with F-basis $x - 1,\ x^2 - 1, \ldots, x^{p-1} - 1$. $f\colon A \to F$ given by $(\sum \lambda_i(x^i - 1))f = \sum \lambda_i i$ is a well-defined $F[G]$-map. If F is $F[G]$-injective f can be extended to $F[G]$. That is, $\exists c \in F \ni 1 = (x-1)f = c \cdot (x-1) = 0$, a contradiction.

Theorem 1([3], [7]). *F is $F[G]$-injective if and only if G is locally finite and the order of no element of G vanishes in F.*

Proof. Suppose F is $F[G]$-injective and $g_1, \ldots, g_n \in G \ni \langle g_1, \ldots, g_n \rangle$ is infinite. Define $f\colon F[G] \to F[G]^{(n)}$ by $(r)f = ((g_1 - 1)r, \ldots, (g_n - 1)r)$, an $F[G]$-map. By [4, p. 105] f is injective.

Complete



and set $\lambda_i = (0, \ldots, 0, 1, 0, \ldots, 0)\varphi$ with ith coordinate 1. $1 = (1)\varepsilon = (1)f\varphi = (g_1 - 1, \cdots, g_n - 1)\varphi = \Sigma\lambda_i \cdot (g_i - 1) = 0$, a contradiction. Thus G is locally finite; the other condition follows from the lemmas.

Conversely suppose G is such a group, $0 \neq I$ is a right ideal of $F[G]$ and $0 \neq f\colon I \to F$ is an $F[G]$-map. If $u \in I \ni (u)f \neq 0$ let $G_0 = \langle \text{supp } u \rangle$, a finite group with $I \cap F[G_0] \neq 0$. By Lemma 2, $f \mid I \cap F[G_0]$ can be extended to $F[G_0]$; i.e. $\exists\lambda \in F \ni a \in I \cap F[G_0] \Rightarrow (a)f = \lambda \cdot a = \lambda(a)\varepsilon$. Now let $b \in I$. There is a finite subgroup G_1 containing G_0 and supp b. As above $\exists\lambda' \in F \ni (b)f = \lambda'(b)\varepsilon$ and $(u)f = \lambda'(u)\varepsilon$. Then $(\lambda - \lambda')(u)\varepsilon = 0$. But $(u)\varepsilon \neq 0$ since $(u)f \neq 0$. Therefore $\lambda = \lambda'$. Thus $(b)f = \lambda \cdot b$ and f can be extended.

2. **Locally Wedderburn algebras.** If all simple $F[G]$-modules are injective then Theorem 1 in conjunction with Maschke's theorem implies that $F[H]$ is a finite dimensional semisimple subalgebra for every finitely generated subgroup $H \subseteq G$. We isolate this situation.

Definition. Let A be an algebra with 1 over F. A is a *locally Wedderburn algebra* if every finite set of elements in A is contained in a semisimple subalgebra finite dimensional over F.

Recall that, in general, if V is an A-module and T is a subset of A then the annihilator $l(T) = \{v \in V \mid vt = 0 \forall t \in T\}$.

Lemma 3. *Let A be a locally Wedderburn algebra. Simple modules finite dimensional over their commuting rings are injective.*

Proof. Let V be a simple A-module finite dimensional over its commuting ring C. If V is not injective there is a right ideal $I \subseteq A$ and an A-map f: $I \to V$ which cannot be lifted to A. Setting $D(B) = \{v \in V \mid vi = (i)f \forall i \in I \cap B\}$ for any subalgebra $B \subseteq A$, our assumption is $\bigcap_{B \in \zeta} D(B) = \varnothing$ where the intersection is taken over all finite dimensional semisimple subalgebras B. Since all modules over such an algebra are injective, $D(B) \neq \varnothing \forall B \in \zeta$.

Choose $B_0 \in \zeta \ni d = \dim_C l(I \cap B_0)$ is minimal. By the empty intersection $\exists B_1 \in \zeta \ni D(B_0) \nsubseteq D(B_1)$. If $B_2 \in \zeta$ contains a basis for B_0 and B_1 then $B_0 \subseteq B_2$ implies $\varnothing \neq D(B_2) \subsetneq D(B_0)$. Thus if $\omega \in D(B_2)$, $D(B_2) = \omega + l(I \cap B_2) \subsetneq \omega + l(I \cap B_0) = D(B_0)$, contradicting the minimality of d.

The next four technical lemmas are needed to prove a converse to Lemma 3. The first is essentially due to D. S. Passman (private communication).

Lemma 4. *Let A be any countable union of F-algebras* $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ (*with the same* 1) *and let V be an injective A-module of countable dimension over F. If I is any right ideal of A then the sequence* $l(I \cap A_1) \supseteq l(I \cap A_2) \supseteq \cdots$ *terminates.*

Proof. Suppose that for some fixed I the sequence does not terminate and write $L = \bigcap_i l(I \cap A_i)$. Let W be the F-space of all countably infinite sequences in F. Then $\dim_F W$ is uncountable. We will construct a linear transformation T: $W \to V/L$ which is one-to-one. Since $\dim_F V/L$ is countable this will yield the desired contradiction.

By renumbering we can assume that the annihilators strictly decrease. Choose $v_i \in l(I \cap A_i) \backslash l(I \cap A_{i+1})$ for $i = 1, 2, \ldots$. Clearly $v_1, \ldots, v_n$ are F-linearly independent modulo $l(I \cap A_{n+1})$. Fix $\omega = (\alpha_1, \alpha_2, \ldots) \in W$. The map f_n: $A_n \to V$ given by $1 \to \sum_{i=1}^n \alpha_i v_i$ is an A_n-map. If $k \leq n$ and $a_k \in I \cap A_k$ then

$$(a_k)f_n = \left(\sum_{i=1}^n \alpha_i v_i\right)a_k = \left(\sum_{i=1}^k \alpha_i v_i\right)a_k = (a_k)f_k.$$

Thus there is an A-map f: $I \to V$ where $f \mid I \cap A_n = f_n \mid I \cap A_n$.

Since V is injective f lifts to A. Set $(1)f = \hat{\omega} \in V$. By construction $\hat{\omega} - \sum_{i=1}^n \alpha_i v_i \in l(I \cap A_n)$. The correspondence $\omega \to \hat{\omega}$ induces a function T: $W \to V/L$. Regarding V/L as a subspace of $\prod_{n=1}^\infty V/l(I \cap A_n)$ we see

$$(\alpha_1, \alpha_2, \ldots) \to (\overline{\alpha_1 v_1}, \overline{\alpha_1 v_1 + \alpha_2 v_2}, \ldots).$$

Thus T is an F-linear map. $\omega T = 0 \Rightarrow$ for all $n \geq 2$, $\sum_{i=1}^n \alpha_i v_i \in l(I \cap A_n) \Rightarrow$ for all $n \geq 2$, $\alpha_1, \ldots, \alpha_{m-1} = 0$ by linear independence mod A_n. That is, $\omega = 0$ and T is one-to-one.

Lemma 5. *Let* $R \subseteq S$ *be rings and* f: $W \to V$ *be an R-monomorphism from the* *simple R-module W (with commuting ring C) to the simple S-module V (with commuting ring D). If $\omega_1, \ldots, \omega_n \in W$ are independent over C then $\omega_1 f, \ldots, \omega_n f$ are independent over D.*

Proof. If not, $\exists d_i \in D \ni \omega_1 f = \sum_{i>1} d_i(\omega_i f)$. By density $\exists r \in R \ni \omega_1 r = \omega_1$ and $\omega_i r = 0$ for $i > 1$.

$$\omega_1 f = (\omega_1 r)f = (\omega_1 f)r = \sum d_i(\omega_i r)f = 0 \Rightarrow \omega_1 = 0,$$

a contradiction.

Lemma 6. *Let A be a countable dimensional locally Wedderburn algebra and let $\mathcal{X}$ be a simple A-module infinite dimensional over its commuting ring D. Then there are finite dimensional semisimple subalgebras $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and subspaces $V_1, V_2, V_3, \ldots$ of $\mathcal{X} \ni$*

(i) $A = \bigcup A_i$.

(ii) *V_i is a simple A_i-submodule with commuting ring C_i.*

(iii) *There are A_i-monomorphisms π_i: $V_i \to V_{i+1}$.*

(iv) $\dim_{C_{i+1}} V_{i+1} > \dim_{C_i} V_i$.

Proof. Let $\{b_1, b_2, \ldots\}$ be a basis for A. We proceed by induction, proving that $b_n \in A_n$. Let A_1 be any finite dimensional semisimple subalgebra containing b_1 and let V_1 be any simple A_1-submodule of $\mathcal{X}$. Assume A_m and V_m have been chosen. Let $u \in \mathcal{X}$ be D-independent of V_m (such an element exists by a dimension argument). Since V_m is finite dimensional, by density $\exists a \in A \ni V_m a = 0$ and $0 \neq ua \in V_m$. Let $A_{m+1} \subseteq A$ be a finite dimensional semisimple subalgebra containing A_m, a, and b_{m+1}. $\mathcal{X}$ decomposes as some direct sum $\bigoplus_\alpha U_\alpha$ of simple A_{m+1}-submodules; let π_α: $\mathcal{X} \to U_\alpha$ be the canonical A_{m+1}-projection. Clearly $\exists \beta \ni (ua)\pi_\beta \neq 0$.

Set $V_{m+1} = U_\beta$. Then π_β: $V_m \to V_{m+1}$ is an A_m monomorphism by the transitive action of A_m on V_m. If C_{m+1} is the commuting ring of V_{m+1}, then $u\pi_\beta$ and $V_m \pi_\beta$ are independent over C_{m+1} since π_β is an A_{m+1}-map and $a \in A_{m+1}$ annihilates V_m and not u. Dimensions over the commuting rings go up by Lemma 5.□

Let A be a countable dimensional locally Wedderburn algebra and $\mathcal{V}$ be a simple A-module with commuting ring D. As in Lemma 6, we see $A = \bigcup A_i$ where each A_i is semisimple. Since A_i has only finitely many isomorphism classes of simple modules we know that $\mathcal{V} = \bigoplus \sum_{j=1}^{t_i} V_{ij}$ where V_{ij} is a homogeneous component for a suitable simple A_i-module. Furthermore each V_{ij} is clearly a D-subspace.

Lemma 7. *Let $A = \bigcup A_i$ be a countable dimensional locally Wedderburn algebra and let $\mathcal{V}$ be a simple A-module infinite dimensional over its commuting ring D. Then for each i there exists a simple A_i-module U_i and A_i-monomorphism f_i: $U_i \to U_{i+1}$ such that the U_i homogeneous component of $\mathcal{V}$ (as an A_i-module) is infinite dimensional over D.*

Proof. For each i, $V = \bigoplus \sum_{j=1}^{t_i} V_{ij}$ and $\dim_D V = \infty$ whence for some j, $\dim_D V_{ij} = \infty$. Take as U_1 any simple A_1-submodule with infinite dimensional homogeneous component. Given U_{n-1} then $U_{n-1} \subseteq \bigoplus \sum_{j=1}^{t_n} V_{nj}$ as A_{n-1} modules. If U_{n-1} is contained isomorphically in only the D finite dimensional V_{nj}, then the infinite dimensional homogeneous component of U_{n-1} must be included in this finite direct sum of finite dimensional subspaces, a contradiction. Hence U_{n-1} is contained isomorphically in some infinite dimensional V_{nj}. Since V_{nj} is a direct sum of simple A_n-modules, by looking at the projection maps we see that U_{n-1} is contained isomorphically in some U_n with homogeneous component V_{nj}.

Theorem 2. *Let A be a countable dimensional locally Wedderburn algebra. All simple A-modules are injective if and only if all simple A-modules are finite dimensional over their commuting rings.*

Proof. One direction has already been proved in Lemma 3. Thus we assume A has a simple module infinite dimensional over its commuting ring and construct a simple module which is not injective. Following Lemma 6 we may assume that the infinite dimensional simple module is $\mathcal{V} = \bigcup V_i$ with $V_1 \subseteq V_2 \subseteq \cdots$. If C is the commuting ring of $\mathcal{V}$ then $\dim_C \mathcal{V} = \infty$ by Lemma 5.

We first show that there exists an integer N such that for all $n \geqslant N$, the homogeneous component of V_n has finite C-dimension.

Suppose not.

Fix $0 \neq u \in V_1$. As a C_i-space $V_i = C_i u \oplus V_1'$. Define $a_i \in A_i$ with action $\left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$ and set $I = a_1 A + a_2 A + \cdots$. We first show that $l_{\mathcal{V}}(I) = Cu$. Clearly $ua_i = 0 \forall i$ and so $Cu \subseteq l(I)$. If $v \in l(I)$ and if u and v are C-independent then $\exists b \in A \ni v \cdot b = v$ and $u \cdot b = 0$. Choose $n \ni b \in A_n$ and $v \in V_n$. $va_n = 0$ $\Rightarrow v$ and u are C_n-dependent. Thus $ub = 0 \Rightarrow vb = 0 \Rightarrow v = 0$, a contradiction.

By Lemma 4, there exists j such that $\dim_C l(I \cap A_j) = 1$. As an A_j-module the homogeneous component T of $\mathcal{V}$ corresponding to V_j is the internal direct sum $\sum_{n=1}^{\infty} V_{jn}$ where $V_{j1} = V_j$ and the subscript map $V_j \to V_{jn}$ is an A_j-isomorphism. $\{u_n \mid n = 1, 2, \ldots\} \subseteq l(I \cap A_j)$ since $u \in l(I)$. Thus there exists c_n in C such that $c_n u = u_n$. If v is in V_{jn} then there exists x in A_j such that $v = u_n x = c_n ux$ in CV_j. Thus $\dim_C T = \dim_{C_j} V_j < \infty$, a contradiction.

Pick U_n as in Lemma 7. For $n \geqslant N$, U_n is not isomorphic to V_n. By taking a subsequence we can further assume $U_n \subseteq V_{n+1}$.

Define $\varphi_i \colon U_i \oplus V_i \to U_{i+1} \oplus V_{i+1}$ by $(u, v)\varphi_i = ((u)f_i, u + v)$, an A_i-monomorphism. The direct limit $\mathcal{W}$ of this system is an A-module under the obvious action and the map $v \to (0, v)$ is an A-injection of $\mathcal{V}$ into $\mathcal{W}$. A typical nonzero element of $\mathcal{W}$ can be represented by an ordered pair (u, v) where $u \in U_n$ and $0 \neq v \in V_n$. Since U_n and V_n are nonisomorphic simple A_n-modules $\exists a \in A_n$ whose action on V_n is the identity and whose action on U_n is zero. That is, $(u, v) \cdot a = (0, v) \in \mathcal{V}$ and so $\mathcal{W}$ is a proper essential extension of $\mathcal{V}$.

Corollary 1. *Let F be a field and $G \neq 1$ a countable locally finite group with no* *elements of order* char *F. If G has no nonidentity finite normal subgroups, then* $F[G]$ *has a simple module which is not injective.*

Proof. Suppose all simple $F[G]$-modules are injective. By [2], $F[G]$ is a primitive ring and hence, by Theorem 2, $F[G]$ is a simple Artinian ring. By [4, p. 7], G is finite and hence G is a finite nonidentity normal subgroup contradicting the hypothesis.

3. Group algebras.

Lemma 8. *Let A be a locally Wedderburn algebra whose finite dimensional semisimple subalgebras can be chosen so that each is a direct sum of matrix algebras over* (*commutative*) *fields. If V is a simple A-module finite dimensional over its commuting ring, then* $\mathrm{Hom}_A(V, V)$ *is a field.*

Proof. It suffices to assume $A \simeq D_n$ for some division ring D and prove that D is a field. There is an idempotent $e \in A \ni eAe \simeq D$. If $x, y \in eAe$ choose a semisimple subalgebra B containing x, y and e. eBe is the direct sum of algebras fK_mf where f is an idempotent in K_m and K is a field. However

$$\exists r \leq m \quad \ni \begin{pmatrix} K_r & 0 \\ 0 & 0 \end{pmatrix} \simeq fK_mf \subseteq eAe$$

has no zero divisors. Thus fK_mf is isomorphic to K or 0 and eBe is commutative. In particular, x and y commute.

Lemma 9. (i) *If* char $F = p > 0$ *and H is a finite group without p-elements then* $F[H]$ *is a sum of matrix algebras over fields.*

(ii) *If* char $F = 0$, *F has all roots of unity, and H is a finite group, then* $F[H]$ *is a sum of matrix algebras over fields.*

Proof. (i) By Maschke's theorem $F[G]$ is locally Wedderburn. Let $P \subseteq F$ be the prime field and let H be a finite subgroup of G. $P[H]$ is a direct sum of matrix algebras over fields by Wedderburn's theorem on finite division algebras. Since no radical appears when tensoring up, the result follows.

(ii) This is an immediate consequence of the Brauer splitting theorem for finite groups [1]. □

Let $\Delta(G) = \{g \in G\colon [G\colon C(g)] < \infty\}$.

Lemma 10. *Let* $H = \Delta(H)$ *be a linear group. Then* $[H\colon Z(H)] < \infty$.

Proof. Choose $h_1, h_2, \cdots, h_n \in H$ to span the linear span of H and set $Z = \bigcap_{i=1}^n C(h_i)$. Then $[H\colon Z] < \infty$. Z centralizes the h_i and hence their linear span. Thus $Z = Z(H)$.

Theorem 3. *Let F be either a field of* char 0 *with all roots of unity or a field of* char $p > 0$. *Let G be a countable group. Then all simple* $F[G]$*-modules are injective if and only if G is locally finite with no elements of order* char *F and G has an abelian subgroup of finite index.*

Proof. If G has an abelian subgroup of finite index, then $F[G]$ has bounded representation degree [4] and so a simple $F[G]$-module is finite dimensional over its commuting ring. But $F[G]$ is locally Wedderburn and hence by Theorem 2 all simple $F[G]$-modules are injective.

Conversely, if all simple $F[G]$-modules are injective then, by Theorem 1, G is locally finite with no elements of order char F. We can write $G = \bigcup_{n=1}^{\infty} G_n$ where $G_n \subsetneq G_{n+1}$ and each G_n is finite. If there exist simple $F[G_i]$-modules V_i and $F[G_i]$-monomorphisms $f_i\colon V_i \to V_{i+1}$ and if $\dim_{C_i} V_i$ is unbounded (C_i is the commuting ring of V_i), then as in Lemmas 5 and 6, there exists a simple module infinite dimensional over its commuting ring. This contradicts Theorem 2. Hence there must exist an n and a simple $F[G_n]$-module V such that if $m > n$, W a simple $F[G_m]$-module and $f\colon V \to W$ a $F[G_n]$-monomorphism, then $\dim_{C(V)} V = \dim_{C(W)} W$.

Let e be the central idempotent of $F[G_n]$ corresponding to V. Let $r = \dim_{C(V)} V$. Then $eF[G_n] = C(V)_r$ (the $r \times r$ matrix ring over $C(V)$). Let $m \geq n$. $F[G_m] = S_1 \oplus \cdots \oplus S_t$ where S_i's are minimal ideals. Write $e = a_1 + \cdots + a_t$. If $a_i \neq 0$, then projection onto S_i gives a monomorphism $C(V)_r \to S_i$ and hence there exists a $F[G_n]$-monomorphism from $V \to V_i$ where V_i is a simple S_i-module. It follows that S_i is also an $r \times r$ matrix ring and a_i is the identity of S_i. Therefore e is in the center of $F[G_m]$ and hence in the center of $F[G]$.

$eF[G]$ satisfies a polynomial identity. To see this let W be a simple $eF[G]$-module. Since $We \neq 0$, there exists an $F[G_n]$-monomorphism $V \to W$. $\dim_{C(W)} W = r$; otherwise as in the proof of Lemma 6, there exists G_m and X a simple $F[G_m]$-module with $\dim_{C(X)} X > r$ and an $F[G_n]$-monomorphism $V \to X$, a contradiction. By Lemma 8, $C(W)$ is a field whence $eF[G]$ is a subdirect sum of $r \times r$ matrix rings over fields. Thus $eF[G]$ satisfies the standard identity of degree $2r$.

By Theorem 3.3 of [5], $[G\colon \Delta] < \infty$ and $|\Delta'| < \infty$. For each $x \in \Delta'$, not 1, $1 - x \notin JK[G] = 0$; hence there exists an irreducible representation ρ with $\rho(x) \neq 1$. If $\bar{\Delta} = \rho(\Delta)$ and $\bar{G} = \rho(G)$, then $\bar{G}$ is a linear group. By Lemma 10, $[\bar{\Delta}\colon Z(\bar{\Delta})] < \infty$. Observe that $Z(\bar{\Delta}) \lhd \bar{G}$. If A_x is the complete inverse image of $Z(\bar{\Delta})$ in G, then $A_x \subseteq \Delta$, $[\Delta\colon A_x] < \infty$, and $A_x \lhd G$. Since $\bar{A}_x$ is abelian and $x \neq 1$, it follows that $x \notin (A_x)'$. Set $A = \bigcap\{A_x\colon x \in \Delta', x \neq 1\}$. Then $A \lhd G$ and $[G\colon A] < \infty$. A is abelian since $A \subseteq \Delta$ and $A \cap \Delta' = 1$.

Remark. Alternatively, one may prove the last theorem by showing that all simple $K[G]$-modules are also finite dimensional over their commuting rings whenever K is an algebraically closed field of char F. Standard techniques in finite group representation theory can be invoked to show that $C[G]$ shares the same property. Finally, one applies an analytic result of Thoma [6] to show that G is abelian-by-finite.

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