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AN INDUCTION PRINCIPLE FOR SPECTRAL AND REARRANGEMENT INEQUALITIES(1)

BY

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ABSTRACT. In this paper, expressions of the form $f \prec g$ or $f \prec\prec g$ (where $\prec$ and $\prec\prec$ denote the Hardy-Littlewood-Pólya spectral order relations) are called spectral inequalities. Here a general induction principle for spectral and rearrangement inequalities involving a pair of n-tuples in R^n as well as their decreasing and increasing rearrangements is developed. This induction principle proves that such spectral or rearrangement inequalities hold iff they hold for the case when $n = 2$, and that, under some mild conditions, this discrete result can be generalized to include measurable functions with integrable positive parts. A similar induction principle for spectral and rearrangement inequalities involving more than two measurable functions is also established. With this induction principle, some well-known spectral or rearrangement inequalities are obtained as particular cases and additional new results given.

Introduction. In [2], characterizations in terms of spectral inequalities are given for the uniform integrability or relatively weak compactness of a family of integrable functions. With these characterizations, the present author proved an extension and a 'converse' of the classical Lebesgue's dominated convergence theorem where domination in the usual partial order sense ($\leq$) can now be replaced by domination in the weak spectral order sense ($\prec\prec$) or in the strong spectral order sense ($\prec$) under some mild restrictions (see [2, §5]).

In [1, Chapters V and VII, pp. 156–238], spectral inequalities are shown to be fundamental tools for the study of rearrangement inequalities (i.e. inequalities involving the equimeasurable rearrangements of functions) which in turn are essential for the investigations concerning rearrangement invariant Banach function spaces and interpolations of operators. Moreover, by means of an extended form of a rearrangement theorem of Hardy, Littlewood and Pólya given in [3, Theorem 2.5], it is proven in [3, Theorem 3.5] that one of the most important classical

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inequalities, viz. Jensen's inequality, can be obtained as a direct consequence of a simple spectral inequality. Furthermore, via the spectral inequalities established in [3], [4] and [5], it is shown that the Hardy-Littlewood-Pólya-type rearrangement theorems obtained by the present author in [3, Theorems 2.1, 2.3 and 2.5] serve as a unifying thread connecting many well-known rearrangement inequalities such as those of Jensen (cf. [3, Theorem 3.5]), Hardy-Littlewood-Luxemburg (cf. [4, Corollary 5.3]), London (cf. [4, Theorem 6.1]), Jurkat and Ryser (cf. [5, Theorem 3.1]) and others.

In a subsequent paper, we show that spectral inequalities also play a fundamental role in our new approach to the study of martingales through the theory of equimeasurable rearrangements of functions.

In this paper, we investigate the methods by which the spectral inequalities given in [4] and [5] were obtained, and develop some very simple criterion from which to derive spectral inequalities of this type. In view of the analogy between the methods used in [4] and [5], we realize that the basic principle is to start with a (continuous) real-valued function Ψ of two real variables u, v (e.g. $\Psi(u, v) = uv$, $u \geq 0$, $v \geq 0$) and then to establish a certain "spectral" relation between any two pairs of real numbers involving the function Ψ (e.g. $(u_1^* v_1', u_2^* v_2') \prec\prec (u_1' v_1', u_2' v_2')$, i.e., $\Psi(\mathbf{u}^*, \mathbf{v}') \prec\prec \Psi(\mathbf{u}', \mathbf{v}')$ where $\mathbf{u}^* = (u_1^*, u_2^*)$, $\mathbf{v}' = (v_1', v_2')$ respectively denotes the decreasing and increasing rearrangement of $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$). The "spectral" relation thus obtained is then easily extended to any pair of n-vectors and also to integrable functions through some limiting process (cf. the proofs given in [4, Lemmas 3.1, 3.2 and Theorem 3.3] and [5, Lemmas 2.1, 2.2 and Theorem 2.3]). This procedure is summarized in Theorem 2.1 below which turns out to be an induction principle.

1. **Preliminaries.** Let (X, Λ, μ) be a finite measure space, i.e. X is a nonempty point set provided with a countably additive nonnegative measure μ on a σ-algebra Λ of subsets of X such that $\mu(X) < \infty$. Whenever it is clear from the context, we shall often write $\int \cdot \, d\mu$ for integration over X. By $M(X, \mu)$ we denote the set of all extended real-valued measurable functions on X. Two functions $f, g \in M(X, \mu)$ are said to be *equimeasurable* (written $f \sim g$) whenever

$$\mu(\{x: f(x) > t\}) = \mu(\{x: g(x) > t\})$$

for all real t. If $f \sim g$ and if (X', Λ', μ') is any other measure space with $\mu'(X') = \mu(X)$, it is not hard to see that $f \circ \sigma \sim g \circ \sigma$ whenever $\sigma: X' \to X$ is a measure-preserving map, i.e., $\sigma^{-1}(E) \in \Lambda'$ and $\mu'(\sigma^{-1}(E)) = \mu(E)$ for all $E \in \Lambda$.

If $f \in M(X, \mu)$, it is well known that there exists a unique right continuous nonincreasing function δ_f on the interval $[0, \mu(X)]$, called the *decreasing rearrangement* of f, such that δ_f and f are equimeasurable. In fact,

$$\delta_f(s) = \inf\{t \in R:\ \mu(\{x:\ f(x) > t\}) \leq s\}$$

for all $s \in [0, \mu(X)]$. Moreover, there also exists a unique right continuous nondecreasing function $\iota_f = -\delta_{-f}$, called the *increasing rearrangement* of f, such that $\iota_f \sim f$.

In [4, Theorem 1.1 and Corollary 3.5], it is shown that the operation of decreasing or increasing rearrangement preserves a.e. pointwise convergence, convergence in measure and all L^p convergence, $1 \leq p \leq \infty$, i.e., if $f_n \to f$ pointwise a.e., in measure or in L^p as $n \to \infty$, then $\delta_{f_n} \to \delta_f$ and $\iota_{f_n} \to \iota_f$ in the same sense as $n \to \infty$.

In what follows, we denote the Lebesgue measure on the real line R by m.

If $f, g \in M(X, \mu)$ and $f^+, g^+ \in L^1(X, \mu)$ where $\mu(X) = a < \infty$, then we write $f \prec\prec g$ whenever $\int_0^t \delta_f\, dm \leq \int_0^t \delta_g\, dm$, $t \in [0, a]$, and $f \prec g$ whenever $f \prec\prec g$ and $\int_0^a \delta_f\, dm = \int_0^a \delta_g\, dm$.

In [6, Proposition 10.2(x)], it is shown that, if $f \prec g$ and if $g \in L^\infty$, then $f \in L^\infty$ and $\|f\|_\infty \leq \|g\|_\infty$.

In the sequel, expressions of the form $f \prec g$ (respectively $f \prec\prec g$) are called *strong* (respectively *weak*) *spectral inequalities*. The spectral inequality $f \prec g$ (respectively $f \prec\prec g$) is said to be *strictly strong* (respectively *strictly weak*) if *f and g are not equimeasurable* (respectively if *f and g do not have equal total integrals*).

In establishing the spectral inequalities to be given below, we need the following results proved earlier in [3].

Theorem 1.1. (Chong [3, Theorems 2.3, 2.8, 3.1 and Corollaries 2.4 and 3.2]). *Suppose (X, Λ, μ) is a finite measure space. Suppose $f, g \in M(X, \mu)$ with integrable positive parts, then $f \prec\prec g$ if and only if $\int \Phi(f)\, d\mu \leq \int \Phi(g)\, d\mu$ for all nondecreasing convex functions $\Phi: R \to R$ or, equivalently, $\Phi(f) \prec\prec \Phi(g)$ for all nondecreasing convex functions $\Phi: R \to R$ such that $\Phi^+(g) \in L^1(X, \mu)$.*

If $f \prec\prec g$ and if $\Phi: R \to R$ is strictly convex increasing such that $\Phi(g) \in L^1(X, \mu)$, then $\int \Phi(f)\, d\mu = \int \Phi(g)\, d\mu$ if and only if $f \sim g$.

Moreover, if $\Phi: R \to R$ is strictly increasing convex and if $f, g \in L^1(X, \mu)$ are such that $f \prec\prec g$, then the strong spectral inequality $f \prec g$ holds whenever the integrals $\int \Phi(f)\, d\mu$ and $\int \Phi(g)\, d\mu$ are finite and equal.

Theorem 1.2 (Hardy, Littlewood and Pólya [6, Theorem 10, p. 152] and Chong [3, Theorem 2.5 and Corollary 2.6]). *Suppose $f, g \in L^1(X, \mu)$, where $\mu(X) < \infty$, then $f \prec g$ if and only if $\int \Phi(f)\, d\mu \leq \int \Phi(g)\, d\mu$ for all convex functions $\Phi: R \to R$ or, equivalently, $\Phi(f) \prec\prec \Phi(g)$ for all convex functions $\Phi: R \to R$ such that $\Phi^+(g) \in L^1(X, \mu)$.*

If $f \prec g$ *and if* $\Phi: R \to R$ *is strictly convex such that* $\Phi(g) \in L^1(X, \mu)$, *then the equality* $\int \Phi(f) d\mu = \int \Phi(g) d\mu$ *holds if and only if* $f \sim g$.

2. An induction principle for spectral and rearrangement inequalities. In what follows, we assume without comment that for any given n-tuple $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in R^n$, the n-tuples $\mathbf{a}^* = (a_1^*, a_1^*, \cdots, a_n^*)$, $\mathbf{a}' = (a_1', a_2', \cdots, a_n')$ will always denote respectively the decreasing and increasing rearrangements of $\mathbf{a}$; here we have regarded $\mathbf{a}$ as a measurable function defined on a discrete finite measure space with n atoms of equal measures.

Theorem 2.1 (a general induction principle). *Let* $\Psi: R \times R \to R$ *be a continuous function. Let* $I \subset R$ *be an interval. Let* $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ *be any two n-tuples in* $I^n \subset R^n$ *where* $n \in N$. *Then each of the following three statements holds* (*for all* $\mathbf{a}, \mathbf{b} \in I^n$) *for any* $n \in N$ *if and only if it holds for the case* $n = 2$.

(i)
$$\sum_{i=1}^{n} \Psi(a_i', b_i^*) \equiv \sum_{i=1}^{n} \Psi(a_i^*, b_i') \le \sum_{i=1}^{n} \Psi(a_i, b_i) \le \sum_{i=1}^{n} \Psi(a_i', b_i') \equiv \sum_{i=1}^{n} \Psi(a_i^*, b_i^*),$$

(ii)
$$\Psi(\mathbf{a}', \mathbf{b}^*) \sim \Psi(\mathbf{a}^*, \mathbf{b}') \prec\prec \Psi(\mathbf{a}, \mathbf{b}) \prec\prec \Psi(\mathbf{a}', \mathbf{b}') \sim \Psi(\mathbf{a}^*, \mathbf{b}^*),$$

(iii)
$$\Psi(\mathbf{a}', \mathbf{b}^*) \sim \Psi(\mathbf{a}^*, \mathbf{b}') \prec \Psi(\mathbf{a}, \mathbf{b}) \prec \Psi(\mathbf{a}', \mathbf{b}') \sim \Psi(\mathbf{a}^*, \mathbf{b}^*).$$

In other words, (i), (ii) *and* (iii) *are respectively equivalent to the following*:

(ia) $\Psi(u_1^*, v_1') + \Psi(u_2^*, v_2') \le \Psi(u_1', v_1') + \Psi(u_2', v_2')$,

(iia) $\Psi(\mathbf{u}^*, \mathbf{v}') \prec\prec \Psi(\mathbf{u}', \mathbf{v}')$ *which is equivalent to condition* (ia) *plus* (ia'): $\Psi(u_1^*, v_1') \vee \Psi(u_2^*, v_2') \le \Psi(u_1', v_1') \vee \Psi(u_2', v_2')$,

(iii*a*) $\Psi(\mathbf{u}^*, \mathbf{v}') \prec \Psi(\mathbf{u}', \mathbf{v}')$,

where $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ *are any 2-tuples in* I^2.

If the inequality in (ia) *or the weak spectral inequality in* (iia) *is strict for any vectors* $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ *in* I^2 *satisfying* $u_1 \ne u_2$, $v_1 \ne v_2$, *then equality in* (i) *or strong spectral inequality in* (ii) *holds on the left, i.e.,* $\sum_{i=1}^n \Psi(a_i^*, b_i') = \sum_{i=1}^n \Psi(a_i, b_i)$ *or* $\Psi(\mathbf{a}^*, \mathbf{b}') \prec \Psi(\mathbf{a}, \mathbf{b})$ (*respectively on the right, i.e.,* $\sum_{i=1}^n \Psi(a_i, b_i) = \sum_{i=1}^n \Psi(a_i', b_i')$ *or* $\Psi(\mathbf{a}, \mathbf{b}) \prec \Psi(\mathbf{a}', \mathbf{b}')$), *if and only if* $\Psi(\mathbf{a}^*, \mathbf{b}') \sim \Psi(\mathbf{a}, \mathbf{b})$ (*respectively* $\Psi(\mathbf{a}, \mathbf{b}) \sim \Psi(\mathbf{a}', \mathbf{b}')$) *or, equivalently, the sequences* $\{(a_i^*, b_i')\}_{i=1}^n$ *and* $\{(a_i, b_i)\}_{i=1}^n$ (*respectively* $\{(a_i, b_i)\}_{i=1}^n$ *and* $\{(a_i', b_i')\}_{i=1}^n$) *of ordered pairs are rearrangements of each other.*

In general, let (X, Λ, μ) *be any finite measure space with* $\mu(X) = a$. *Then* *condition* (ia) *and condition* (iia) *are respectively both necessary and sufficient that*

(ib) $\int_0^a \Psi(\delta_f, \iota_g)\,dm \le \int_X \Psi(f, g)\,d\mu \le \int_0^a \Psi(\delta_f, \delta_g)\,dm$, *and*

(iib) $\Psi(\delta_f, \iota_g) \prec\prec \Psi(f, g) \prec\prec \Psi(\delta_f, \delta_g)$ *for all* $f, g \in L^\infty(X, \mu)$. *Moreover, the spectral inequalities in* (iib) *are strong, i.e.*,

(iiib) $\Psi(\delta_f, \iota_g) \prec \Psi(f, g) \prec \Psi(\delta_f, \delta_g)$, *if condition* (iiia) *holds.*

If $I = R$, *then the assertions concerning* (ib), (iib) *and* (iiib) *are also true for all* $f, g \in L^1(X, \mu)$ *provided that one of the following two conditions holds*:

(I) Ψ *is bounded on* R^2;

(II) $\Psi(h_n, k_n) \to \Psi(h, k)$ *in* L^1 *whenever* $h_n \to h$ *and* $k_n \to k$ *in* L^1 *where* $h_n, k_n, h, k \in L^1(X, \mu)$ *or* $L^1([0, a], m)$, $n \in N$.

Moreover, if $R^+ \subset I \subset R$ *and if* Ψ *is nondecreasing or nonincreasing in both variables on* I^2, *then condition* (ia) *is both necessary and sufficient for* (ib) *and* (iib) *to hold and equality in* (ia) *is sufficient for* (iiib) *to hold for all* $0 \le f, g \in L^1(X, \mu)$ *such that* $\Psi^+(\delta_f, \delta_g) \in L^1([0, a], m)$ *in* (iib) *and* (iiib), *provided that* $|\Psi(0, 0)| < \infty$ (*respectively* $\Psi^+(0, 0) < \infty$) *if* Ψ *is nondecreasing* (*respectively nonincreasing*) *in both variables.*

Finally, if $I = R$, *if* Ψ *is nondecreasing in both variables on* R^2 *and if* (ib), (iib) *or* (iiib) *holds with* $f - t$, $g - t$ *replacing* f, g *for all* $t \in R$ *and for all* $0 \le f, g \in L^1(X, \mu)$, *then* (ib), (iib) *or* (iiib) *also holds for all* $f, g \in M(X, \mu)$ *such that* $f^+, g^+ \in L^1(X, \mu)$ *and* $\Psi^+(\delta_f, \delta_g) \in L^1([0, a], m)$ *provided that* $\Psi^+(h, 0)$, $\Psi^+(0, k) \in L^1(X, \mu) \cup L^1([0, a], m)$ *whenever* $0 \le h, k \in L^1(X, \mu) \cup L^1([0, a], m)$.

Proof. For the first part of the theorem, the necessity of the conditions is clear. For the sufficiency of the conditions, we first prove that (ia)$\Rightarrow$(i) and we need only prove this for the left-hand inequality, i.e. $\sum_{i=1}^n \Psi(a_i^*, b_i') \le \sum_{i=1}^n \Psi(a_i, b_i)$; the rest is analogous.

Without loss of generality, we may assume that $a_i = a_i^*$, $i = 1, 2, \cdots, n$. In this case, if $1 \le i < j \le n$ and $b_i > b_j$, then condition (ia) implies that the sum $\sum_{i=1}^n \Psi(a_i, b_i)$ is never increased on interchanging b_i and b_j. Thus $\sum_{i=1}^n \Psi(a_i^*, b_i')$ is the smallest possible value attainable by $\sum_{i=1}^n \Psi(a_i, b_i)$ as $\mathbf{b}$ ranges through all its rearrangements. With virtually the same principle, mutatis mutandis, we can prove the implications (iia)$\Rightarrow$(ii) and (iiia)$\Rightarrow$(iii). For, by a theorem of Hardy, Littlewood and Pólya [7, Theorem 10] (cf. [3, Corollary 1.7]), statements (iia) and (ii) are respectively equivalent to requiring that

$$[\Psi(u_1^*, v_1') - t]^+ + [\Psi(u_2^*, v_2') - t]^+ \le [\Psi(u_1', v_1') - t]^+ + [\Psi(u_2', v_2') - t]^+$$

and

$$\sum_{i=1}^n [\Psi(a_i^*, b_i') - t]^+ \le \sum_{i=1}^n [\Psi(a_i, b_i) - t]^+ \le \sum_{i=1}^n [\Psi(a_i', b_i') - t]^+$$

hold for all $t \in R$.

Next, we consider the cases of equalities. The sufficiency of the conditions is clear. For the necessity of the conditions, here again, the general case for $n \geq 3$ follows from the case that $n = 2$. In fact, suppose $\Sigma_{i=1}^n \Psi(a_i^*, b_i') = \Sigma_{i=1}^n \Psi(a_i, b_i)$ in (i) or $\Psi(\mathrm{a}^*, \mathrm{b}') \prec \Psi(\mathrm{a}, \mathrm{b})$ in (ii). Suppose by contradiction that $\{(a_i^*, b_i')\}_{i=1}^n$ is not a rearrangement of $\{(a_i, b_i)\}_{i=1}^n$. Without loss of generality (in fact, by renumbering), we may assume that $a_1, a_2, \cdots, a_n$ are already arranged in decreasing order, i.e., $a_i = a_i^*$, $i = 1, 2, \cdots, n$. Then at least one, say (a_1^*, b_1'), of the ordered pairs of $\{(a_i^*, b_i')\}_{i=1}^n$ does not belong to $\{(a_i, b_i)\}_{i=1}^n$. Thus $b_1' < b_1$ and $b_1' = b_j$ for some $1 < j \leq n$. Clearly, $a_1^* > a_j^*$, otherwise the pair (a_1^*, b_1') would have belonged to $\{(a_i, b_i)\}_{i=1}^n$. By hypothesis, $\Psi(a_1^*, b_1) + \Psi(a_j^*, b_j) > \Psi(a_1^*, b_j) + \Psi(a_j^*, b_1)$ implying that

$$\sum_{i=1}^{n} \Psi(a_i^*, b_i) > \Psi(a_1^*, b_j) + \Psi(a_2^*, b_2) + \cdots + \Psi(a_{j-1}^*, b_{j-1})$$

$$+ \Psi(a_j^*, b_1) + \Psi(a_{j+1}^*, b_{j+1}) + \cdots + \Psi(a_n^*, b_n).$$

But the latter sum is not less than $\Sigma_{i=1}^n \Psi(a_i^*, b_i')$ to which the former sum is equal, a contradiction. The remaining case is treated similarly.

To prove the result for measurable functions f and g, we first assume that (X, Λ, μ) is nonatomic. Then there exist two sequences $\{f_n\}_{n=1}^\infty, \{g_n\}_{n=1}^\infty$ of simple functions f_n, g_n with the same number (say 2^n) of common sets of constancy such that $f_n \prec f$, $g_n \prec g$ and $f_n \to f$, $g_n \to g$ both pointwise μ-a.e. and in L^1(cf. the functions f_n, g_n constructed in the proof of Lemma 3.2 in [4]). Since the operation of decreasing or increasing rearrangements preserves a.e. pointwise convergence, convergence in measure and all L^p convergence, $1 \leq p \leq \infty$ [4, Theorem 1.1], we see that δ_{f_n} and ι_{g_n} also converge in L^1 to δ_f and ι_g respectively. Moreover, since $f_n \prec f$ and $g_n \prec g$, we have $\|f_n\|_\infty \leq \|f\|_\infty$, $\|g_n\|_\infty \leq \|g\|_\infty$, $n \in N$ (see [6, Proposition 10.2(x)]). Thus, if $f, g \in L^\infty(X, \mu)$, let

$$M = \max\{|\Psi(u, v)|: (u, v) \in I^2, \ |u| \leq \|f\|_\infty \text{ and } |v| \leq \|g\|_\infty\};$$

then $M < \infty$ since Ψ is continuous. It is not hard to see that $|\Psi(\delta_{f_n}, \iota_{g_n})| \leq M$, $|\Psi(f_n, g_n)| \leq M$ and $|\Psi(\delta_{f_n}, \delta_{g_n})| \leq M$, $n \in N$. Since the assertions concerning (ib), (iib) and (iiib) are true for the functions f_n and g_n, $n \in N$, they are also true for all $f, g \in L^\infty(X, \mu)$, by Lebesgue's dominated convergence theorem.

If (X, Λ, μ) is not nonatomic, we can imbed it into a nonatomic measure space $(\overline{X}, \overline{\Lambda}, \overline{\mu})$. (For details of this device, see [6, pp. 52–54] or [10].) Then $f, g \in L^1(X, \mu)$ can be identified with $\overline{f}, \overline{g} \in L^1(\overline{X}, \overline{\mu})$ by a map $h \to \overline{h}$ satisfying $h \sim \overline{h}$ and $\overline{\Psi(f, g)} = \Psi(\overline{f}, \overline{g})$. Thus the case that (X, Λ, μ) is not nonatomic follows directly from that case that (X, Λ, μ) is nonatomic.

The case that (I) or (II) holds is treated similarly.

If Ψ is nondecreasing or nonincreasing in both variables, we first observe that condition (ia′) is automatically satisfied since then either $\Psi(u_2', v_2')$ or $\Psi(u_1', v_1')$ is the largest of the four terms involved in that inequality. Next, it is easily seen that (iia) is equivalent to (ia) plus (ia′). Thus, if (ia) holds and if $0 \leq f, g \in L^1(X, \mu)$, then (ib) and (iib) are true for the functions $f \wedge n$, $g \wedge n$, $n \in N \cup \{0\}$, by the preceding result, and hence for the functions f, g by Lebesgue's monotone convergence theorem for the case that Ψ is increasing in both variables and Levi's monotone convergence theorem for the case that Ψ is decreasing in both variables. The assertion concerning (iiib) is similarly proven.

The last part follows directly from [3, Corollary 1.11], [4, Theorem 3.7] and the preceding result using the fact that the hypotheses imply that $\Psi^+(\delta_{f^+}, \delta_{g^+})$, $\Psi^+(f^+, g^+)$ and $\Psi^+(\delta_{f^+}, \iota_{g^+}) \in L^1(X, \mu) \cup L^1([0, a], m)$.

We note that with Theorem 2.1 we are able to obtain any spectral inequality or more general rearrangement inequality involving any pair of n-vectors in R^n merely by verifying its validity for any pair of 2-vectors in R^2. Since 2-vectors are much more easily dealt with than any other n-vectors, Theorem 2.1 thus proves to be a very powerful tool in this respect. However, owing to the complications involved in the limiting processes, a more general theorem, which works for all integrable functions (or, more generally, for measurable functions with integrable positive parts) and for any function Ψ satisfying condition (ia′) in Theorem 2.1, does not seem to be readily feasible, though martingale theory may sometimes be very useful in this regard, especially when dealing with some special individual cases. Take, for example, Theorem 5.1 in [4], where it is proven that

$$\Phi(\delta_f + \iota_g) \ll \Phi(f + g) \ll \Phi(\delta_f + \delta_g)$$

for all convex functions $\Phi: R \to R$ and for all $f, g \in L^1(X, \mu)$ such that $\Phi^+(\delta_f + \delta_g)$ is integrable. In this case, Theorem 2.1 is not immediately applicable to proving the desired result, since the function $\Psi: R \times R \to R$ defined by $\Psi(u, v) = \Phi(u + v)$ for some convex $\Phi: R \to R$ is not necessarily increasing in both variables on $R \times R$ (unless Φ is increasing convex on R). In this particular instance, some martingale theory or some indirect procedure resembling the one given for the proof of [3, Theorem 2.5] will help overcome this difficulty. Nevertheless, Theorem 2.1 is most effectively used in conjunction with Theorems 1.1 and 1.2. In this way, most of the problems arising in connection with the limiting processes are readily solvable and we are thus able to derive from the discrete case many spectral or more general rearrangement inequalities for any pair of measurable functions with integrable positive parts, and also to deal with the cases of equalities or strong spectral inequalities accordingly.

If the function Ψ has continuous second partial derivatives, it is often useful to observe that condition (ia) of Theorem 2.1 is equivalent to requiring $\partial^2\Psi/\partial u\partial v \geq 0$. This fact can be easily verified after interchanging any two terms between both sides of (ia) (cf. G. G. Lorentz [9, condition (3a), p. 176]). For example, the functions $(u, v) \mapsto uv$, $(u, v) \to u + v$ are of this type and are increasing in both variables and so Theorem 2.1 applies to give the spectral inequalities obtained in [4, Theorems 3.8 and 4.1]. Of course, the results given in [4, Theorem 3.10] and [5, Theorem 2.3 and Corollary 2.4] are also related to Theorem 2.1. The following theorem gives a further illustration of the applications of our general induction principle. We include a particular case of [4, Theorem 5.1], since its proof serves as an illustrative example.

Theorem 2.2. *Let $\Phi: R \to R$ (respectively $\Phi: R^+ \to R$) be any convex function. Then*

(I) $$\Phi(\mathbf{a}^* + \mathbf{b}') \sim \Phi(\mathbf{a}' + \mathbf{b}^*) \ll \Phi(\mathbf{a} + \mathbf{b}) \ll \Phi(\mathbf{a}^* + \mathbf{b}^*) \sim \Phi(\mathbf{a}' + \mathbf{b}')$$

(respectively

(II) $$\Phi(\mathbf{a}^* + \mathbf{b}') - \Phi(\mathbf{a}^*) - \Phi(\mathbf{b}') \ll \Phi(\mathbf{a} + \mathbf{b}) - \Phi(\mathbf{a}) - \Phi(\mathbf{b}) \ll \Phi(\mathbf{a}^* + \mathbf{b}^*) - \Phi(\mathbf{a}^*) - \Phi(\mathbf{b}^*)).$$

for all n-vectors $\mathbf{a}, \mathbf{b} \in R^n$ (respectively $0 \leq \mathbf{a}, \mathbf{b} \in R^n$).

In general, if $0 \leq f,\ g \in L^1(X, \mu)$ where $\mu(X) = a < \infty$, such that $[\Phi(\delta_f + \delta_g) - \Phi(\delta_f) - \Phi(\delta_g)]^+ \in L^1([0, a], m)$, then

(III) $$\Phi(\delta_f + \iota_g) - \Phi(\delta_f) - \Phi(\iota_g) \ll \Phi(f + g) - \Phi(f) - \Phi(g) \ll \Phi(\delta_f + \delta_g) - \Phi(\delta_f) - \Phi(\delta_g)$$

and if Φ is also nondecreasing, then, for all $f,\ g \in L^1(X, \mu)$ satisfying $\Phi^+(\delta_f + \delta_g) \in L^1([0, a])$,

(IV) $$\Phi(\delta_f + \iota_g) \ll \Phi(f + g) \ll \Phi(\delta_f + \delta_g).$$

If, in addition, Φ is strictly convex, then strong spectral inequalities hold in (I) *iff $\Phi(\mathbf{a}^* + \mathbf{b}') \sim \Phi(\mathbf{a} + \mathbf{b}) \sim \Phi(\mathbf{a}^* + \mathbf{b}^*)$ or, equivalently, iff $\mathbf{a}^* + \mathbf{b}' \sim \mathbf{a} + \mathbf{b} \sim \mathbf{a}^* + \mathbf{b}^*$, and strong spectral inequality holds in either side of* (II) *or* (III) *iff both sides of the corresponding spectral inequality are equimeasurable, provided they are integrable. If $\Phi(\delta_f + \delta_g), \Phi(\delta_f), \Phi(\delta_g) \in L^1([0, a])$, then strong spectral inequality holds on the left (respectively on the right) of* (III) *iff $\delta_f + \iota_g \sim f + g$ (respectively $f + g \sim \delta_f + \delta_g$) and similarly for* (IV).

Proof. By Theorem 2.1, we need only prove (I) and (II) for the case that $n = 2$. To prove (I), let $\Psi(u, v) = \Phi(u + v)$ where $u, v \in R$. If $s < t \leq t'$ and $s \leq s' < t'$, then using the convexity of Φ it is not hard to see that

(1) $$\Phi(t) \vee \Phi(s') \leq \Phi(s) \vee \Phi(t'),$$

(2) $$\frac{\Phi(t) - \Phi(s)}{t - s} \leq \frac{\Phi(t') - \Phi(s')}{t' - s'}.$$

Now, for any pair of vectors $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2) \in R^2$ such that either $u_1 \neq u_2$ or $v_1 \neq v_2$, let $s = u_2^* + v_2^*$, $t = u_1^* + v_1'$, $s' = u_2^* + v_2'$, $t' = u_1^* + v_1^*$. Then, clearly, $s < t \leq t'$, $s \leq s' < t'$. Moreover, conditions (ia$'$) and (ia) of Theorem 2.1 are respectively direct consequences of inequalities (1) and (2) above. Thus (I) follows immediately.

To prove (II), let $\Psi(u, v) = \Phi(u + v) - \Phi(u) - \Phi(v)$ for $u \geq 0$, $v \geq 0$. When $u_1 \leq u_2$ then, by (2) above, we have

$$\frac{\Phi(u_1 + v) - \Phi(u_1)}{(u_1 + v) - u_1} \leq \frac{\Phi(u_2 + v) - \Phi(u_2)}{(u_2 + v) - u_2}$$

which implies that $\Psi(u_1, v) \leq \Psi(u_2, v)$ or Ψ is increasing in the first variable and hence in both variables by symmetry. Thus condition (ia$'$) of Theorem 2.1 is automatically satisfied. Moreover, condition (ia) of Theorem 2.1 is an obvious consequence of the proof given for (I). Hence (II) follows.

Since the function $(u, v) \mapsto \Phi(u + v) - \Phi(u) - \Phi(v)$ is increasing in both variables, (III) follows immediately from Theorem 2.1, so does (IV) when $f, g \in L^1(X, \mu)$ are nonnegative. Since the function $u \mapsto \Phi(u - 2t)$ remains convex in $u \in R$ for any $t \in R$, (IV) still holds true with f, g respectively replaced by $f - t$, $g - t$ whenever $0 \leq f, g \in L^1(X, \mu)$. Hence, by Theorem 2.1, (IV) holds for all $f, g \in L^1(X, \mu)$ whenever $\Phi^+(\delta_f + \delta_g) \in L^1([0, a], m)$.

Finally if Φ is strictly convex, then it is easily seen that the inequality (2) is strict whenever $s < t < t'$, $s < s' < t'$ and this fact in turn implies that the inequality in condition (ia) of Theorem 2.1 is strict for the function $\Psi(u, v) = \Phi(u + v)$ or $\Phi(u + v) - \Phi(u) - \Phi(v)$ when $u_1 \neq u_2$ and $v_1 \neq v_2$. Hence the result follows from Theorems 1.1, 1.2 and 2.1.

By assigning different convex functions for Φ in (III) of Theorem 2.2, various new spectral inequalities can be derived, the simplest example of which is given by $\Phi(u) = u^2$, $u \geq 0$, in which case the spectral inequalities obtained are precisely those given in [4, Theorem 4.1]. Further examples are obtained as follows.

Examples 2.3

(i) $$\delta_f \iota_g(\delta_f + \iota_g) \ll fg(f + g) \ll \delta_f \delta_g(\delta_f + \delta_g).$$

(ii) $$\frac{\delta_f \iota_g}{\delta_f + \iota_g} \ll \frac{fg}{f + g} \ll \frac{\delta_f \delta_g}{\delta_f + \delta_g}, \quad f, g \geq 1.$$

(iii) $$\delta_f \iota_g - (\delta_f + \iota_g) \ll fg - (f+g) \ll \delta_f \delta_g - (\delta_f + \delta_g), \qquad f, g \geq 1.$$

(iv) $$\log \frac{\delta_f \iota_g}{\delta_f + \iota_g} \ll \log \frac{fg}{f+g} \ll \log \frac{\delta_f \delta_g}{\delta_f + \delta_g}, \qquad f, g \geq 1.$$

(v) $$(\delta_f + \iota_g)^p - (\delta_f^p + \iota_g^p) \ll (f+g)^p - (f^p + g^p) \ll (\delta_f + \delta_g)^p - (\delta_f^p + \delta_g^p), \qquad p > 1.$$

(vi) $$(\delta_f^p + \iota_g^p) - (\delta_f + \iota_g)^p \ll (f^p + g^p) - (f+g)^p \ll (\delta_f^p + \delta_g^p) - (\delta_f + \delta_g)^p, \qquad 0 < p < 1.$$

etc., where $0 \leq f$, $g \in L^1(X, \mu)$, $\mu(X) < \infty$ and the last function of each set of spectral inequalities is integrable.

To see this, apply to Theorem 2.2 the convex functions $\Phi(u) = u^3$ for (i), $\Phi(u) = -\log u$ for (iv), $\Phi(u) = u^p$ for (v) and $\Phi(u) = -u^p$ for (vi), where $u \geq 0$. Again, (ii) is obtained from (iv) by exponentiating and (iii) is obtained first by replacing the function f, g by $\log f$, $\log g$ in (III) of Theorem 2.2 and then apply the convex function $\Phi(u) = e^u$, $u \geq 0$.

Observe that the spectral inequalities in Examples 2.3 are strong iff the corresponding sides are equimeasurable, and that in some cases the conditions for strong spectral inequalities can be simplified, e.g., in (iii), strong spectral inequality holds on the left (respectively on the right) iff $\delta_f \iota_g \sim fg$ (respectively $fg \sim \delta_f \delta_g$).

In [9, p. 176], G. G. Lorentz proved a general rearrangement theorem for any finite sequence of bounded nonnegative measurable functions defined on the unit interval (0, 1). Using a concept similar to the one developed in Theorem 2.1, we can generalize his theorem for the spectral orders $\prec$ and $\prec\prec$ (see Theorems 2.4 and 2.5 below).

Let $\Psi(x, u_1, u_2, \cdots, u_n)$ be a continuous function defined for $0 < x < a$, $a < \infty$, $u_k \geq 0$, $k = 1, 2, \cdots, n$. Following G. G. Lorentz [9], in any expression involving Ψ, we shall omit those variables which merely repeat themselves throughout the expression.

Theorem 2.4. *Suppose* $f_1, f_2, \cdots, f_n \in L^\infty((0,a), m)$ *are positive functions. Let* I *denote the identity map of* $(0, a)$. *Then*

$$\Psi(I, f_1, \cdots, f_n) \ll \Psi(I, \delta_{f_1}, \cdots, \delta_{f_n})$$

if and only if

(1) $$(\Psi(u_i + h, u_j), \Psi(u_i, u_j + h)) \prec\prec (\Psi(u_i, u_j), \Psi(u_i + h, u_j + h))$$

and

(2) $$\int_0^c \{\Psi_s(x - t, u_i + h) + \Psi_s(x + t, u_i) - \Psi_s(x + t, u_i + h) - \Psi_s(x - t, u_i)\}\, dt \geq 0$$

for all $0 < x < a$, $u_k \geq 0$, $k = 1, \cdots, n$, $h > 0$, $0 < c < x$, $c < a - x$, $i \neq j$ *and* $\Psi_s: t \mapsto (\Psi(t) - s)^+$, $t \in R^+$, $s \in R$. *If* Ψ *has continuous second partial derivatives with respect to all variables, then the necessary and sufficient conditions reduce to*

$$\text{(1a)} \quad \frac{\partial^2 \Psi_s}{\partial u_i \partial u_j} \geq 0 \quad \textit{and} \quad \text{(2a)} \quad \frac{\partial^2 \Psi_s}{\partial x \partial u_i} \leq 0.$$

Proof. By Hardy, Littlewood and Pólya's Theorem [7, p. 152] (cf. [2, Theorem 1.1] and [3, Corollary 1.7]), condition (1) is equivalent to

$$\Psi_s(u_i + h, u_j) + \Psi_s(u_i, u_j + h) \leq \Psi_s(u_i, u_j) + \Psi_s(u_i + h, u_j + h)$$

for all $s \in R$. Thus by Lorentz's Theorem [9, p. 176], conditions (1) and (2) are equivalent to

$$\int_0^a \Psi_s(I, f_1, \cdots, f_n)\, dm \leq \int_0^a \Psi_s(I, \delta_{f_1}, \cdots, \delta_{f_n})\, dm$$

for all $s \in R$, i.e. $\Psi(I, f_1, \cdots, f_n) \prec\prec \Psi(I, \delta_{f_1}, \cdots, \delta_{f_n})$ by Hardy, Littlewood and Pólya's Theorem again.

Remark. Condition (1) in Theorem 2.4 is equivalent to

(1′) $$\Psi(u_i + h, u_j) \vee \Psi(u_i, u_j + h) \leq \Psi(u_i, u_j) \vee \Psi(u_i + h, u_j + h)$$

and

(1″) $$\Psi(u_i + h, u_j) + \Psi(u_i, u_j + h) \leq \Psi(u_i, u_j) + \Psi(u_i + h, u_j + h).$$

Thus if Ψ is increasing (or decreasing) in each variable, then condition (1′) is trivially satisfied.

Using a combination of the arguments given in Theorems 2.1 and 2.4, we can generalize part of Theorem 2.1 in the following direction, thus obtaining a similar induction principle for spectral and rearrangement inequalities involving more than two functions.

Theorem 2.5. *Suppose* $\Psi: R^p \to R$ *is a continuous function of* p *variables,* $p \in N$. *Let* $I \subset R$ *be an interval. Suppose* $\mathbf{a}_i = (a_{i1}, \cdots, a_{in})$, $i = 1, 2, \cdots, p$, *are n-tuples in* I^n. *Then*

(i) $\sum_{j=1}^n \Psi(a_{1j}, \cdots, a_{pj}) \leq \sum_{j=1}^n \Psi(a_{1j}^*, \cdots, a_{pj}^*)$

[*respectively*

(ii) $\Psi(\mathbf{a}_1, \cdots, \mathbf{a}_p) \prec\prec \Psi(\mathbf{a}_1^*, \cdots, \mathbf{a}_p^*)$,

(iii) $\Psi(\mathbf{a}_1, \cdots, \mathbf{a}_p) \prec \Psi(\mathbf{a}_1^*, \cdots, \mathbf{a}_p^*)$]

if and only if

(ia) $\Psi(u_i, u_j + b) + \Psi(u_i + b, u_j) \le \Psi(u_i, u_j) + \Psi(u_i + b, u_j + b)$

[*respectively*

(iia) $(\Psi(u_i, u_j + b), \Psi(u_i + b, u_j)) \prec\prec (\Psi(u_i, u_j), \Psi(u_i + b, u_j + b))$,

(iiia) $(\Psi(u_i, u_j + b), \Psi(u_i + b, u_j)) \prec (\Psi(u_i, u_j), \Psi(u_i + b, u_j + b))$]

for all $u_k, u_k + b \in I$, $k = 1, 2, \cdots, n$, $b > 0$ *and* $i \neq j$.

If the inequality in (ia) *or the weak spectral inequality in* (iia) *is strict, then equality in* (i) *or strong spectral inequality in* (ii) *holds if and only if* $\Psi(\mathbf{a}_1, \cdots, \mathbf{a}_p) \sim \Psi(\mathbf{a}_1^*, \cdots, \mathbf{a}_p^*)$ *or, equivalently, the sequences* $\{(a_{1j}, a_{2j}, \cdots, a_{pj})\}_{j=1}^n$ *and* $\{(a_{1j}^*, a_{2j}^* \cdots, a_{pj}^*)\}_{j=1}^n$ *of ordered p-tuples are rearrangements of each other.*

In general, let (X, Λ, μ) *be any finite measure space with* $\mu(X) = a$. *Then* (ia), (iia) *and* (iiia) *are respectively the necessary and sufficient conditions that*

(i′) $\int_X \Psi(f_1, f_2, \cdots, f_p)d\mu \le \int_0^a \Psi(\delta_{f_1}, \delta_{f_2}, \cdots, \delta_{f_p})dm$,

(ii′) $\Psi(f_1, f_2, \cdots, f_p) \prec\prec \Psi(\delta_{f_1}, \delta_{f_2}, \cdots, \delta_{f_p})$,

(iii′) $\Psi(f_1, f_2, \cdots, f_p) \prec \Psi(\delta_{f_1}, \delta_{f_2}, \cdots, \delta_{f_p})$

hold either for all $f_1, f_2, \cdots, f_p \in L^\infty(X, \mu)$ *or for all* $0 \le f_1, f_2, \cdots, f_p \in L^1(X, \mu)$ *such that* $\Psi^+(\delta_{f_1}, \delta_{f_2}, \cdots, \delta_{f_p}) \in L^1([0, a], m)$, *provided that, in the latter case,* Ψ *is nondecreasing in each variable on* I^p, *where* $R^+ \subset I \subset R$.

If Ψ *is nondecreasing in each variable on* R^p *and if* (i′), (ii′) *or* (iii′) *holds with* $f_i - t$ *replacing* f_i, $i = 1, 2, \cdots, p$, *for all* $t \in R$ *and for all* $0 \le f_1, f_2, \cdots, f_p \in L^1(X, \mu)$, *then* (i′), (ii′) *or* (iii′) *also holds for all* $f_i \in M(X, \mu)$ *such that* $f_i^+ \in L^1(X, \mu)$, $i = 1, 2, \cdots, p$, *and* $\Psi^+(\delta_{f_1}, \delta_{f_2}, \cdots, \delta_{f_p}) \in L^1([0, a], m)$, *provided that* $\Psi^+(g_{i_1}, g_{i_2}, \cdots, g_{i_p}) \in L^1(X, \mu) \cup L^1([0, a], m)$, *where* $(i_1, i_2, \cdots, i_p)$ *is a permutation of* $(1, 2, \cdots, p)$ *whenever* $0 \le g_i \in L^1(X, \mu) \cup L^1([0, a], m)$, $i = 1, 2, \cdots, p - 1$ *and* $g_p = 0$.

Proof. This follows as in Theorems 2.1 and 2.4.

Remark. Some of the results given earlier in [4, Corollary 3.6, Theorem 3.8, Corollary 3.9 and Theorem 4.1] and [5, Theorem 2.3] can be obtained directly from Theorem 2.5.

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