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# UNITARY MEASURES ON LCA GROUPS

BY

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ABSTRACT. A unitary measure on a locally compact Abelian (LCA) group $G$ is a complex measure whose Fourier transform is of absolute value 1 everywhere. The problem of finding all such measures is known to be closely related to that of finding all invertible measures on $G$. In this paper, we find all unitary measures when $G$ is the circle or a discrete group. If $G$ is a torsion-free discrete group, the characterization generalizes a theorem of Bohr.

**1. Introduction.** Let $G$ be a locally compact group, and let $\mu$ be a finite (complex-valued) regular Borel measure on $G$. Define $\mu^\sim$ by $\mu^\sim(E) = \mu(E^{-1})$ for all Borel sets $E$. We call $\mu$ unitary if $\mu^\sim = \mu^{-1}$ (i.e., $\mu^\sim * \mu = \mu * \mu^\sim = \delta_e$). If $G$ is Abelian, this condition is equivalent to saying that $|\hat{\mu}(\delta)| = 1$ for every $\gamma \in \Gamma$, the dual group of $G$.

We investigate the problem of finding all unitary measures on a locally compact Abelian group. The key tools are results of J. Taylor [7] and the Arens-Royden theorem (one form of which is given as Proposition 4.1 of [7]). We obtain complete results when $G$ is discrete (Theorem 5); in the particular case where $G$ is also torsion-free, the answer has a nicer form (Theorem 4), and this result generalizes a theorem of Bohr [1]. We also obtain all unitary measures on the circle group $T$; this was the original aim of the paper. In view of Corollaries 4.6 and 4.7 of [7], these results give characterizations of the measures in $\mathfrak{M}(G)^\times$, the multiplicative group of invertible measures on $G$, when $G$ is one of the above groups.

Before getting down to serious work, we make a few preliminary remarks. If $\mu$ is unitary, write $\mu = \mu_d + \mu_c$, where $\mu_d$ is discrete and $\mu_c$ is continuous. Then (see [2]) $\mu_d$ is also unitary, and so $\mu = \mu_d * (\delta_e + \mu_c * \mu_d^\sim)$. Thus classifying the unitary measures amounts to classifying the discrete ones and those which are (continuous + $\delta_e$). We call the latter continuous unitary measures.

Next, if $\nu$ is a measure on $\nu$ satisfying $\nu^\sim = -\nu$ (for Abelian $G$, this $\Leftrightarrow$ $\hat{\nu}$ is purely imaginary), then $\exp \nu = \delta_e + \nu + (\nu * \nu)/2! + \cdots$ is unitary. (Conversely, if $\exp \nu$ is unitary, then $\nu^\sim = -\nu$.) Unitary measures of this form are precisely the ones lying in the connected component of the identity of $\mathfrak{M}(G)^\times$. We gener-

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ally regard these measures as trivial and look for others. Corollaries 4.6 and 4.7 of [7] show that if $\mu \in \mathfrak{M}(G)^\times$, then $\exists$ a measure $\nu_0 \in \mathfrak{M}(G)$ such that $(\exp \nu_0)^\frown(\gamma) = |\mu(\gamma)|$, $\forall \gamma \in \Gamma$; it is this result that makes the problem of classifying unitary measures equivalent to the problem of classifying elements of $\mathfrak{M}(G)^\times$.

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**2. Continuous unitary measures.** The main purpose of this section is to prove the following theorem for $T$.

**Theorem 1.** *If $\mu$ is a continuous unitary measure on $T$, $\mu = \exp \nu$ for some continuous $\nu$.*

**Proof.** Theorem 3 of [7] says that there are measures $\mu_1, \cdots, \mu_n$, $\nu_0 \in \mathfrak{M}(T)$, topologies $\mathcal{T}_1, \cdots, \mathcal{T}_n$ on $T$ (all at least as fine as the usual topology, and all making $T$ into a locally compact group), and complex numbers $\lambda_1, \cdots, \lambda_n$ such that $\mu_j - \lambda_j \delta_e$ is absolutely continuous with respect to Haar measure on $(T, \mathcal{T}_j)$ and $\mu = \mu_1 * \cdots * \mu_n * \exp(\nu_0)$. But the only topologies on $T$ satisfying the condition are the usual one and the discrete topology.(1) Hence $\mu = \mu_1 * \mu_2 * \exp(\nu_0)$, where $\mu_1$ is discrete and $\mu_2 - \lambda_2 \delta_e$ is absolutely continuous. Write $\nu_0 = \nu_1 + \nu_2$, where $\nu_1$ is discrete and $\nu_2$ is continuous; then $\mu = (\mu_1 * \exp \nu_1) * (\mu_2 * \exp \nu_2)$; the first expression in parentheses is a discrete measure, and the second is (except for a multiple of $\delta_e$) continuous. As $\mu - \delta_e$ is continuous, the first term must be a multiple of $\delta_e$.

We have reduced the theorem to the following: If $\mu$ is unitary and $\mu - \delta_e$ is absolutely continuous, then $\mu = \exp \nu$ for some continuous $\nu$. In fact, one can even pick $\nu$ absolutely continuous, as we shall see. We work in $\mathfrak{L}^1(T) \oplus \mathbf{C}\delta_e$; it suffices to show that $\mu$ is in the connected component of $\delta_e$ in $\mathfrak{M}(T)^\times$. If $\hat{\mu}(m) \neq -1$, $\forall m$, then $\delta_e + t(\mu - \delta_e)$ never has zero transform, and hence is invertible for all $t$. In general, $\lim_{k\to\infty} \hat{\mu}_0(k) = 0$; hence $\hat{\mu}$ is $-1$ on a finite set. Suppose $\hat{\mu}(k) = -1$ for $k = m_1, \cdots, m_l$; let $\sigma$ be the measure given by $d\sigma(z) = i(z^{m_1} + \cdots + z^{m_l})dz$ ($dz$ = Harr measure). Then $(\mu + t\sigma)^\wedge(k) = \hat{\mu}(k)$ except for $k = m_1, \cdots, m_l$; at those points, $(\mu + t\sigma)^\wedge(k) = -1 + it$. Hence $\mu$ and $\mu + \sigma$ are connected by a line of invertible measures and, as in the first part, $\mu + \sigma$ is in the connected component of $\delta_e$. That proves the theorem.

The last part of the proof gives the following result.

(1) This follows from structure theory. Another proof: If $G = (T, \mathcal{T})$, then the identity map of $G \to T$ gives (by duality) a dense map of $\mathbf{Z}$ into $\hat{G}$. Hence $\hat{G}$ is monothetic. All monothetic groups are $\mathbf{Z}$ or compact [5, Theorem 2.3.2]; hence $G$ is $T$ or discrete.

**Corollary 1.** *Suppose that $G$ is compact. Then if $\mu = \delta_e + \mu_0$ is unitary and $\mu_0$ is absolutely continuous, $\mu = \exp \nu$ for some absolutely continuous $\nu$.*

The argument is the same as above. Suppose $\hat{\mu} = -1$ at $\gamma_1, \cdots, \gamma_l$; one lets $d\sigma(x) = i(\gamma_1(x) + \cdots + \gamma_l(x))\,dx$ and reasons the same way. (The set where $\hat{\mu} = -1$ is finite because $\hat{\mu}_0$ is 0 at $\infty$.)

The argument of the first half of the theorem reduces the problem for general Abelian $G$ to finding absolutely continuous unitary measures on various other groups (viz., $G$ with finer topologies). But that still leaves a good deal of work in most cases.

A similar attack does, however, work in at least one other case.

**Theorem 2.** *If $\mu$ is a unitary measure on a discrete torsion group $G$, then $\mu = e^{\nu}$ for some $\nu \in \mathfrak{M}(G)$.*

**Proof.** Since $|\hat{\mu}| = 1$ and $\Gamma$ is totally disconnected, $\hat{\mu}(\gamma) = e^{a(\gamma)}$ for some continuous function $a$. Now the Arens-Royden theorem says that $\mu = \exp(\nu)$ for some $\nu \in \mathfrak{M}(G)$.

**3. Delta measures; unitary measures on torsion-free discrete groups.** Theorem 2 implies that if $z \in T$ is of finite order, then $\delta_z$, the point mass at $z$ with mass 1, is of the form $\exp(\nu)$. Here is a converse.

**Theorem 3.** *If $z \in T$ has infinite order, then $\delta_z \neq \exp(\nu)$ for any measure $\nu$.*

**Proof.** Let $z = e^{i\theta}$. Then $\hat{\delta}_z(n) = e^{-in\theta}$; if, therefore, $\exp(\nu) = \delta_z$, then $\hat{\nu}(n) = -in\theta + 2\pi i k_n$, $k_n \in \mathbf{Z}$. Let $K = \|\nu\|$, so that $K \geq |2\pi k_n - n\theta|$, and let $p$ be an integer $> K$. Let $y = e^{-i\theta/p}$. Then $\exp(\nu/p) * \delta_y$ has a Fourier-Stieltjes transform whose range consists of $p$th roots of unity. Let the roots be $\omega_1, \cdots, \omega_p$; then the set $S_j$ on which the transform is $\omega_j$ is, according to results on idempotent measures (see, e.g., [5, p. 61 ff.]), a union of arithmetic progressions (with finitely many exceptions). The idea in what follows is that the irrationality of $\theta/\pi$ makes it impossible for the $S_j$ to be so orderly.

Let $N$ be large enough so that the variations in the progressions have been ironed out by then; pick $S_j$ so that it contains an infinite arithmetic progression, with common difference $r$, say. Since $|2\pi k_n - n\theta| < K$, we have $(n\theta - K)/2\pi \leq k_n \leq (n\theta + K)/2\pi$.

Replace $n$ by $n + mr$; we get

$$\frac{(n+mr)\theta - K}{2\pi} \leq k_{n+mr} \leq \frac{(n+mr)\theta + K}{2\pi}, \quad \text{or} \quad \frac{mr\theta - 2K}{2\pi} \leq k_{n+mr} - k_n \leq \frac{mr\theta + 2K}{2\pi}.$$

But $k_n - k_{n+mr}$ is a multiple of $p$; thus there is a multiple of $p$ between $(mr\theta - 2K)/2\pi$ and $(mr\theta + 2K)/2\pi$, $\forall m$. That means that for all $m$, there is an integer between $(mr\theta - 2K)/2\pi p$ and $(mr\theta + 2K)/2\pi p$, or that $-mr\theta/2\pi p$ is congruent (mod 1) to a number between $-2K/2\pi p > -1/\pi$ and $2K/2\pi p < 1/\pi$. As $mr\theta/2\pi p$ is irrational, this is impossible; in fact, the numbers $-mr\theta/2\pi p$ are dense.

**Corollary 2.** *If $G$ is any locally compact group and $x \in G$ has infinite order, then $\delta_x$ is not an exponential.*

**Proof.** We may as well assume that $G$ is discrete, since if $\delta_x = \exp \nu$ and $\nu = \nu_1 + \nu_2$, with $\nu_1$ discrete and $\nu_2$ continuous, then $\delta_x = \exp \nu_1$ also. As $T$ is divisible, we can extend the map $\alpha : nx \mapsto e^{in}$ to a homomorphism (also called $\alpha$) of $G$ into $T$. If $\delta_x = \exp(\nu)$, then it is easily checked that $\delta_{\alpha(x)} = \exp(\alpha_x \nu)$, where $\alpha_x \nu(E) = \nu(\alpha^{-1}(E))$. But Theorem 3 makes this impossible.

Theorem 3 makes it possible to find all the unitary measures on any torsion-free discrete group.

**Theorem 4.** *Let $G$ be discrete and torsion-free. Then every unitary measure $\mu$ on $G$ is of the form $\delta_x * \exp \nu$, for some $x \in G$ and some measure $\nu$ on $G$ with $\nu^\sim = -\nu$; moreover, $x$ is uniquely determined by $\mu$.*

**Proof.** It suffices to show that any function $f : \Gamma \to \mathbf{C}^x$ is homotopic to a character $X_x : \gamma \mapsto (x, \gamma)$, $x \in G$. For then we can choose $x \in G$ such that $\hat{\mu} \cdot X_{-x}$ is homotopic to the trivial map. Since $X_{-x} = \hat{\delta}_{-x}$, we can use Arens-Royden to show that $\mu * \delta_{-x} = \exp(\nu)$ for some $\nu$, and the theorem follows. The uniqueness of $x$ follows from Theorem 3.

Now we prove the homotopy result. Since $\Gamma$ is compact, we can use Stone-Weierstrass to approximate $f$ by a finite linear combination of characters, $f \approx \Sigma_{j=1}^n a_j \chi_{x_j} = g$, say, so that $\|f - g\|_\infty < \frac{1}{2} \inf_{\gamma \in \Gamma} |f(\gamma)|$. Then $f$ and $g$ are homotopic. Let $\Gamma_0$ be the intersection of the kernels of the $\chi_{x_j}$. Then $g$ is constant on $\Gamma_0$-cosets, and therefore we can define $\tilde{g}$ on $\Gamma/\Gamma_0$ by $\tilde{g}(x\Gamma_0) = g(x)$. $(\Gamma/\Gamma_0)^\wedge$ is the group generated by the $x_j$: therefore $\Gamma/\Gamma_0$ is isomorphic to a torus. But it is well known (see, e.g., [3, Theorem II. 7.1]) that the characters of $T^m$ represent the homotopy classes of maps on $T^m$; hence $\tilde{g}$ is homotopic to a character $\tilde{\chi}$ of $\Gamma/\Gamma_0$. Pull $\tilde{\chi}$ back to $\Gamma$, getting $\chi$; then $g$ and $\chi$ are homotopic, as desired.

A corollary of the proof is

**Corollary 3.** *If $\Gamma$ is a connected compact group, then $H^1(\Gamma, \mathbf{Z}) \cong G$. (The cohomology is Čech cohomology.)*

**Proof.** From [4], $H^1(\Gamma, \mathbf{Z}) \cong$ group of homotopy classes of maps from $\Gamma$ to $T$. Since $\Gamma$ is connected, $G$ is torsion-free [5, Theorem 2.5.6]; the last part of the above proof does the rest. (The result is dual to one of Steenrod's: $H_1(G, T) \cong G$. See [6, Theorem 15]).

**4. Unitary measures on arbitrary discrete groups and $T$.** We have still not solved the problem of finding the unitary measures on $T_d$, since $T_d$ has torsion elements. The following example shows that we can actually find other unitary measures besides $\delta$-measures $*$exponentials.

Let $z_0 \in T$ have infinite order, and let $\mu = \frac{1}{2}(\delta_1 + \delta_{z_0} + \delta_{-1} - \delta_{-z_0})$. Then if $n$ is even, $\hat{\mu}(n) = \frac{1}{2}(1 + z_0^{-n} + (-1)^{-n} - (-z_0)^{-n}) = 1$, while if $n$ is odd, $\hat{\mu}(n) = z_0^{-n}$. Suppose now that $\mu = \delta_{z_1} * \exp(\nu)$ for some $z \in T$ and some measure $\nu$. Then $\mu * \delta_{z_1^{-1}} = \exp(\nu)$. Let $\mu_0 = \frac{1}{2}(\mu * \delta_{z_1^{-1}}) * (\delta_1 + \delta_{-1})$, $\mu_1 = \frac{1}{2}(\mu * \delta_{z_1^{-1}}) * (\delta_1 - \delta_{-1})$. Then

$$\mu_0 + \mu_1 = \mu * \delta_{z_1^{-1}}, \quad \text{and} \quad \hat{\mu}_0(n) = \begin{cases} (\mu * \delta_{z_1^{-1}})\hat{\ }(n), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Let $\alpha: T \to T$ take $z \mapsto z^2$, and let $\mu_0^* = \mu \circ \alpha$. Then $\hat{\mu}_0^*(n) = \hat{\mu}_0(2n) = z_1^{2n}$; also, $\hat{\mu}_0(2n) = \exp \hat{\nu}(2n) = \exp \hat{\nu} * (n)$. Hence $\delta_{z_1^{-2}} = \mu_0^* = \exp(\nu^*)$. It follows that $z_1$ has finite order in $T$.

On the other hand,

$$\hat{\mu}_1(n) = \begin{cases} (\mu * \delta_{z_1^{-1}})\hat{\ }(n) = (z_1 z_0^{-1})^n, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}$$

and $z_2 = z_1 z_0^{-1}$ has infinite order. Also, $\hat{\mu}_1(n) = \exp \hat{\nu}(n)$ whenever $n$ is odd. We can now use the same reasoning as in the proof of Theorem 3 to get a contradiction. Let $z_2 = e^{i\theta}$, and define $K$, $p$, and $y$ as in Theorem 3. Then $\exp \nu/p * \delta_y * \frac{1}{2}(\delta_1 - \delta_{-1})$ has a Fourier-Stieltjes transform whose range consists of $p$th roots of unity and 0; the value is 0 on $2\mathbf{Z}$. Define the $S_j$ as in Theorem 3, and the rest of the argument in Theorem 3 goes through. It follows that $\mu$ is not a point measure convolved with an exponential.

What this argument says is that Theorem 4 is false for $G = Z \oplus \mathbf{Z}/2\mathbf{Z}$. ($G$ is embedded in $T$ as $z_0\mathbf{Z} \oplus \{-1, 1\}$.) Then $\Gamma \cong T \oplus \mathbf{Z}/2\mathbf{Z}$; $\hat{\mu}$ is 1 on one circle and $z$ on the other. This construction generalizes.

Let $G$ be any discrete group, and let $G_1$ be a finite subgroup (of order $n$, say). Then $\Gamma_1 = G_1^{\perp}$ is of index $n$; let $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ be the cosets. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be the idempotent measures whose Fourier-Stieltjes transforms are the characteristic functions of $\Gamma_1, \cdots, \Gamma_n$ respectively. Let $x_1, \cdots, x_n$ be elements of $G$, and let $\nu$ be a measure on $G$ with $\nu^{\sim} = -\nu$. Then $\mu = (\sum_{j=1}^n \delta_{x_j} * \sigma_j) * \exp \nu$ is unitary.

**Theorem 5.** *Every unitary measure on a discrete group $G$ arises in this way.*

Proof. Let $\mu_1$ be a unitary measure. As in Theorem 3, it suffices to show that $\hat{\mu}_1$ and $\hat{\mu}$ are homotopic for some $\mu = (\Sigma_{j=1}^n \delta_{x_j} * \sigma_j) * \exp(\nu)$. Again, as in Theorem 3, $\hat{\mu}_1$ is homotopic to a linear combination of finitely many characters: $\hat{\mu}_1 \sim \Sigma_{j=1}^m a_j \chi_{y_j} = f$, say. Let $\Gamma_0$ be the common kernel of $\chi_{y_1}, \cdots, \chi_{y_m}$; $f$ gives $\bar{f}$ on $\Gamma/\Gamma_0$. We may assume from now on that $\Gamma_0 = \{1\}$, since from now on everything will be constant on $\Gamma_0$-cosets. Note that $f$ is the transform of a measure; thus $f^{-1}\hat{\mu}_1 = (\exp \nu_0)\hat{}$.

Given our assumption, $G$ is generated by $y_1, \cdots, y_m$; hence $G \cong Z^k \oplus G_1$, where $G_1$ is finite of order $n$. Thus $\Gamma_1, \cdots, \Gamma_n$ are $k$-tori. Hence there are elements $x_1, \cdots, x_n$ such that $\chi_{x_j}$ and $f$ are homotopic on $\Gamma_j$. It follows that if $\mu_0 = \Sigma_{j=1}^n \delta_{x_j} * \sigma_j$, then $\hat{\mu}_0^{-1} f$ is homotopic to the trivial map on each component. Hence $\hat{\mu}_0^{-1} f = (\exp \nu_1)\hat{}$, and the theorem follows.

In the case of $T_d$, $G_1$ is necessarily cyclic. A more careful analysis along the lines of the example shows that if one picks $m$ as small as possible, then each $x_j$ is determined modulo the torsion group of $T_d$.

Theorems 1 and 5 together determine all the unitary measures on $T$. As noted earlier, they also determine all the connected components of $\mathfrak{M}(T)^\times$. We state the result here for completeness.

Corollary 4. *Let $\mu$ be an invertible measure on $T$. Then there are an integer $m$, elements $z_1, \cdots, z_m \in T$, and a measure $\nu$ on $T$ such that $\mu = \exp(\nu) * (\Sigma_{j=1}^m \delta_{z_j} * \sigma_{m,j})$, where $\sigma_{m,j}$ is the idempotent measure on $T$ whose Fourier-Stieltjes transform is 1 on $mZ + j$ and 0 elsewhere. If $\mu$ is unitary, $\mu$ can be expressed in the same form, but with $\tilde{\nu} = -\nu$.*

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