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MAJORIZATION-SUBORDINATION THEOREMS FOR LOCALLY UNIVALENT FUNCTIONS. III

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ABSTRACT. A quantitative majorization-subordination result of Goluzin and Tao Shah for univalent functions is generalized to $\mathfrak{U}_\alpha$, the linear invariant family of locally univalent functions of finite order α. If $f(z)$ is subordinate to $F(z)$ in the open unit disc, $f'(0) \geq 0$, and $F(z)$ is in $\mathfrak{U}_\alpha$, $1.65 \leq \alpha < \infty$, then $f'(z)$ is majorized by $F'(z)$ in $|z| \leq (\alpha + 1) - (\alpha^2 + 2\alpha)^{1/2}$. The result is sharp.

I. Introduction. Let $\mathfrak{S}$ denote the set of all normalized analytic univalent functions in the open unit disc D. Let $f(z)$, $F(z)$ and $\varphi(z)$ be analytic in $|z| < r$. We say that $f(z)$ is *majorized* by $F(z)$ in $|z| < r$, if $|f(z)| \leq |F(z)|$ in $|z| < r$. We say that $f(z)$ is *subordinate* to $F(z)$ in $|z| < r$ if $f(z) = F(\varphi(z))$ where $|\varphi(z)| \leq |z|$ in $|z| < r$.

Let $\mathfrak{U}_\alpha$ be the set of all locally univalent ($f'(z) \neq 0$) analytic functions in D with order $\leq \alpha$ which are of the form $f(z) = z + \cdots$. The family $\mathfrak{U}_\alpha$ is known as the universal linear invariant family of order α [4]. A concise summary and introduction to properties of linear invariant families which relate to the following material is contained in [1]. The present paper concludes the proof of results announced in [1].

Majorization-subordination theory begins with Biernacki who showed in 1936 that if $f(z)$ is subordinate in D to $F(z)$ ($F(z) \in \mathfrak{S}$), then $f(z)$ is majorized by $F(z)$ in $|z| < 1/4$. In the succeeding years Goluzin, Tao Shah, Lewandowski and MacGregor examined various related problems but always under the stipulation that the dominant function $F(z)$ is in $\mathfrak{S}$ (for greater detail see [1]).

In 1951 Goluzin showed that if $f(z)$ is majorized by a univalent function $F(z)$, then $f'(z)$ would be majorized by $F'(z)$ in $|z| < 0.12$. He conjectured that majorization would always occur for $|z| < 3 - \sqrt{8}$ and this was proved by Tao Shah in 1958.

In this paper we show that the result is actually true for functions in $\mathfrak{U}_\alpha$ and obtain the sharp radius of majorization as $\alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$ for $1.65 \leq \alpha < \infty$. This yields $3 - \sqrt{8}$ for the case $\alpha = 2$.

Our investigation shows that the important datum for majorization-subordination theory is *not* univalence, but the order of a linear invariant family. In particular, many classically derived estimates for univalent functions are true for functions of *infinite valence*.

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The method of proof uses a considerable number of estimates. Because of these estimates it remains an open question as to whether the result of Theorem 1 is true for $1 \leq \alpha < 1.65$. We conjecture that Theorem 1 is true in this range, and therefore conjecture that for convex univalent functions ($F(z) \in \mathfrak{U}_1$) the radius of majorization of the derivative should be $2 - \sqrt{3}$.

II. **Statement and proof of the theorem.** We first state and prove an improved form of the Schwarz lemma for unimodular analytic functions which is due to Tao Shah [5]. We then state a weaker form due to Goluzin.

Lemma 1. *Let* $\varphi(z) = az + \cdots$, $a \geq 0$, $|\varphi(z)| \leq 1$, *be analytic in* $|z| < 1$. *Then*

$$\varphi(z) = z \cdot \frac{a + \omega(z)}{1 + a\omega(z)}, \tag{1}$$

where $\omega(z)$ *is analytic and satisfies* $|\omega(z)| \leq |z|$ *in* $|z| < 1$. *Moreover, for any* z_0 *in* $|z| < 1$, *if we let* $\omega(z_0) = c$, *then*

$$|\varphi'(z_0)| \leq \left|\frac{a + 2c + ac^2}{(1 + ac)^2}\right| + \frac{1 - a^2}{|1 + ac|^2} \cdot \frac{|z_0|^2 - |c|^2}{1 - |z_0|^2}. \tag{2}$$

Proof. Since $|\varphi(z)/z| \leq 1$ in $|z| < 1$, the function

$$\omega(z) = \frac{\varphi(z)/z - a}{1 - a\varphi(z)/z} \tag{3}$$

satisfies the Schwarz lemma. Solving (3) for $\varphi(z)$ yields (1).

Fix a point z_0 in D and let $\omega(z_0) = c$. The derivative of $\varphi(z)$ at z_0 is

$$\varphi'(z_0) = (z_0\omega'(z_0) - c)\frac{(1 - a^2)}{(1 + ac)^2} + \frac{a + c}{1 + ac} + \frac{c(1 - a^2)}{(1 + ac)^2}. \tag{4}$$

It therefore suffices to show

$$|z_0\omega'(z_0) - c| \leq (|z_0|^2 - |c|^2)/(1 - |z_0|^2).$$

The function

$$f(\zeta) = \left\{\omega\left(\frac{\zeta + z_0}{1 + \bar{z}_0\zeta}\right) - \omega(z_0)\right\} \Big/ \left\{1 - \overline{\omega(z_0)}\omega\left(\frac{\zeta + z_0}{1 + \bar{z}_0\zeta}\right)\right\}$$

satisfies the Schwarz lemma in $|\zeta| < 1$ and $f(-z_0) = -c$. Let $g(\zeta) = f(\zeta)/\zeta$ and $h(\zeta) = (g(\zeta) - f'(0)) \cdot (1 - \overline{f'(0)}g(\zeta))^{-1}$. Since $h(\zeta)$ also satisfies the Schwarz lemma we obtain

$$|h(-z_0)| = \left|\frac{c - z_0 f'(0)}{z_0 - c\overline{f'(0)}}\right| \leq |z_0|. \tag{5}$$

However, $f'(0) = (1 - |z_0|^2)(1 - |c|^2)^{-1}\omega'(z_0)$ and therefore upon squaring both sides of (5) and noting that

$$|z_0|^2|\omega'(z_0)|^2 + |c|^2 - |z_0\omega'(z_0) - c|^2 = \overline{\omega'(z_0)}\bar{z}_0 c + \bar{c} z_0 \omega'(z_0),$$

we obtain

$$(1 - |z_0|^2)^2(|z_0\omega'(z_0) - c|^2 - |c|^2) \le (1 - |c|^2)(|z_0|^4 - |c|^2).$$

Hence

$$(1 - |z_0|^2)^2(|z_0\omega'(z_0) - c|^2) \le (|z_0|^2 - |c|^2)^2,$$

or, equivalently,

$$|z_0\omega'(z_0) - c| \le (|z_0|^2 - |c|^2)/(1 - |z_0|^2),$$

which concludes the lemma.

Lemma 2. *Under the condition of Lemma* 1,

$$\left|\frac{\varphi(z) - z}{1 - \bar{z}\varphi(z)}\right| \le \frac{|z|(1 - a)}{1 + |z|^2 - |z|(1 + a)}, \qquad z \in D,$$

and

$$|\varphi'(z)| \le \frac{a(1 + |z|^2) + 2|z|}{1 + |z|^2 + 2a|z|} \cdot \frac{1 - |\varphi(z)|^2}{1 - |z|^2}, \qquad z \in D.$$

Proof. The proof of this lemma can be found within a proof by Goluzin [3, pp. 331–332].

Theorem. *Let $f(z)$ be subordinate to $F(z)$ in D with $f'(0) \ge 0$. If $F(z) \in \mathfrak{U}_\alpha$, $1.65 \le \alpha < \infty$, then $f'(z)$ is majorized by $F'(z)$ in $|z| \le \alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$ and the result is best possible.*

Proof. Since $f(z)$ is subordinate to $F(z)$ in D with $f'(0) \ge 0$ we have $f(z) = F(\varphi(z))$ where $\varphi(z)$ satisfies Lemma 1. Choose and fix an arbitrary z_0 in $|z| \le (\alpha + 1) - (\alpha^2 + 2\alpha)^{1/2}$. Our goal is to show that $|f'(z_0)/F'(z_0)| \le 1$.

Since $f(z) = F(\varphi(z))$ we have

$$|f'(z_0)/F'(z_0)| = |F'(\varphi(z_0))/F'(z_0)|\,|\varphi'(z_0)|. \tag{6}$$

For any **a** and **b** in D and any function F in $\mathfrak{U}_\alpha$ we have [4, Lemma 2.1]

$$\left|\frac{F'(a)}{F'(b)}\right| \le \frac{1 - |b|^2}{1 - |a|^2}\left(\frac{|1 - \bar{a}b| + |a - b|}{|1 - \bar{a}b| - |a - b|}\right)^\alpha. \tag{7}$$

We therefore obtain our fundamental inequality

$$\left|\frac{f'(z_0)}{F'(z_0)}\right| \le \frac{1 - |z_0|^2}{1 - |\varphi(z_0)|^2}\left(\frac{|1 - \overline{\varphi(z_0)}z_0| + |\varphi(z_0) - z_0|}{|1 - \overline{\varphi(z_0)}z_0| - |\varphi(z_0) - z_0|}\right)^\alpha |\varphi'(z_0)|. \tag{8}$$

Our proof now proceeds in two different directions depending on whether $f'(0)$ is large or small in relation to α. We first consider the case of small $f'(0)$; namely,

$$0 \leq f'(0) \leq 3/20 \quad (\text{if } 1.65 \leq \alpha \leq 2),$$
$$0 \leq f'(0) \leq 1/6 \quad (\text{if } 2 \leq \alpha \leq 3),$$
$$0 \leq f'(0) \leq 1/10 \quad (\text{if } 3 \leq \alpha < \infty).$$

If we apply Lemma 2 to our fundamental inequality (8) we obtain

$$\left|\frac{f'(z_0)}{F'(z_0)}\right| \leq \frac{ba+1}{b+a}\left(\frac{b-a}{b-1}\right)^{\alpha} \equiv k(a,\alpha,b) \tag{9}$$

where $b = (1 + |z_0|^2)/2|z_0|$ and $a = f'(0)$. Note that b is always bounded below by $\alpha + 1$ since $r_0 = |z_0|$ is bounded above by $\alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$.

It is quite easy to show that $k(a,\alpha,b)$ is the product of two positive decreasing functions in **b** and hence is itself a decreasing function in **b**. We now show that $k(a,\alpha,\alpha+1)$ is increasing in **a**. Since

$$\frac{\partial k(a,\alpha,\alpha+1)}{\partial a} = \frac{(\alpha+1-a)^{\alpha-1}}{(\alpha)^{\alpha}} \frac{P(a,\alpha)}{(\alpha+1+a)^2},$$

where $P(a,\alpha) = -\alpha(\alpha+1)a^2 - (\alpha^3 + 3\alpha^2 + 4\alpha)a + \alpha(\alpha+1)^2$, we are reduced to establishing $P(a,\alpha) \geq 0$. But $P(a,\alpha)$ is a quadratic in **a** with negative leading coefficient and $P(0,\alpha) > 0$. Therefore, if $P(.4,\alpha)$ is greater than 0, $P(a,\alpha)$ will be greater than zero for any **a** in $[0,.4]$ which will conclude the argument that $P(a,\alpha)$ is nonnegative for small $f'(0)$.

A computation shows $P(.4,\alpha+1) = \alpha(.6\alpha^2 + .64\alpha - .76) > 0$ which therefore concludes the demonstration that $k(a,\alpha,\alpha+1)$ is increasing in **a**. Thus $|f'(z_0)/F'(z_0)| \leq k(a,\alpha,b) \leq k(a,\alpha,\alpha+1)$, and, since $k(a,\alpha,\alpha+1)$ is increasing in **a**, in order to conclude the proof of the theorem for 'small' values of a, it suffices to show that

(a) $k(3/20,\alpha,\alpha+1) \leq 1$ when $1.65 \leq \alpha \leq 2$,
(b) $k(1/6,\alpha,\alpha+1) \leq 1$ when $2 \leq \alpha \leq 3$,
(c) $k(1/10,\alpha,\alpha+1) \leq 1$ when $3 \leq \alpha < \infty$.

Subcase a. Since

$$\begin{aligned}
&(d/d\alpha)k(.15,\alpha,\alpha+1) \\
&\quad = \left(\frac{\alpha+1-.15}{\alpha}\right)^{\alpha}\left[\frac{(.15)^2-1}{(\alpha+.15+1)^2}\right. \\
&\qquad\qquad \left. + \frac{(\alpha+1).15+1}{\alpha+.15+1}\left(\log\frac{\alpha+1-.15}{\alpha} + \frac{.15-1}{\alpha+1-.15}\right)\right] \\
&\quad \leq \left(\frac{\alpha+1-.15}{\alpha}\right)^{\alpha}\left[\frac{(.15)^2-1}{(2+.15+1)^2} + \frac{(1.65+1).15+1}{1.65+.15+1}\left(\log\frac{1.65+1-.15}{1.65}\right)\right. \\
&\qquad\qquad \left. + \left(\frac{(2+1).15+1}{2+.15+1}\right)\left(\frac{.15-1}{2+1-.15}\right)\right] \\
&\quad < 0
\end{aligned}$$

for $1.65 \leq \alpha \leq 2$, therefore $k(.15, \alpha, \alpha + 1)$ is a decreasing function of α in this range. It therefore suffices to check by a routine computation that $k(.15, 1.65, 2.65) \leq 1$.

Subcase b. Since $[(\alpha + 1)a + 1]/(\alpha + a + 1)$ is decreasing with α, and $((\alpha + b)/\alpha)^\alpha$ is an increasing function of α for all $\alpha > 0$ and all $b > 0$, we can state

$$k(1/6, \alpha, \alpha + 1) \leq \frac{(2 + 1)(1/6) + 1}{2 + 1/6 + 1}\left(\frac{3 + 1 - 1/6}{3}\right)^3.$$

It is easy to verify that this latter quantity is indeed less than one.

Subcase c. This is the easiest case since, as above,

$$k(.10, \alpha, \alpha + 1) \leq e\left(\frac{(3 + 1).10 + 1}{3 + .10 + 1}\right) < 1.$$

Thus for small values of $f'(0)$ we have shown that $f'(z)$ is majorized by $F'(z)$ in $|z| < (\alpha + 1) - (\alpha^2 + 2\alpha)^{1/2}$.

We now consider the case that $f'(0)$ is large. Returning to our fundamental inequality (8), we note that in the language of Lemma 1

$$\varphi(z_0) = z_0(a + c)/(1 + ac), \qquad c = re^{i\theta}, \ |z_0| = r_0, \text{ and}$$

$$(1 - |z_0|^2)(1 - |\varphi(z_0)|^2)$$

$$= |1 + ac|^{-2}(|1 + ac - r_0^2(a + c)| + r_0(1 - a)|1 - c|) \tag{10}$$

$$\cdot(|1 + ac - r_0^2(a + c)| - r_0(1 - a)|1 - c|).$$

Therefore (8), (10) and Lemma 1 together imply that

$$\left|\frac{f'(z_0)}{F'(z_0)}\right| \leq (1 - r_0^2)\frac{(|1 + ac - r_0^2(a + c)| + r_0(1 - a)|1 - c|)^{\alpha-1}}{(|1 + ac - r_0^2(a + c)| - r_0(1 + a)|1 - c|)^{\alpha+1}} \tag{11}$$

$$\cdot[|a + 2c + ac^2|(1 - r_0^2) + (r_0^2 - |c|^2)(1 - a^2)].$$

Lemma 3, at the end of the paper, shows that when $f'(0)$ is 'large' the right-hand side of (11) as a function of $c = re^{i\theta}$ has its maximum at $\theta = 0$. Noting that

$$1 + ar - r_0^2(a + r) \pm r_0(1 - a)(1 - r) = (1 \pm r_0)[1 + ar \mp r_0(a + r)],$$

we infer

$$\left|\frac{f'(z_0)}{F'(z_0)}\right| \leq \left(\frac{1 + r_0}{1 - r_0}\right)^\alpha \frac{[1 + ar - r_0(a + r)]^{\alpha-1}}{[1 + ar + r_0(a + r)]^{\alpha+1}} \tag{12}$$

$$\cdot[(1 - r_0^2)(a + 2r + ar^2) + (r_0^2 - r^2)(1 - a^2)].$$

Let $L(r, r_0, a)$ denote the right-hand side of (12). The proof of majorization will be concluded if we can show that L is an increasing function of a since $L(r, r_0, 1) \equiv 1$.

However,

$$\frac{dL}{da} = \left(\frac{1+r_0}{1-r_0}\right)^{\alpha} \frac{[1+ar-r_0(a+r)]^{\alpha-2}}{[1+ar+r_0(a+r)]^{\alpha+2}} \cdot R(a)$$

where

$$\begin{aligned} R(a) = {} & [(1+ar)^2 - r_0(a+r)^2] \cdot [1 - r_0^2 r^2 - (2a+1)(r_0^2 - r^2)] \\ & -[(1-r_0^2)(a+2r+ar^2) + (1-a^2)(r_0^2 - r^2)] \\ & \cdot [2r(1-r_0^2) + 2\alpha r_0(1-r^2) - 2a(r_0^2 - r^2)]. \end{aligned}$$

The problem then is to show $R(a)$ is nonnegative. Since

$$\begin{aligned} R'(a) &= 2(r_0^2 - r^2)[2\alpha a r_0(1-r^2) + (1-r)^2(1-r_0^2)(a-1)] \\ &\quad - 2\alpha r_0 \cdot (1-r_0^2)(1-r^4) \\ &\leq 2\alpha r_0[2r_0^2 - (1-r_0^2)(1-r_0^4)], \end{aligned}$$

we can conclude that $R(a)$ is a decreasing function if we note that $2r_0^2 - (1-r_0^2)(1-r_0^4) < 0$ since $r_0 \leq (\alpha+1) - (\alpha^2+2\alpha)^{1/2} \leq 1/2$. Thus $R(a) \geq R(1)$. However,

$$\begin{aligned} R(1) &= (1+r)^2(1-r_0^2)(1-r)[(1-r)(1-r_0^2) - 2\alpha r_0(1+r)] \\ &\geq (1+r)^2(1-r_0^2)(1-r)(1+r_0)[1-r_0^2-2\alpha r_0] \\ &\geq 0, \end{aligned}$$

since $(1-r_0^2) - 2\alpha r_0 \geq 0$ for $r_0 \leq (\alpha+1) - (\alpha^2+2\alpha)^{1/2}$.

Thus for large $f'(0)$, $|f'(z_0)/F'(z_0)| \leq 1$; that is, $f'(z)$ is majorized by $F'(z)$ in $|z| \leq (\alpha+1) - (\alpha^2+2\alpha)^{1/2}$.

We now show that this result cannot be improved. This means that for any real number $m' > m(\alpha) \equiv (\alpha+1) - (\alpha^2+2\alpha)^{1/2}$ we must find analytic functions $f(z)$ and $F(z)$ such that $f(z)$ is subordinate to $F(z)$, $f'(0) \geq 0$, $F(z) \in \mathfrak{U}_\alpha$, but for which $|f'(z)| \leq |F'(z)|$ for all $|z| < m'$ is false.

Let

$$F(z) = \frac{1}{2\alpha}\left\{1 - \left(\frac{1-z}{1+z}\right)^{\alpha}\right\} \quad \text{and} \quad f(z,a) = F(\varphi(z))$$

where $\varphi(z) = z(a+z)/(1+az)$, $0 \leq a \leq 1$. Then $f(z,a)$ is subordinate to $F(z)$ in D for any a, $0 \leq a \leq 1$, $F(z)$ is in $\mathfrak{U}_\alpha$, and $f'(0) \geq 0$. A computation shows

$$\left.\frac{\partial}{\partial z} f(z,a)\right|_{z=r} = \frac{(1-r^2)^{\alpha-1}}{2^{\alpha} r^{\alpha}} \cdot \frac{ab+1}{(b+a)^{\alpha+1}}$$

where $b = (1+r^2)/2r$, and

(13) $$\frac{\partial}{\partial a}\left[\frac{\partial}{\partial z}f(z,a)\bigg|_{z=r}\right]\bigg|_{a=1} = \frac{(1-r^2)^{\alpha-1}}{2^\alpha r^\alpha}\cdot\frac{[b-(\alpha+1)]}{(b+1)^{\alpha+1}}.$$

Thus if we let $z = r$, $m < r < m'$, then $b = (1 + r^2)/2r < \alpha + 1$ and (13) implies that $\partial f(z, a)/\partial z|_{z=r}$ is a decreasing function of a for such a value of r. Therefore for a sufficiently close to 1,

$$f'(r,a) \equiv \partial f(z, a)/\partial z|_{z=r} > \partial f(z, 1)/\partial z|_{z=r} \equiv F'(r) > 0.$$

Therefore f' is not majorized by F' in $|z| < m'$.

This concludes the proof of the theorem.

Corollary 1. *If $f(z)$ is majorized by $F(z)$ in $\mathfrak{U}_2$ and $f'(0) \geq 0$, then $f'(z)$ is majorized by $F'(z)$ in $|z| \leq 3 - \sqrt{8}$ and the result is sharp.*

Corollary 1 is an improvement on Tao Shah's result for $F(z)$ in $\mathfrak{S}$ since $\mathfrak{S}$ is a proper subset of $\mathfrak{U}_2$. The same estimates therefore hold even for the functions of infinite valence which lie in $\mathfrak{U}_2$.

III. Statement and proof of Lemma 3.

Lemma 3. *If $1.65 \leq \alpha \leq 2$ and $3/20 \leq a \leq 1$, or if $2 \leq \alpha \leq 3$ and $1/6 \leq a \leq 1$, or if $3 \leq \alpha \leq \infty$ and $1/10 \leq a \leq 1$, then, as a function of θ, the maximum of*

(14) $$\frac{(|1 + ac - r_0^2(a + c)| + r_0(1 - a)|1 - c|)^{\alpha-1}}{(|1 + ac - r_0^2(a + c)| - r_0(1 - a)|1 - c|)^{\alpha+1}} \cdot \{|a + 2c + ac^2|(1 - r_0^2) + (r_0^2 - |c|^2)(1 - a^2)\},$$

where $c = re^{i\theta}$ and $0 \leq r \leq r_0 \leq \alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$, occurs at $\theta = 0$.

Proof. Let $I(\theta)$ denote the quantity in (14). In order to compute $dI/d\theta$ we first write I as $I = A^{\alpha-1}C/B^{\alpha+1}$ where

$$A \equiv |1 - ar_0^2 + c(a - r_0^2)| + r_0(1 - a)|1 - c| \equiv D + E,$$

$$B \equiv |1 - ar_0^2 + c(a - r_0^2)| - r_0(1 - a)|1 - c| \equiv D - E,$$

$$C \equiv |a + 2c + ac^2|(1 - r_0^2) + (r_0^2 - r^2)(1 - a^2).$$

Then

(15) $$\frac{dI}{d\theta} = \frac{A^{\alpha-2}}{B^{\alpha+2}}\left[\left\{AB\frac{dC}{d\theta} + 2C\left(\frac{EdE}{d\theta} - \frac{DdD}{d\theta}\right)\right\} + 2\alpha C\left(\frac{DdE}{d\theta} - \frac{EdD}{d\theta}\right)\right].$$

Since

$$\frac{dD}{d\theta} = \frac{-r\sin\theta}{D}(1 - ar_0^2)(a - r_0^2),$$

$$\frac{dE}{d\theta} = \frac{r\sin\theta}{E},$$

$$\frac{dC}{d\theta} = \frac{-2ar\sin\theta}{|a+2c+ac^2|}(1-r_0^2)(1+r^2+2ar\cos\theta),$$

$$AB = (1-r_0^2)[1-a^2r_0^2+r^2(a^2-r_0^2)+(1-r_0^2)2ar\cos\theta],$$

$$E\frac{dE}{d\theta} - D\frac{dD}{d\theta} = a(1-r_0^2)^2 r\sin\theta,$$

$$D\frac{dE}{d\theta} - E\frac{dD}{d\theta} = \frac{rr_0\sin\theta(1-a^2)}{DE}(1-r_0^2)(1-ar_0^2+r^2(a-r_0^2)),$$

we can verify that

$$\frac{dI}{d\theta} = \frac{A^{\alpha-2}}{B^{\alpha+2}}\left(\frac{-2ar\sin\theta(1-r_0^2)^2(1-a^2)}{|a+2c+ac^2|}\right)\{I_1+I_2+I_3\},$$

where

$$I_1 = (1-r^2)(1+r^2r_0^2) - 2r^2(1-r_0^2) + 2ar(r_0^2-r^2)\cos\theta,$$

$$I_2 = -(r_0^2-r^2)|a+2c+ac^2|,$$

$$I_3 = \frac{-\alpha r_0}{a(1-r_0^2)}\left\{\frac{1-ar_0^2+r^2(a-r_0^2)}{|1-c|\,|1-ar_0^2+c(a-r_0^2)|}\right\}$$
$$\cdot\{(1-a^2)(r_0^2-r^2)|a+2c+ac^2| + |a+2c+ac^2|^2(1-r_0^2)\}.$$

Clearly it now suffices to verify that $I_1+I_2+I_3>0$ in order to prove the maximum of $I(\theta)$ occurs at $\theta=0$.

We first determine an estimate for I_3. The expression in the denominator of I_3 satisfies

$$|1-c|\,|1-ar_0^2+c(a-r_0^2)| \geq (1-r)(1-ar_0^2+r(a-r_0^2)).$$

This is most easily seen by squaring both expressions, removing the common factors and noting that $0<(a-r_0^2)/(1-ar_0^2)\leq 1$ since $a\geq .10>(\alpha+1-(\alpha^2+2\alpha)^{1/2})^2\geq r_0^2$. Thus,

$$|I_3| \leq \frac{\alpha r_0[1-ar_0^2+r^2(a-r_0^2)](a+2r+ar^2)}{a(1-r_0^2)(1-r)(1-ar_0^2+r(a-r_0^2))} \tag{16}$$
$$\cdot\{(1-a^2)(r_0^2-r^2)+(a+2r+ar^2)(1-r_0^2)\}.$$

The denominator of (16) is a decreasing function of r and the numerator of (16) is the product of three increasing functions in r. Consequently we obtain,

$$|I_3| \leq \frac{\alpha r_0}{a}\frac{(1+r_0^2)(1+r_0)}{(1-r_0)}\frac{[a(1+r_0^2)+2r_0]^2}{(1+r_0^2+r_0(1+a))} = J_3.$$

However,

$$\frac{d}{dr_0}\left(\frac{a(1+r_0^2)+2r_0}{1+r_0^2+r_0(1+a)}\right)=\frac{(1-r_0^2)[2-a(1+a)]}{(1+r_0^2+r_0(1+a))^2}\geq 0.$$

Therefore J_3 is the product of monotone increasing functions in r_0 and so $J_3 \leq J_3|_{r_0=m}$, where $m = m(\alpha) = \alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$. Upon substituting this value for r_0 into J_3 and noting that $1 + m^2 = 2(\alpha + 1)m$, we obtain

$$|I_3| \leq \frac{8(\alpha+1)}{a}\alpha m^3\frac{(1+m)}{1-m}\frac{[1+a(\alpha+1)]^2}{(3+a+2\alpha)} = J_4.$$

We now turn to a lower estimate on $I_1 + I_2$. Clearly,

$$I_1 + I_2 \geq J_5 \equiv (1-r^2)(1+r_0^2r^2) - 2r^2(1-r_0^2) - (r_0^2-r^2)(1+4r+r^2).$$

To obtain a lower estimate for J_5 we note that

$$\begin{aligned} dJ_5/dr &= -4[r_0^2 + r(1 - r^2 - r_0^2 + r^2r_0^2 - 3r)] \\ &< -4[r(1-2r_0^2-3r_0)+r_0^2]. \end{aligned}$$

Since $1 - 2r_0^2 - 3r_0 \geq 1 - 2m^2 - 3m > .03$, it follows that J_5 is a decreasing function of r. Therefore upon noting $(1-m)^2 = 2\alpha m$ and $(1-m^2)^2 = 4m^2(\alpha^2 + 2\alpha)$ we can conclude that

$$J_5(r) \geq J_5(r_0) = (1-r_0^2)^3 \geq (1-m^2)^3 = 8m^3(\alpha^3+2\alpha^2)\frac{1+m}{1-m}.$$

Therefore,

$$I_1 + I_2 + I_3 \geq J_5 - J_4 \geq \frac{8\alpha m^3(1+m)}{(1-m)a(3+a+2\alpha)}\cdot Q(a,\alpha),$$

where $Q(a,\alpha) = a^2(-\alpha^3 - 2\alpha^2 - \alpha - 1) + a(2\alpha^3 + 5\alpha^2 + 2\alpha - 2) - (\alpha + 1)$. We are therefore reduced to showing $Q(a,\alpha)$ is positive for all possible cases of a and α in the hypothesis of the lemma.

We first note that $Q(a,\alpha)$ is an increasing function of α since

$$\begin{aligned} \frac{\partial Q}{\partial \alpha} &= \alpha^2(6a-3a^2) + \alpha(10a-4a^2) - 1 + 2a - a^2 \\ &> .57\alpha^2 + .96\alpha - .81 > 0. \end{aligned}$$

We next note that Q is a quadratic function in a with negative leading coefficient and $Q(1,\alpha) = \alpha^3 + 3\alpha^2 - 4 \geq 0$. Thus if $Q(a_0,\alpha) > 0$ for a_0 in $(0,1)$, then $Q(a,\alpha) > 0$ for all a, $a_0 \leq a \leq 1$. We thus need only show that $Q(3/20, 1.65)$, $Q(1/6, 2)$ and $Q(1/10, 3)$ are each positive. This is true as a routine computation indicates.

This concludes the proof of the lemma.

BIBLIOGRAPHY

1. D. M. Campbell, *Majorization-subordination theorems for locally univalent functions*, Bull. Amer. Math. Soc. **78** (1972), 535–538. MR **45** #8817.

2.———, *Majorization-subordination theorems for locally univalent functions*. II, Canad. J. Math. **25** (1973), 420–425.

3. G. M. Goluzin, *Geometric theory of functions of a complex variable*, GITTL, Moscow, 1952; English transl., Transl. Math. Monographs, vol. 26, Amer. Math. Soc., Providence, R. I., 1969. MR **15**, 112; **40** #308.

4. Ch. Pommerenke, *Linear-invariante Familien analytischer Funktionen*. I, Math. Ann. **155** (1964), 108–154. MR **29** #6007.

5. Tao Shah, *On the radius of superiority in subordination*, Sci. Record **1** (1957), 329–333. MR **20** #6531.

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