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ASYMPTOTIC PROPERTIES OF U-STATISTICS*

BY

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ABSTRACT. Let r be a fixed positive integer. A U-statistic U_n is an average of a symmetric measurable function of r arguments over a random sample of size n. Such a statistic may be expressed as an average of independent and identically distributed random variables plus a remainder term. We develop a Kolmogorov-like inequality for this remainder term as well as examine some of its (a.s.) convergence properties. We then relate these properties to the U-statistic. In addition, the asymptotic normality of U_N, where N is a positive integer-valued random variable, is established under certain conditions.

1. **Introduction.** Let $X_1, \cdots, X_n$ be independent and identically distributed random variables and let $f(x_1, \cdots, x_r)$ be a symmetric function of r arguments. Then Hoeffding [4] defined a U-statistic as

$$U_n = \binom{n}{r}^{-1} \sum^{(n,r)} f(x_{\alpha_1}, \cdots, x_{\alpha_r})$$

where the summation here and in the sequel is over all combinations $(\alpha_1, \cdots, \alpha_r)$ formed from the integers $\{1, 2, \cdots, n\}$ and $n \geqslant r$. The class of U-statistics includes many of the best-known statistics including the sample mean and the sample variance.

Assume $\theta = E\{U_n\} = E\{f(X_1, \cdots, X_r)\}$ exists and define

$$f_c(x_1, \cdots, x_c) = E\{f(x_1, \cdots, x_c, X_{c+1}, \cdots, X_r)\}$$

for $c = 1, 2, \cdots, r$. We interpret $E\{f(x_1, \cdots, x_c, X_{c+1}, \cdots, X_r)\}$ as the expected value of $f(X_1, \cdots, X_r)$ given that $X_1, \cdots, X_c$ are fixed at the

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values $x_1, \cdots, x_c$, respectively. Next, define $\zeta_c = \operatorname{Var}\{f_c(X_1, \cdots, X_c)\}$ for $c = 1, 2, \cdots, r$. In particular $f_1(x_1) = E\{f(x_1, X_2, \cdots, X_r)\}$ and $\zeta_1 = \operatorname{Var}\{f_1(X_1)\}$. From [4] we have

LEMMA 1 (HOEFFDING). *Assume* $E\{f(X_1, \cdots, X_r)\}^2 < \infty$. *Then*

(i) $0 \leqslant \zeta_c/c < \zeta_d/d$ *for* $1 \leqslant c < d \leqslant r$, *and*

(ii) *for* $n \geqslant r$, *the variance of* U_n *is given by*

$$\operatorname{Var}\{U_n\} = \binom{n}{r}^{-1} \sum_{c=1}^{r} \binom{r}{c} \binom{n-r}{r-c} \zeta_c = n^{-1} r^2 \zeta_1 + O(n^{-2}).$$

We now introduce notation used by Hoeffding [5] to develop a decomposition of U_n (the "H-decomposition"), one having great value in establishing properties of U_n in general.(2) Define $g^{(1)}(x_1) = f_1(x_1) - \theta$ and

$$g^{(h)}(x_1, \cdots, x_h) = f_h(x_1, \cdots, x_h) - \theta - \sum_{j=1}^{h-1} \sum^{(h,j)} g^{(j)}(x_{\alpha_1}, \cdots, x_{\alpha_j})$$

for $h = 2, 3, \cdots, r$. For example, if $h = 2, g^{(2)}(x_1, x_2) = f_2(x_1, x_2) - \theta - g^{(1)}(x_1) - g^{(1)}(x_2)$. Then, for $n \geqslant r$ and $h = 1, 2, \cdots, r$, let

$$V_n^{(h)} = \binom{n}{h}^{-1} \sum^{(n,h)} g^{(h)}(x_{\alpha_1}, \cdots, x_{\alpha_h}).$$

In particular $V_n^{(1)} = n^{-1}\Sigma_{i=1}^n g^{(1)}(x_i) = n^{-1}\Sigma_{i=1}^n f_1(x_i) - \theta$. Strictly speaking, $V_n^{(h)}$ is not a U-statistic as it may depend upon unknown functionals. Nevertheless, it does have most of the attributes of a U-statistic. From [5] we have

LEMMA 2 (HOEFFDING). *Assume that* $E\{f(X_1, \cdots, X_r)\}^2 < \infty$ *and let* $\delta_h = \operatorname{Var}\{g^{(h)}(X_1, \cdots, X_h)\}$ *for* $h = 1, 2, \cdots, r$. *Then*

(i) *for* $h = 1, 2, \cdots, r$ *the mean of* $V_n^{(h)}$ *is* 0 *and the variance is* $\binom{n}{h}^{-1}\delta_h$. *Also,*

(ii) *for* $r \leqslant m \leqslant n$.

$$\begin{aligned} \operatorname{Cov}\{V_n^{(h)}, V_m^{(l)}\} &= \operatorname{Var}\{V_n^{(h)}\}, \qquad h = l = 1, 2, \cdots, r, \\ &= 0, \qquad h \neq l = 1, 2, \cdots, r. \end{aligned}$$

A simple relationship exists between the ζ's and the δ's. Clearly $\delta_1 = \zeta_1$. For further details see Hoeffding [4] and Sproule [10]. The following the-

(2) This material has not been formally published by Hoeffding, and is presented here with his permission.

orem given in [5] introduces the H-decomposition.

THEOREM 1 (HOEFFDING). *Assume that* $E\{|f(X_1, \cdots, X_r)|\} < \infty$. *A U-statistic may be decomposed into a linear combination of uncorrelated U-statistics, specifically,*

$$U_n = \theta + \sum_{h=1}^{r} \binom{r}{h} V_n^{(h)} = \theta + rV_n^{(1)} + R_n, \tag{1.1}$$

where $R_n = \Sigma_{h=2}^r \binom{r}{h} V_n^{(h)}$ *and* Correlation $\{V_n^{(1)}, R_n\} = 0$. *Further,* $S_n^{(h)} = \binom{n}{h} V_n^{(h)}$ *forms a martingale sequence for* $h = 1, 2, \cdots, r$.

Theorem 1 states that U_n is a linear combination of U-statistics, mutually uncorrelated (by Lemma 2) and each successive term having variance of smaller order. It shows that a U-statistic is essentially the sum of an average of I. I. D. random variables $V_n^{(1)}$ and a zero-mean remainder term R_n, and that the two are uncorrelated. From Lemma 2 we see that $\operatorname{Var}\{R_n\} = O(n^{-2})$.

Hoeffding [5] uses the H-decomposition to show that, under the assumption that $E\{|f(X_1, \cdots, X_r)|\} < \infty$, a U-statistic converges to its mean almost surely as $n \longrightarrow \infty$. Berk [2] contains a rather simple proof of the almost sure convergence of a U-statistic by recognizing that U-statistics are reverse martingales.

The asymptotic normality of U_n, first proved by Hoeffding [4], follows directly from the H-decomposition by recognizing that $r\sqrt{n}\, V_n^{(1)}$ is asymptotically $N(0, r^2\zeta_1)$, by the Lindberg-Lévy central limit theorem, and that

$$\lim_{n \to \infty} E\{\sqrt{n} R_n\}^2 = 0.$$

The usefulness of the H-decomposition is further demonstrated in this paper.

2. **Kolmogorov inequalities.** Theorem 1 states that, for each $h = 1, 2, \cdots, r$, $S_n^{(h)} = \binom{n}{h} V_n^{(h)}$ forms a martingale sequence. This fact is used to prove

LEMMA 3. *Assume that* $0 < \delta_h < \infty$ *for some* $h = 1, 2, \cdots, r$. *Then the following Kolmogorov-like inequality holds: for* $\lambda > 0$ *and* $n \geq r$,

$$P\left\{\max_{h \leq \alpha \leq n} |S_\alpha^{(h)}| \geq \lambda \delta_h^{1/2} \binom{n}{h}^{1/2}\right\} \leq \lambda^{-2}. \tag{2.1}$$

PROOF. By Lemma 2, $E\{S_n^{(h)2}\} = \binom{n}{h}\delta_h$. Thus, by the Kolmogorov inequality for martingales, for any $\epsilon > 0$,

$$P\left\{\max_{h \leq \alpha \leq n} |S_\alpha^{(h)}| \leq \epsilon\right\} \leq \epsilon^{-2} \binom{n}{h} \delta_h.$$

Putting $\epsilon = \lambda\delta_h^{1/2}\binom{n}{h}^{1/2}$ completes the proof of (2.1).

We now use Lemma 3 to derive a Kolmogorov-like inequality for a U-statistic. From Theorem 1,

$$S_n = \binom{n}{r}\theta + \binom{n}{r}\sum_{h=1}^{r}\binom{r}{h}\binom{n}{h}^{-1} S_n^{(h)},$$

where we have set $S_n = \binom{n}{r}U_n$ for $n \geqslant r$.

THEOREM 2. *Assume* $E\{f(X_1, \cdots, X_r)\}^2 < \infty$ *and* $\delta_1 > 0$, *and let* $\delta = \Sigma_{h=1}^{r}\binom{r}{h}\delta_h^{1/2}$. *Then*

$$P\left\{\max_{r\leqslant\alpha\leqslant n}\left|S_\alpha - \binom{\alpha}{r}\theta\right| \geqslant \lambda\delta n^{-1/2}\binom{n}{r}\right\} \leqslant r\lambda^{-2} \tag{2.2}$$

for $\lambda > 0$.

PROOF. First note that $\delta_h < \infty$ for $h = 1, 2, \cdots, r$ as a consequence of our assumption. Lemma 1 (i) and the Schwarz inequality. Let E be the event in (2.2). Define the events

$$E_h = \left\{\max_{r\leqslant\alpha\leqslant n} |S_\alpha^{(h)}| \geqslant \lambda\delta_h^{1/2}\binom{n}{h}^{1/2}\right\}$$

for $h = 1, 2, \cdots, r$. Then $E \subseteq \bigcup_{h=1}^{r} E_h$, so that by Lemma 3, $P(E) \leqslant P(\bigcup_{h=1}^{r} E_h) \leqslant \Sigma_{h=1}^{r} P(E_h) \leqslant r\lambda^{-2}$, which completes the proof.

The Kolmogorov inequality for U-statistics (Theorem 2) first appeared in Sproule [10]. Miller and Sen [7] obtain similar results in the course of proving their Lemma 2.5.

3. Strong convergence results.

The main theorem is

THEOREM 3. *Let* $\{b_n\}_2^\infty$ *be a positive increasing sequence of real numbers with* $\lim_{n\to\infty} b_n = \infty$. *If, for some* $h = 1, 2, \cdots, r$, $0 < \delta_h < \infty$ *and*

$$\sum_{j=1}^{\infty} 2^{hj} b_{2^j}^{-2} < \infty, \tag{3.1}$$

then $b_n^{-1}S_n^{(h)}$ *converges almost surely to* 0 *as* $n \longrightarrow \infty$.

PROOF. From Lemma 3, for any $\epsilon > 0$,

$$P\left\{\max_{h\leqslant\alpha\leqslant n} |S_\alpha^{(h)}| \geqslant \epsilon b_n\right\} \leqslant \epsilon^{-2} b_n^{-2}\delta_h\binom{n}{h}. \tag{3.2}$$

Then (3.1), (3.2) and the Borel-Cantelli lemma imply that

$$\lim_{j\to\infty} b_{2^j}^{-1} S_{2^j}^{(h)} = 0 \qquad \text{(a.s.)}. \tag{3.3}$$

Next define $T_j = \max_{2^j \leq n < 2^{j+1}} |S_n^{(h)} - S_{2^j}^{(h)}|$ for $j = 1, 2, \cdots$ and $Y_n = S_{2^j+n}^{(h)} - S_{2^j}^{(h)}$ for $n = 1, 2, \cdots$. Then $\{Y_n\}_1^\infty$ is a martingale sequence, so that, by the Kolmogorov inequality for martingales,

$$P\{T_j \geq \epsilon b_{2^j}\} \leq \epsilon^{-2} b_{2^j}^{-2} E\{Y_{2^j}\}^2. \tag{3.4}$$

Now, since $E\{S_{2^{j+1}}^{(h)} S_{2^j}^{(h)}\} = E\{S_{2^j}^{(h)}\}^2$, then

$$E\{Y_{2^j}\}^2 = E\{S_{2^{j+1}}^{(h)}\}^2 - E\{S_{2^j}^{(h)}\}^2 = \delta_h \left[\binom{2^{j+1}}{h} - \binom{2^j}{h}\right]. \tag{3.5}$$

A little computation shows that $\binom{2^{j+1}}{h} - \binom{2^j}{h} \leq K 2^{hj}$ for some constant $0 < K < \infty$. Thus (3.1), (3.4), (3.5) and the Borel-Cantelli lemma imply that

$$\lim_{j\to\infty} b_{2^j}^{-1} T_j = 0 \qquad \text{(a.s)}. \tag{3.6}$$

Now, for each n, let j be the positive integer such that $2^j \leq n < 2^{j+1}$. Then, since $\{b_n\}_2^\infty$ is positive increasing.

$$b_n^{-1} |S_n^{(h)}| \leq b_{2^j}^{-1} |S_{2^j}^{(h)}| + b_{2^j}^{-1} T_j \tag{3.7}$$

for $n = h, h+1, \cdots$. Combining (3.3), (3.6) and (3.7) completes the proof of the theorem.

COROLLARY. *Assume* $0 < \delta_h < \infty$ *for some* $h = 1, 2, \cdots, r$.

(i) *If* $\gamma < h/2$, *then* $n^\gamma V_n^{(h)}$ *converges almost surely to* 0 *as* $n \to \infty$.

(ii) *If* $\gamma < 1$, *then* $n^\gamma R_n$ *converges almost surely to* 0 *as* $n \to \infty$, *where* R_n *is defined by* (1.1).

PROOF. To prove (i) let $b_n = n^{h-\gamma}$. Then, since $h - 2\gamma > 0$, (3.1) becomes $\Sigma_{j=1}^\infty 2^{-j(h-2\gamma)} < \infty$. Thus $n^{\gamma-h} S_n^{(h)}$ converges almost surely to 0 as $n \to \infty$ which is equivalent to (i). Part (ii) follows directly from (i).

Theorem 3 is a strong result and leads to the law of the iterated logarithm for U-statistics, that is,

THEOREM 4. *Assume* $E\{f(X_1, \cdots, X_r)\}^2 < \infty$ *and* $\zeta_1 > 0$. *Then*

$$\limsup_{n\to\infty} n^{1/2}(U_n - \theta)/(2r^2\zeta_1 \log\log n\zeta_1)^{1/2} = 1 \qquad (a.s).$$

The lim inf *as* $n \to \infty$ *equals* -1 *(a.s.).*

PROOF. Let $t_n = (2 \log \log n\zeta_1)^{1/2}$. From (1.1),

$$(r\zeta_1^{1/2}t_n)^{-1}n^{1/2}(U_n - \theta) = (n^{1/2}\zeta_1^{1/2}t_n)^{-1}S_n^{(1)} + (r\zeta_1^{1/2}t_n)^{-1}n^{1/2}R_n.$$

The result then follows from the law of the iterated logarithm for independent and identically distributed random variables and corollary (ii) of Theorem 3.

THEOREM 5. *Assume* $E\{f(X_1, \cdots, X_r)\}^2 < \infty$ *and if* $\gamma < 1/2$, *then* $n^\gamma(U_n - \theta)$ *converges almost surely to* 0 *as* $n \longrightarrow \infty$.

PROOF. The result follows directly from the H-decomposition (1.1) and corollary (i) of Theorem 3.

4. The asymptotic normality of U_N. Let $\sigma^2 = r^2\zeta_1$. Throughout this section we assume that $E\{f(X_1, \cdots, X_r)\}^2 < \infty$ and $\delta_1 > 0$. Let $\{n_s\}$ be an increasing sequence of positive integers tending to ∞ as $s \longrightarrow \infty$ and $\{N_s\}$ a sequence of proper random variables taking on positive integer values. $\Phi(x)$ represents the standard normal c.d.f. Anscombe's theorem [1] on the asymptotic normality of averages of a random number of I.I.D. random variables extends to U-statistics as follows.

THEOREM 6. *Assume that*

$$p\text{-}\lim_{s\to\infty} n_s^{-1}N_s = 1. \tag{4.1}$$

Then

$$\lim_{s\to\infty} P\{(U_{N_s} - \theta) \leqslant N_s^{-1/2}x\sigma\} = \Phi(x). \tag{4.2}$$

PROOF. A sequence of random variables $\{Y_n\}$ satisfies condition C2 of Anscombe [1] if: given $\epsilon > 0$ and $\eta > 0$ there exists a large $V_{\epsilon,\eta}$ and a small $c > 0$ such that for any $n > V_{\epsilon,\eta}$

$$P\{|Y_{n'} - Y_n| < \epsilon n^{-1/2}\sigma \text{ for all } n' \text{ such that } |n' - n| < cn\} \geqslant 1 - \eta.$$

Since U_n is asymptotically normal, the theorem follows from Theorem 1 of Anscombe [1] if $\{U_n\}$ satisfies C2. Now $\{rV_n^{(1)}\}$ satisfies C2 by Theorem 3 of Anscombe [1]. Also, by corollary (ii) of Theorem 3 we have $\lim_{n\to\infty}n^{1/2}R_n = 0$ (a.s.) which implies that $\{R_n\}$ satisfies C2. Thus $\{U_n\}$ satisfies C2 by the H-decomposition.

Theorem 7 offers the same conclusion as Theorem 6 except that assumption (4.1) is replaced by the weaker assumption (4.4). Theorem 6 is introduced mainly to show that U-statistics satisfy Anscombe's condition C2, a fact used in the proof of Theorem 7. Theorem 6 first appeared in Sproule [10]. Later, in a more general setting, Miller and Sen [7] demonstrates that Theorem 6 follows as a corollary of their Theorem 1.

LEMMA 4. *Suppose that the sequence of I. I. D. random variables* $X_1, X_2,$ $\cdots$ *are defined on a probability space* $[\ ,\mathrm{A},P]$ *and that* Q *is an arbitrary probability measure on* $[\ ,\mathrm{A}]$ *absolutely continuous with respect to* P. *Then* (4.2) *holds with* Q, n *and* $n \longrightarrow \infty$ *in place of* P, N_s *and* $s \longrightarrow \infty$, *respectively.*

LEMMA 4. *Let* $S_n = \binom{n}{r}U_n, c_n = \binom{n}{r}\theta$ and $d_n = \sigma n^{-1/2}\binom{n}{r}$. By the asymptotic normality of U_n, for any real number x we can find a positive integer n_0 such that $P\{(S_k - c_k)/d_k \leqslant x\} > 0$ for any $k > n_0$. By Theorem 1 and 2 of Renyi [8], the theorem follows if we verify that

$$\lim_{n\to\infty} P\{(S_n - c_n)/d_n \leqslant x \mid (S_k - c_k)/d_k \leqslant x\} = \Phi(x) \tag{4.3}$$

for any $k > n_0$. To this end write $S_n = S_{k,n} + S_{k,n}^*$ where $S_{k,n} = \Sigma f(x_{\alpha_1}, \cdots, x_{\alpha_r})$ with the summation over all combinations $(\alpha_1, \cdots, \alpha_r)$ formed from the integers $\{k+1, k+2, \cdots, n\}$ and $S_{k,n}^* = S_n - S_{k,n}$. Now $E\{S_{k,n}^*/d_n\} = O(n^{-1/2})$. Also, using the H-decomposition, Lemma 1(ii) and Lemma 2, a little computation yields $\mathrm{Var}\{S_{k,n}^*/d_n\} = O(n^{-1})$. Thus $S_{k,n}^*/d_n$ converges in probability to 0 as $n \longrightarrow \infty$. Next, $\{(S_n - c_n)/d_n - S_{k,n}^*/d_n \leqslant x\}$ and $\{(S_k - c_k)/d_k \leqslant x\}$ are independent, and so, for any $k > n$,

$$\lim_{n\to\infty} P\{(S_n - c_n)/d_n - S_{k,n}^*/d_n \leqslant x \mid (S_k - c_k)/d_k \leqslant x\}$$
$$= \lim_{n\to\infty} P\{(S_n - c_n)/d_n - S_{k,n}^*/d_n \leqslant x\} = \Phi(x).$$

Thus (4.3), and therefore the lemma holds.

Denote the integral part of the real number x by $[x]$. Following Renyi [9] we prove

LEMMA 5. *Let* λ *be a positive random variable having a discrete distribution. If* $N_s = [n_s\lambda]$ *for* $s = 1, 2, \cdots$ *then* (4.2) *holds.*

PROOF. Assume that λ takes on values $l_1, l_2, \cdots$ with positive probability and that $0 \leqslant l_1 < l_2 < \cdots$. (A slight adjustment is made if λ takes on a finite number of values.) Define the events $A_k = \{\lambda = l_k\}$ for $k = 1, 2, \cdots$. Then, for any $k = 1, 2, \cdots, P\{A_k\} > 0$, and so, using Lemma 4 with $Q\{\cdot\} = P\{\cdot \mid A_k\}$, we obtain

$$\lim_{s\to\infty} P\{U_{[n_s l_k]} - \theta \leqslant x\sigma n_s^{-1/2} \mid A_k\} = \Phi(x)$$

and (4.2) follows from the theorem on total probabilities.

THEOREM 7. *Assume that*

$$\underset{s\to\infty}{p\text{-lim}}\ n_s^{-1}N_s = \lambda \tag{4.4}$$

where λ *is a positive random variable having a discrete distribution. Then* (4.2) *holds.*

PROOF. Write $Z_n = n^{1/2}(U_n - \theta)/\sigma$. Then

$$\begin{aligned} Z_{N_s} &= Z_{[n_s\lambda]} + N_s^{1/2}[n_s\lambda]^{-1/2}\{[n_s\lambda]^{1/2}(U_{N_s} - U_{[n_s\lambda]})/\sigma\} \\ &\quad + Z_{[n_s\lambda]}\{N_s^{1/2}[n_s\lambda]^{-1/2} - 1\}. \end{aligned} \tag{4.5}$$

By Lemma 5, $Z_{[n_s\lambda]}$ has an asymptotic normal distribution as $s \to \infty$. Also, by (4.4), $p\text{-lim}_{s\to\infty} N_s^{1/2}[n_s\lambda]^{-1/2} = 1$. Thus, in order to prove (4.2) we need only verify that

$$\underset{s\to\infty}{p\text{-lim}}\ [n_s\lambda]^{1/2}(U_{N_s} - U_{[n_s\lambda]}) = 0. \tag{4.6}$$

Make the same assumptions on λ that are made in the proof of Lemma 5. Let $m_{sk} = [n_s l_k]$. Define the events

$$E_s = \{[n_s\lambda]^{1/2}|U_{N_s} - U_{[n_s\lambda]}| > \epsilon\}, \qquad C_{sk} = \{m_{sk}^{1/2}|U_{N_s} - U_{m_{sk}}| > \epsilon\}$$

and for $\rho > 0$, $B_s(\rho) = \{|N_s - [n_s\lambda]| < \rho n_s\}$. Then $E_s A_k \leqslant C_{sk}$, so that

$$P\{E_s\} \leqslant \sum_{k=1}^{\infty} P\{C_{sk}B_s(\rho)A_k\} = P\overline{\{B_s(\rho)\}}. \tag{4.7}$$

Now, there exists an $S_{\epsilon,\eta}$ such that $n_s > l_1^{-1}(\nu_{\epsilon,\eta} + 1)$ for any $s > S_{\epsilon,\eta}$. Then $m_{sk} > \nu_{\epsilon,\eta}$ for any $s > S_{\epsilon,\eta}$ and any $k = 1, 2, \cdots$. Recall that U_n satisfies Anscombe's condition C2 (Theorem 6). Thus, for any $s > S_{\epsilon,\eta}$ and any $k = 1, 2, \cdots$,

$$P\left\{\max_{|l-m_{sk}|<cm_{sk}} |U_l - U_{m_{sk}}| > \epsilon m_{sk}^{-1/2}\right\} \leqslant \eta. \tag{4.8}$$

Next, since $l_1 > 0$, we can find a $K > 0$ such that $0 < 1/K < l_1$. Put $\rho = c(l_1 - 1/K)$. Then $\rho > 0$ and, whenever $n_s > K$, we have $\rho n_s \leqslant cm_{sk}$ for any $k = 1, 2, \cdots$. Suppose $s > S_K$ ensures that $n_s > K$. Then, by (4.8), for any $s > \max(S_{\epsilon,\eta}, S_K)$ and any $k = 1, 2, \cdots$,

$$P\left\{\max_{|l-m_{sk}|<\rho n_s} |U_l - U_{m_{sk}}| > \epsilon m_{sk}^{-1/2}\right\} \leqslant \eta. \tag{4.9}$$

Therefore, by (4.9), for s large enough and any $k = 1, 2, \cdots$, we have $P\{C_{sk}B_s(\rho)A_k\} \leqslant \eta$. Then, from (4.7), for s large enough, $P\{E_s\} \leqslant P\{\lambda \geqslant l_M\} + \eta M + P\{\overline{B_s(\rho)}\}$ for any positive integer M. Now, suppose $\delta > 0$. Choose M large enough so that $P\{\lambda \geqslant l_M\} < \delta/3$. Next, let $\eta = \delta/3M$. Choose $S_{\epsilon,\delta}$ such that $P\{\overline{B_s(\rho)}\} < \delta/3$ for any $s > S_{\epsilon,\delta}$. Therefore finally, for any $s > \max(S_{\epsilon,\eta}, S_K, S_{\epsilon,\delta})$ we have $P\{E_s\} < \delta$. This proves (4.6) and the theorem follows.

5. **Examples.** In Examples (1) and (2) we illustrate the H-decomposition (1.1) as well as Theorem 3. Assume that $X_1, X_2, \cdots$ are I. I. D. random variables having a continuous c.d.f. F.

(1) Let $f(x_1, x_2) = 1$ if $x_1 + x_2 > 0$ and 0 if $x_1 + x_2 < 0$. Then

$$\theta = P\{X_1 + X_2 > 0\} \quad \text{and} \quad f_1(x_1) = 1 - F(-x_1).$$

The corresponding U-statistic $U_n = \binom{n}{2}^{-1}\Sigma_{i<j}f(x_i, x_j)$ is closely related to Wilcoxon's signed-rank sum [11]. Assume further that the distribution F is symmetric. Then $\theta = ½, g^{(1)}(x_1) = F(x_1) - ½$, $V_n^{(1)} = n^{-1}\Sigma_{i=1}^n(F(x_i) - ½)$ and $U_n = ½ + 2V_n^{(1)} + R_n$ where R_n is the zero-mean remainder term. By Theorem 3, $n^\gamma R_n$ converges to 0 (a.s) as $n \longrightarrow \infty$ for $\gamma < 1$. Thus, the U-statistic U_n behaves very much like $½ + 2n^{-1}\Sigma_{i=1}^n(F(x_i) - ½)$ whose distribution does not depend on the form of F and indeed, is related to the distribution of the average of a sample drawn from the uniform distribution. See page 258 of Kendall and Stuart [6].

(2) Let $f(x_1, x_2) = |x_1 - x_2|$. Then $\theta = \int\int |x_1 - x_2| dF(x_1)dF(x_2)$ and the corresponding U-statistic is Gini's mean difference [3], $U_n = \binom{n}{2}^{-1}\Sigma_{i<j}|x_i - x_j|$. Let $\mu = E\{X_1\}$. Then $f_1(x_1) = 2\int_{-\infty}^{x_1}F(y)dy + \mu - x_1$. Define $z_i = \int_{-\infty}^{x_i}F(y)dy$ for $i = 1, 2, \cdots, n$ so that $V_n^{(1)} = 2\bar{z}_n - 2\bar{x}_n + \mu - \theta$ where $\bar{z}_n$ and $\bar{x}_n$ denote the averages of the z's and the x's respectively.

It may be noted that σ may be replaced in Theorems 6 and 7 by any consistent estimate of it.

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