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EXTENDING CONTINUOUS LINEAR FUNCTIONALS IN CONVERGENCE VECTOR SPACES

BY

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ABSTRACT. Let (E, τ) be a convergence vector space, M a subspace of E, and φ a linear functional on M continuous in the induced convergence structure. Sufficient and sometimes necessary conditions are given that (1) φ has a continuous linear extension to the τ-adherence $\overline{M}$ of M; (2) φ has a continuous linear extension to E; (3) $\overline{M}$ is τ-closed; (4) every τ-closed convex subset of E is $\sigma(E, E')$-closed. Several examples are included illustrating the extent and limitations of the theory presented.

Introduction. Through introduction of an appropriate notion of local convexity, necessary and sufficient conditions are given in order that a subspace M of a convergence vector space (c.v.s.) (E, τ) (H. R. Fischer, *Limesräume*, Math. Ann. 137 (1959), 269–303) have the Hahn-Banach Property (H.B.P.), namely: Every continuous linear functional φ on M has a continuous linear extension to E. This yields an extension of the Hahn-Banach Theorem to a class of c.v.s. satisfying a local convexity condition. Conditions are given insuring that the τ-closed and weakly closed subsets of E coincide and, in a c.v.s. where this is the case, that a subspace will have the H.B.P. Prerequisite to this last result is the determination of when every continuous linear functional, φ, on M has a continuous linear extension to $\overline{M}$, the τ-adherence of M. The notion of a nearly closed subspace M of (E, τ) is introduced, and it is shown that for nearly closed subspaces, one can always extend φ on M continuously to $\overline{M}$ and that $\overline{M}$ is τ-closed. Subsequently, it is demonstrated that in a strict convergence inductive limit of Fréchet spaces, M is nearly closed if and only if every such φ on M extends continuously to $\overline{M}$ if and only if $\overline{M}$ is τ-closed.

The final section consists of examples illustrating the extent and limitations of the theory presented. In particular, we (1) provide an example of a locally convex convergence space with a closed subspace which does not have the H.B.P.; (2) provide a characterization of those subspaces M of a strict inductive limit of metrizable spaces in which every continuous linear functional on M has a continuous linear extension to $\overline{M}$; and thus (3) characterize those subspaces of a

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strict inductive limit of reflexive Banach spaces which enjoy the H.B.P. by appealing to an extension result for spaces with boundedness [7, p. 66].

We have included in §1, a brief exposition of the parts of the theory of convergence structures given in [5] by Fischer that we use in this paper. This hopefully will be a convenience to the reader and at the same time allows the easy introduction of notation as well as the opportunity to make some simple observations not explicit in Fischer's paper but useful in our work.

1. **Preliminaries.** Let $\mathbf{F}(E)$ denote the set of all filters on a nonempty set E. If $\{\mathcal{F}_\nu: \nu \in I\}$ is an indexed family of filters in $\mathbf{F}(E)$, we denote by $\bigwedge_{\nu\in I}\mathcal{F}_\nu$ the filter $\{H \subset E: H \in \mathcal{F}_\nu \forall \nu \in I\}$. A mapping τ from E into the power set of $\mathbf{F}(E)$ is called a *convergence structure* for E if

(c.s. 1) $\forall x \in E$, $\mathcal{F} \in \tau(x)$ and $\mathcal{G} \in \tau(x) \Rightarrow \mathcal{F} \wedge \mathcal{G} \in \tau(x)$.

(c.s. 2) $\forall x \in E$, $\mathcal{F} \in \tau(x)$ and $\mathcal{G} \in \mathbf{F}(E)$ with $\mathcal{G} \supseteq \mathcal{F} \Rightarrow \mathcal{G} \in \tau(x)$.

(c.s. 3) $\forall x \in E$, $\dot{x}$ (the ultrafilter of all supersets of x) is in $\tau(x)$.

When τ is a convergence structure for E, we call the pair (E, τ) a *convergence space.* In a convergence space (E, τ) the filters in $\tau(x)$ are said to be *convergent to* x. A partial order is defined on $\mathbf{F}(E)$ by $\mathcal{F} \geqslant \mathcal{G} \Leftrightarrow \mathcal{F} \supseteq \mathcal{G}$. If τ_1 and τ_2 are two convergence structures for a set E, then we write $\tau_1 \geqslant \tau_2$ provided $\mathcal{F} \in \tau_1(x) \Rightarrow \mathcal{F} \in \tau_2(x)$ $\forall x \in E$, and, in this case, we say τ_1 is *finer* than τ_2 or τ_2 is *coarser* than τ_1. A convergence space (E, τ) is said to be Hausdorff if $\tau(x) \cap \tau(y) \neq \emptyset$ implies $x = y$.

Let $\mathcal{T}$ denote the class of all convergence structures on E. If $\tau \in \mathcal{T}$, we henceforth denote by $\mathcal{A}(x)$ the filter $\bigwedge_{\mathcal{F}\in\tau(x)}\mathcal{F}$. The class of all $\tau \in \mathcal{T}$, satisfying

(c.s. 4) $\forall x \in E$, $\mathcal{A}(x) \in \tau(x)$

will be denoted by $\mathcal{T}_1$. We call $\mathcal{A}(x)$ the *generating filter* of $\tau(x)$ when $\tau \in \mathcal{T}_1$. Observe in this case, $\mathcal{F} \in \tau(x)$ if and only if $\mathcal{F} \geqslant \mathcal{A}(x)$. By $\mathcal{T}_0$, we denote the class of elements of $\mathcal{T}$ that satisfy (c.s. 1) through (c.s. 4), and in addition

(c.s. 5) For each $x \in E$, $V \in \mathcal{A}(x)$ implies $\exists W \in \mathcal{A}(x)$ such that $y \in W \Rightarrow V \in \mathcal{A}(y)$.

Fischer [5] points out that $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}$ and that $\mathcal{T}_0$ is exactly the class of topologies for E.

In a natural way, to each $\tau \in \mathcal{T}$ is associated an element $\psi\tau$ of $\mathcal{T}_1$ and an element $\bar{\omega}\tau$ of $\mathcal{T}_0$. $\psi\tau$ is defined by $\psi\tau(x) = \{\mathcal{F}\in \mathbf{F}(E): \mathcal{F} \geqslant \mathcal{A}(x)\}$ for each $x \in E$. $\bar{\omega}\tau$ is the class of all τ-open subsets of E, where $A \subset E$ is τ-open if $\forall x \in A$, $\mathcal{F} \in \tau(x) \Rightarrow A \in \mathcal{F}$. The τ-closed sets in E are those whose complements are open in the topology $\bar{\omega}\tau$. It is easy to see that $\bar{\omega}\tau = \bar{\omega}\psi\tau$, and hence the terms $\bar{\omega}\tau$-closed, $\psi\tau$-closed and τ-closed are synonymous. The operators ψ and $\bar{\omega}$ on $\mathcal{T}$ preserve order. That is, if $\tau, \sigma \in \mathcal{T}$ and $\tau \leqslant \sigma$, then $\psi\tau \leqslant \psi\sigma$

and $\bar{\omega}\tau \leqslant \bar{\omega}\sigma$. For a given $\tau \in \mathcal{T}$, $\bar{\omega}\tau$ is the finest topology weaker than τ and $\psi\tau$ is the finest $\mathcal{T}_1$ convergence structure weaker than τ. Consequently, if $\tau \in \mathcal{T}_0$, $\bar{\omega}\tau = \tau$ and if $\tau \in \mathcal{T}_1$, $\psi\tau = \tau$.

Let (E, τ), (F, σ) be convergence spaces, $\varphi: E \longrightarrow F$. We say φ is *continuous at* $x \in E$ if $\forall \mathcal{F} \in \tau(x)$, $\varphi(\mathcal{F}) \in \sigma(\varphi(x))$, where $\varphi(\mathcal{F})$ is the filter generated by the filter-base $\{\varphi(F): F \in \mathcal{F}\}$. If φ is continuous at each $x \in E$, we say φ is *continuous.*

If (E, τ) is a convergence space and $\emptyset \neq A \subset E$, we can define the *induced convergence structure* τ_A on A by $\tau_A(x) = \{\mathcal{F} \in \mathbf{F}(A): F_E(\mathcal{F}) \in \tau(x)\}$ where $F_E(\mathcal{F})$ is the filter in E generated by the filter base (in E) $\mathcal{F}$. τ_A is the weakest limit structure on A under which the natural injection $i_A: A \longrightarrow E$ is continuous. If $x \in A$, $\mathcal{F} \in \tau(x)$, and $F \cap A \neq \emptyset$ $\forall F \in \mathcal{F}$, then the filter $\mathcal{F}_A = \{F \cap A: F \in \mathcal{F}\}$ is defined and belongs to $\tau_A(x)$, and every $\mathcal{F} \in \tau_A(x) = \mathcal{G}_A$ for some $\mathcal{G} \in \tau(x)$. The use of the notation $\mathcal{F}_A$ will be understood implicitly to imply that the filter $\mathcal{F}_A$ is defined.

If (E_ν, τ_ν), $\nu \in I$, is a family of convergence spaces the *product convergence structure* $\Pi\tau_\nu$ is defined to be the coarsest convergence structure for ΠE_ν under which the natural projections are continuous.

In this paper, we focus attention on convergence structures defined on a vector space E over the reals, R. Let $V = \{(-\epsilon, \epsilon): \epsilon > 0\}$, $\mathcal{F}, \mathcal{G} \in \mathbf{F}(E)$, and $\lambda \in R$. By $\mathcal{F} + \mathcal{G}, \lambda\mathcal{F}, V \cdot \mathcal{F}$ are meant the filters generated respectively by $\{F + G: F \in \mathcal{F}, G \subset \mathcal{G}\}$, $\{\lambda F: F \in \mathcal{F}\}$, and $\{G = \bigcup_{|\lambda| < \epsilon} \lambda F: \epsilon > 0, F \in \mathcal{F}\}$. If τ is a convergence structure for E, we denote by $\tau(0) + \tau(0)$, $\lambda \cdot \tau(0)$ and $V \cdot \tau(0)$ the collections of filters, respectively, $\{\mathcal{F} + \mathcal{G}: \mathcal{F}, \mathcal{G} \in \tau(0)\}$, $\{\lambda \cdot \mathcal{F}: \mathcal{F} \in \tau(0)\}$, $\{V \cdot \mathcal{F}: \mathcal{F} \in \tau(0)\}$. A convergence structure τ for E will be said to be *translation invariant* if $\forall x \in E$, $\tau(x) = x + \tau(0) = \{\mathcal{G} \in \mathbf{F}(E): \mathcal{G} = x + \mathcal{F}, \mathcal{F} \in \tau(0)\}$ where $x + \mathcal{F} = \{x + F: F \in \mathcal{F}\}$. If τ is a convergence structure for E such that addition and scalar multiplication are continuous, we say τ is *compatible* with the algebraic structure of E, or, simply, τ is compatible. In this case (E, τ) is called a *convergence vector space* (c.v.s.). Fischer [5] shows that τ is compatible if and only if

(c.v.s. 1) $\tau(0) + \tau(0) \subset \tau(0)$.

(c.v.s. 2) $\lambda \cdot \tau(0) \subset \tau(0)\ \forall \lambda \in R$.

(c.v.s. 3) $V \cdot \tau(0) \subset \tau(0)$.

(c.v.s. 4) $\forall x \in E$, $V \cdot x\ (= V \cdot \dot{x}) \in \tau(0)$.

Note that if (E, τ) is a c.v.s., τ is translation invariant. Fischer and Cook [3] observe that even if (E, τ) is a c.v.s., $(E, \psi\tau)$ may not be. Indeed $(E, \bar{\omega}\tau)$ may not be a c.v.s. either. It is comforting and useful to note, however, that every compatible convergence structure τ for a vector space E such that $\tau \in \mathcal{T}_1$ is, in fact, a topology.

If (E, τ) is a c.v.s., we associate with τ the locally convex, compatible topology $\psi^\circ\tau$ determined on E by the family of continuous seminorms on (E, τ). It is the finest locally convex vector space topology for E coarser than τ. Moreover, if $\sigma \leqslant \tau$ are two compatible convergence structures for E, $\psi^\circ\sigma \leqslant \psi^\circ\tau$. If E is a v.s. and τ a convergence structure for E, we denote by $(E, \tau)'$ the vector space of τ-continuous linear functionals for E. Fischer showed that $(E, \psi^\circ\tau)' = (E, \tau)'$. It follows from the easily verifiable inequalities $\psi^\circ\tau \leqslant \bar{\omega}\tau \leqslant \psi\tau \leqslant \tau$ that $(E, \psi^\circ\tau)' = (E, \bar{\omega}\tau)' = (E, \psi\tau)' = (E, \tau)'$. We will denote by $\sigma(E, E')$ the weak topology induced on E by $(E, \psi^\circ\tau)'$.

Finally, we observe that in a convergence space (E, τ) every $\mathcal{F} \in \tau(x)$ is finer than a filter $\mathcal{G} \in \tau(x)$ having the property that $x \in G$ for all $G \in \mathcal{G}$. Indeed, we may take $\mathcal{G} = \mathcal{F} \wedge \dot{x} \in \tau(x)$. Consequently, in many cases one may assume without loss of generality that when $\mathcal{F} \in \tau(x)$, $\mathcal{F}$ has the property indicated for $\mathcal{G}$ above. We shall take advantage of this from time to time.

2. Local convexity.

The notion of local convexity is a familiar and useful tool in the study of vector space topologies and bornologies [7]. As was observed in the introduction, the concept of a general locally convex convergence vector space has neither been defined nor studied. In this section we give some basic definitions and results surrounding the notion of a locally convex convergence structure for a vector space. Throughout the remainder of the paper all vector spaces are over the real number field, R.

DEFINITION 2.1. Let E be a vector space, τ a convergence structure for E. τ is *locally convex* if $\forall x \in E$, $\mathcal{F} \in \tau(x)$ implies $\exists \mathcal{G} \in \tau(x)$ with $\mathcal{F} \geqslant \mathcal{G}$ such that $\mathcal{G}$ has a filter base of convex sets. In case τ is locally convex, and compatible, (E, τ) will be called a *locally convex convergence vector space* (1.c.c.v.s.).

DEFINITION 2.2. Let E be a vector space and $\mathcal{F} \in \mathbf{F}(E)$. If $A \subset E$, denote the *convex hull of* A by ΓA. We define $\Gamma\mathcal{F}$ to be the filter generated by the filter base $\{\Gamma A : A \in \mathcal{F}\}$.

DEFINITION 2.3. Let E be a vector space, and τ a convergence structure for E. For each $x \in E$, we define $\Gamma\tau(x) = \{\mathcal{F} \in \mathbf{F}(E) : \exists \mathcal{G} \in \tau(x)$ such that $\mathcal{F} \geqslant \Gamma\mathcal{G}\}$.

PROPOSITION 2.1. *If E is a vector space and τ a convergence structure for E, then $x \longmapsto \Gamma\tau(x)$ defines a locally convex convergence structure $\Gamma\tau$ for E such that $\Gamma\tau \leqslant \tau$. Moreover, $\Gamma\tau$ is the finest locally convex convergence structure for E coarser than τ.*

PROOF. To see that $\Gamma\tau$ is a convergence structure is easy. (c.s. 2) and (c.s. 3) are obvious, while (c.s. 1) follows readily from the fact that, if $\mathcal{F}_1$ and

$\mathcal{F}_2$ are in $\mathbf{F}(E)$, then $\Gamma\mathcal{F}_1 \wedge \Gamma\mathcal{F}_2 \geqslant \Gamma(\mathcal{F}_1 \wedge \mathcal{F}_2)$. That $\Gamma\tau$ is locally convex is immediate from the Definitions (2.1) through (2.3). Suppose $\sigma \leqslant \tau$, where σ is a locally convex convergence structure for E. If $\mathcal{F} \in \Gamma\tau(x)$, then $\mathcal{F} \geqslant \Gamma\mathcal{G}$ for some $\mathcal{G} \in \tau(x)$. $\sigma \leqslant \tau$ implies $\mathcal{G} \in \sigma(x)$ and therefore $\Gamma\mathcal{G} \in \sigma(x)$. Hence, by (c.s. 2) $\mathcal{F} \in \sigma(x)$ and thus $\sigma \leqslant \Gamma\tau$. □

PROPOSITION 2.2. *If E is a vector space, Γ preserves order in $\mathbf{F}(E)$ and $\mathcal{T}$. Specifically, if $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(E)$ and $\mathcal{F}_1 \geqslant \mathcal{F}_2$, then $\Gamma\mathcal{F}_1 \geqslant \Gamma\mathcal{F}_2$; if $\tau, \sigma \in \mathcal{T}$ and $\tau \geqslant \sigma$ then $\Gamma\tau \geqslant \Gamma\sigma$.*

PROOF. Let $A \in \Gamma\mathcal{F}_2$, then $A \supset \Gamma B$ for some $B \in \mathcal{F}_2$. But $\mathcal{F}_1 \geqslant \mathcal{F}_2$ implies $B \in \mathcal{F}_1$, and hence $\Gamma B \in \Gamma\mathcal{F}_1$. Thus $A \in \Gamma\mathcal{F}_1$ and $\Gamma\mathcal{F}_1 \geqslant \Gamma\mathcal{F}_2$. Now since $\tau \geqslant \sigma \geqslant \Gamma\sigma$, it follows from Proposition 2.1 that $\Gamma\tau \geqslant \Gamma\sigma$. □

Before stating our next result, we observe that it follows from Proposition 2.1 that τ is a locally convex convergence structure for a vector space E if and only if $\Gamma\tau = \tau$.

PROPOSITION 2.3. *If (E, τ) is a c.v.s., then $(E, \Gamma\tau)$ is a c.v.s.*

PROOF. In view of what has preceded, we need only show that compatibility of τ implies compatibility of $\Gamma\tau$. For any $\mathcal{F}, \mathcal{G} \in \mathbf{F}(E), \lambda \in R$, $x \in E$, we observe:

(1) $\Gamma\mathcal{F} + \Gamma\mathcal{G} = \Gamma(\mathcal{F} + \mathcal{G})$ since, if $F \in \mathcal{F}, G \in \mathcal{G}, \Gamma F + \Gamma G = \Gamma(F + G)$.

(2) $\lambda \cdot \Gamma\mathcal{F} = \Gamma(\lambda \cdot \mathcal{F})$ since if $F \in \mathcal{F}, \lambda \cdot \Gamma F = \Gamma(\lambda \cdot F)$.

(3) $V \cdot \Gamma\mathcal{F} \geqslant \Gamma(V \cdot \mathcal{F})$ since if $F \in \mathcal{F}, \epsilon > 0$, then $\Gamma((-\epsilon, \epsilon)F) \supset (-\epsilon, \epsilon)\Gamma F$.

(4) $\Gamma(V \cdot x) = V \cdot x$ since $(-\epsilon, \epsilon) \cdot x$ is convex $\forall \epsilon > 0$.

The compatibility conditions (c.v.s. 1) through (c.v.s. 4) follow respectively from (1) through (4) above without difficulty. □

PROPOSITION 2.4. *If τ is a convergence structure for a vector space E, then $(E, \tau)' = (E, \Gamma\tau)'$.*

PROOF. By Proposition 2.1 it is clear that $\psi^\circ\tau \leqslant \Gamma\tau \leqslant \tau$, and since $(E, \psi^\circ\tau)' = (E, \tau)'$ the result is immediate. □

DEFINITION 2.4. Let E be a vector space, $\mathcal{F} \in \mathbf{F}(E)$. We say $\mathcal{F}$ is *stable* if $F \in \mathcal{F}, \lambda > 0$ implies $\lambda F \in \mathcal{F}$. A translation invariant convergence structure τ for E is said to be a *stable convergence structure* if $\forall \mathcal{F} \in \tau(0)$, $\exists \mathcal{G} \in \tau(0)$ such that $\mathcal{G}$ is stable and $\mathcal{F} \geqslant \mathcal{G}$. If (E, τ) is a c.v.s. such that τ is stable, we will say (E, τ) is a *stable c.v.s.*

THEOREM 2.1. *If (E, τ) is a stable c.v.s., then $\Gamma(\psi\tau) = \Gamma(\bar{\omega}\tau) = \psi^\circ\tau$.*

PROOF. Since τ is translation invariant, it follows from

$$x + \psi\tau(0) = x + \bigwedge_{\mathcal{F}\in\tau(0)} \mathcal{F} = \bigwedge_{\mathcal{F}\in\tau(0)} (x + \mathcal{F}) = \bigwedge_{\mathcal{G}\in\tau(x)} \mathcal{G} = \psi\tau(x)$$

that $\psi\tau$ also is translation invariant. Let $\mathcal{A}(0)$ be the generating filter for $\psi\tau(0)$. We shall show that (1) $\mathcal{A}(0)$ is stable, (2) $\lambda \cdot \mathcal{A}(0) \geqslant \mathcal{A}(0)\ \forall\lambda \in \mathbf{R}$, (3) $V \cdot \mathcal{A}(0) \geqslant \mathcal{A}(0)$ and (4) $V \cdot x \geqslant \mathcal{A}(0)\ \forall x \in E$. It then follows that $\psi\tau$ is stable and satisfies (c.v.s. 2), (c.v.s. 3) and (c.v.s. 4). Let $A \in \mathcal{A}(0)$. Since for each $\mathcal{F} \in \tau(0)$, $\alpha \cdot \mathcal{F} \in \tau(0)$ for all $\alpha \in \mathbf{R}$ and, in particular, $\alpha > 0$, it follows that $A \in 1/\lambda \cdot \mathcal{F}$ for all $\lambda > 0$. Hence $\lambda A \in \mathcal{F}$ for all $\lambda > 0$. Since $\mathcal{F} \in \tau(0)$ was arbitrary, $\mathcal{A}(0)$ is stable. Now given $\mathcal{F} \in \tau(0), \lambda \cdot \mathcal{F} \in \tau(0)\ \forall\lambda \in \mathbf{R}$. Hence, $A \in \lambda \cdot \mathcal{F}\ \forall\lambda \in \mathbf{R}$, and therefore for each $\lambda \in \mathbf{R}\ \exists F_{\mathcal{F}} \in \mathcal{F}$ such that $\lambda F_{\mathcal{F}} \subset A$. Let $B_\lambda = \bigcup_{\mathcal{F}\in\tau(0)} \lambda F_{\mathcal{F}}$. Then $A \supset B_\lambda = \lambda \bigcup_{\mathcal{F}\in\tau(0)} F_{\mathcal{F}}$. Letting $A' = \bigcup_{\mathcal{F}\in\tau(0)} F_{\mathcal{F}}$, we see $A' \in \mathcal{A}(0)$ and $A \supset \lambda A'$. Hence, $\lambda \cdot \mathcal{A}(0) \geqslant \mathcal{A}(0)$ for each $\lambda \in \mathbf{R}$. It is easy to see that if $\mathcal{F} \in \tau(0)$ is stable, then $\mathcal{F} \geqslant V \cdot \mathcal{F}$. Let $A \in \mathcal{A}(0)$. For each stable $\mathcal{F} \in \tau(0)$, then, $\exists F'_{\mathcal{F}} \in \mathcal{F}$ such that for some $\epsilon > 0$, $(-\epsilon, \epsilon)F'_{\mathcal{F}} \subset A$. Let $F_{\mathcal{F}} = (-\epsilon, \epsilon)F'_{\mathcal{F}}$. Let A' be the union over all stable $\mathcal{F}$ in $\tau(0)$ of the sets $F_{\mathcal{F}}$. A', then, is balanced and consequently $[-1, 1]A' = A' \in V \cdot \mathcal{A}(0)$. By construction, $A' \subset A$ and therefore $V \cdot \mathcal{A}(0) \geqslant \mathcal{A}(0)$. (4) above is trivial. We now observe that (c.v.s. 2), (c.v.s. 3) and (c.v.s. 4) all carry over to $\Gamma(\psi\tau)$ according to (2), (3), and (4) in Proposition 2.3. It only remains, then, to show (c.v.s. 1) for $\Gamma(\psi\tau)$ in order to deduce that $\Gamma(\psi\tau)$ is compatible. Since it is clear that $\Gamma(\psi\tau)$ is stable, if $A' \in \Gamma\mathcal{A}(0)$, $\exists$ a convex $A \in \mathcal{A}(0)$ with $A \subset A'$. But then $\frac{1}{2}A \in \mathcal{A}(0)$. Consequently $A = \frac{1}{2}A + \frac{1}{2}A \in \Gamma\mathcal{A}(0) + \Gamma\mathcal{A}(0)$ and (c.v.s. 1) follows. The foregoing proves that $\Gamma(\psi\tau)$ is a locally convex, compatible T_1 convergence structure for E. But compatible T_1 convergence structures are topologies and hence $\Gamma(\psi\tau) \leqslant \psi^\circ\tau$. But $\psi^\circ\tau = \Gamma(\psi^\circ\tau) \leqslant \Gamma(\psi\tau)$ since $\psi^\circ\tau \leqslant \psi\tau$. Hence $\Gamma(\psi\tau) = \psi^\circ\tau$. The remainder of the assertion in the theorem follows now by applying Γ through $\psi^\circ\tau \leqslant \overline{\omega}\tau \leqslant \psi\tau$. □

REMARK 2.1. Theorem 2.4 is of considerable interest since it gives a constructive description of $\psi^\circ\tau$ for a large class of convergence structures. The definition of $\psi^\circ\tau$ in terms of continuous seminorms on (E, τ) is rather unwieldy by comparison as, in general, it would be difficult to identify those seminorms. Moreover, we observe the simple corollary that if τ is a stable c.v.s. and $\psi\tau$ is locally convex, then $\psi\tau$ is $\psi^\circ\tau$ and therefore a locally convex topology for E.

3. Let (E, τ) be a convergence vector space. In this section, we are primarily concerned with the relationships of the various convergence structures τ_M, $\psi(\tau_M)$, $\overline{\omega}(\tau_M)$, $(\psi\tau)_M$ and $(\overline{\omega}\tau)_M$ for a subspace M of E and some consequences of these relationships. Importantly, we provide a sufficient condition

under which the continuous linear functionals on M are the same for every convergence structure listed. In §4 (Example 4), we show that in fact this condition is also necessary in a wide class of convergence vector spaces. Finally, we give conditions on (E, τ) under which extension of continuous linear functionals from every subspace of E is possible and under which the τ-closed convex sets coincide with the weakly closed convex sets.

DEFINITION 3.1. Let (E, τ) be a c.v.s., $A \subset E$. The τ-adherence $\bar{A}$ of A is defined by $\bar{A} = \{x \in E: \exists \mathcal{F} \in \tau(0) \ni \forall F \in \mathcal{F}, (x + F) \cap A \neq \emptyset\}$. We remark that $\bar{A}$ need not necessarily be τ-closed. It *is* the case, however, that A is τ-closed if and only if $\bar{A} = A$. This phenomenon is discussed somewhat in [7]. If (E, τ) is a c.v.s., M a subspace of E and $\mathcal{F} \in \tau(x)$, we will say that $\mathcal{F}$ *leaves a trace on* M to describe the situation that for all $F \in \mathcal{F}$, $F \cap M \neq \emptyset$. Let (E, τ) be a c.v.s. and $\{\mathcal{F}_v\}_{v \in I} \subset \tau(0)$. We say that $\{\mathcal{F}_v\}_{v \in I}$ is a *fundamental family for* τ if for each $\mathcal{F} \in \tau(0)$ there exists $\alpha \in I$ *such* that $\mathcal{F} \geqslant \mathcal{F}_\alpha$.

DEFINITION 3.2. A subspace M of a c.v.s. (E, τ) is said to be *nearly closed* if there is a fundamental family $\{\mathcal{F}_v\}_{v \in I}$ for τ such that for each $v \in I$ there exists $\mu \in I$ and $F \in \mathcal{F}_v$ such that if $x \in F \cap \bar{M}$ then $x + \mathcal{F}_\mu$ leaves a trace on M (without loss of generality it may be assumed $\mathcal{F}_\mu \leqslant \mathcal{F}_v$).

THEOREM 3.1. *Let M be a nearly closed subspace of a c.v.s. (E, τ) and φ a τ_M-continuous linear functional on M. Then there exists a $\tau_{\bar{M}}$-continuous linear functional ψ on $\bar{M}$ such that $\psi|_M = \varphi$.*

PROOF. Let $x \in \bar{M}$. Then $\exists \mathcal{F} \in \tau(x)$ such that $\mathcal{F}_M \in \tau_M(x)$ is defined. Moreover, $\mathcal{F}_M - \mathcal{F}_M \in \tau_M(0)$ and since φ is τ_M-continuous, the filter $\varphi(\mathcal{F}_M - \mathcal{F}_M)$ converges to 0 in R. But $\varphi(\mathcal{F}_M) - \varphi(\mathcal{F}_M) \geqslant \varphi(\mathcal{F}_M - \mathcal{F}_M)$, and hence $\varphi(\mathcal{F}_M)$ is a Cauchy filter in R. The limit of $\varphi(\mathcal{F}_M)$ will be called $\psi(x)$, and for brevity we write $\lim \varphi(\mathcal{F}_M) = \psi(x)$. It is a straightforward verification to show that ψ is well defined and linear. To show that ψ is $\tau_{\bar{M}}$-continuous we note (see §1) that it is no restriction to assume $0 \in F$ for all $F \in \mathcal{F} \in \tau(0)$. Since M is nearly closed, there exists a fundamental family $\{\mathcal{F}_v\}_{v \in I}$ satisfying Definition 3.2. Let $\mathcal{F} \in \tau(0)$ and $\epsilon > 0$. Choose $\alpha \in I$ such that $\mathcal{F} \geqslant \mathcal{F}_\alpha$. Let $F_1 \in \mathcal{F}_\alpha$ and $\beta \in I$ be such that $\mathcal{F}_\alpha \geqslant \mathcal{F}_\beta$ and $y \in F_1 \cap \bar{M}$ implies $y + \mathcal{F}_\beta$ leaves a trace on M. Choose $\mathcal{G} \in \tau(0)$ such that $\mathcal{G} \leqslant \mathcal{F}_\beta + \mathcal{F}_\beta$. Since φ is continuous, one can find $G \in \mathcal{G}$ such that $z \in G \cap M$ implies $|\varphi(z)| < \epsilon$. Choose $F_2 \in \mathcal{F}_\beta$ such that $F_2 + F_2 \subset G$. Then $F_2 \cap F_1 \in \mathcal{F}_\alpha$ since $\mathcal{F}_\alpha \geqslant \mathcal{F}_\beta$, and thus there exists $F \in \mathcal{F}$ such that $F \subset F_2 \cap F_1$. Let $x \in F \cap \bar{M}$. Then $x \in F_1 \cap \bar{M}$, and consequently $x + \mathcal{F}_\beta$ leaves a trace on M. If $y \in (x + F_2) \cap M$, then $y \in x + F_2 \subset F_2 + F_2 \subset G$, and thus $|\varphi(y)| < \epsilon$. Thus, $x + \mathcal{F}_\beta$ converges to x, leaves a trace on M, and there exists $F_2 \in \mathcal{F}_\beta$

such that $y \in (x + F_2) \cap M$ implies $|\varphi(y)| < \epsilon$. Hence $|\psi(x)| \leqslant \epsilon$. But x was arbitrary in $F \in \mathcal{F}$, and thus ψ is continuous. $\square$

THEOREM 3.2. *If M is a nearly closed subspace of a c.v.s. (E, τ) then $\overline{M}$ is τ-closed.*

PROOF. It suffices to show that $\overline{\overline{M}} = \overline{M}$. Let $\{\mathcal{F}_\upsilon\}_{\upsilon \in I}$ be a fundamental family for τ satisfying Definition 3.2. If $x \in \overline{\overline{M}}$ there exists $\mathcal{H} \in \tau(0)$ such that $x + \mathcal{H}$ leaves a trace on $\overline{M}$. Let $\mathcal{F} = \mathrm{V} \cdot [(\mathrm{V} \cdot x) \wedge \mathcal{H}] \in \tau(0)$. Then there exists $\mathcal{F}_\alpha \leqslant \mathcal{F} + \mathcal{F}, \beta \in I$ and $F_1 \in \mathcal{F}_\alpha$ such that $\mathcal{F}_\beta \leqslant \mathcal{F}_\alpha$ and $y \in F_1 \cap \overline{M}$ implies $y + \mathcal{F}_\beta$ leaves a trace on M. Choose a balanced $F_2 \in \mathcal{F}$ such that $F_2 + F_2 \subset F_1$. Since $\mathrm{V} \cdot x \geqslant \mathcal{F}$, fix $1 \geqslant \lambda > 0$ such that $\lambda x \in F_2$. Now $x + \mathcal{F}$ leaves a trace on $\overline{M}$, and thus for all balanced $F \in \mathcal{F}, F \cap F_2 \in \mathcal{F}$, and so $(x + [F \cap F_2]) \cap \overline{M} \neq \emptyset$. Therefore, for each balanced $F \in \mathcal{F}$, there exists $b'_F \in F \cap F_2$ such that $x + b'_F \in \overline{M}$, and since $\overline{M}$ is a subspace, $\lambda x + \lambda b'_F \in \overline{M}$. But, since $\lambda \leqslant 1, \lambda b'_F = b_F \in F \cap F_2$, and so $\lambda x + b_F \in \overline{M}$. Moreover $\lambda x + b_F \subset (F_2 + F_2) \cap \overline{M} \subset F_1 \cap \overline{M}$, and therefore since $\mathcal{F}$ has a base of balanced sets, for all $F \in \mathcal{F}$ there exists $b_F \in F \cap F_2$ such that $\lambda x + b_F + \mathcal{F}_\beta$ leaves a trace on M. Hence $\lambda x + \mathcal{F} + \mathcal{F}_\beta$ leaves a trace on M. Choose $\mathcal{G} \in \tau(0)$ such that $\mathcal{F} + \mathcal{F}_\beta \geqslant \mathcal{G}$. Then $\lambda x + \mathcal{G}$ leaves a trace on M and thus $\lambda x \in \overline{M}$. Since $\overline{M}$ is a subspace, $x \in \overline{M}$.

THEOREM 3.3. *Let (E, τ) be a convergence vector space, and M a nonempty subspace of E. Then (1) $\psi\tau_M = (\psi\tau)_M$ and (2) if in addition M is τ-closed, $\bar{\omega}\tau_M = (\bar{\omega}\tau)_M$.*

PROOF. Let $\tau(0) = \{\mathcal{F}_\upsilon : \upsilon \in I\}$. That (1) is valid follows from the fact that if $\mathcal{A}(0)$ is the generating filter for $\psi\tau$ then

$$\mathcal{A}(0) = \left\{\bigcup_{\upsilon \in I} F_\upsilon : F_\upsilon \in \mathcal{F}_\upsilon\right\}, \quad \text{and}$$

$$\mathcal{A}_M(0) = \left\{\bigcup_{\upsilon \in I} (F_\upsilon \cap M) : F_\upsilon \in \mathcal{F}_\upsilon\right\}.$$

To show (2) note that $\bar{\omega}\tau \leqslant \tau$ and thus $(\bar{\omega}\tau)_M \leqslant \tau_M$. Hence,

$$(\bar{\omega}\tau)_M = \bar{\omega}[(\bar{\omega}\tau)_M] \leqslant \bar{\omega}(\tau_M).$$

Now, let $W \subset M$ be open for $\bar{\omega}(\tau_M)$. We need to show that it is open for $(\bar{\omega}\tau)_M$. Let $V = W \cup M^c$, where M^c is the complement of M in G. If $x \in V$, then $x \in W$ or $x \in M^c$. In the first case, if $\mathcal{F} \in \tau(x)$ such that $x \in F$ for all $F \in \mathcal{F}$, then $\mathcal{F}_M$ is defined. But $W \in \mathcal{F}_M$ and, hence, $V \in \mathcal{F}$. If $x \in M^c$, then, since M^c is τ-open, each $\mathcal{F} \in \tau(x)$ contains M^c and therefore V. Thus, V is open for $\bar{\omega}\tau$. However $W = V \cap M$ and consequently is $(\bar{\omega}\tau)_M$-open. $\square$

COROLLARY 3.3.1. *Let* (E, τ) *be a c.v.s. and* M *a subspace of* E. *Then* $(M, \tau_M)' = (M, \psi(\tau_M))' = (M, (\psi\tau)_M)' = (M, \bar{\omega}\tau_M)'$. *If, in addition,* M *is nearly closed, then* $(M, \tau_M)' = (M, (\bar{\omega}\tau)_M)'$.

DEFINITION 3.3. Let E be a vector space with convergence structure τ. A subspace M of E is said to have the *Hahn-Banach Property* (H.B.P.) if every $\varphi \in (M, \tau_M)'$ has a τ-continuous linear extension to E. We say that (E, τ) has the H.B.P. if every subspace of E has the H.B.P.

REMARK 3.1. It is evident from Theorem 3.1 and Theorem 3.2 that a nearly closed subspace M of a c.v.s. (E, τ) has the H.B.P. if the τ-closure of M has the H.B.P.

THEOREM 3.4. *Let* (E, τ) *be a c.v.s. Then a subspace* M *of* E *has the H.B.P. if and only if*

$$(M, \Gamma[(\psi\tau)_M])' = (M, [\Gamma(\psi\tau)]_M)'.$$

PROOF. Assume that a subspace M of (E, τ) has the H.B.P. Then by Theorem 3.3 and Theorem 2.1, $\Gamma[(\psi\tau)_M] = \Gamma(\psi\tau_M) = \psi^\circ\tau_M$. Also $[\Gamma(\psi\tau)]_M = (\psi^\circ\tau)_M$. Set $(E, \tau)'|_M = \{\varphi|_M : \varphi \in (E, \tau)'\}$. Then the H.B.P. implies $(M, \tau_M)' = (E, \tau)'|_M$, and thus

$$\begin{aligned}(M, \Gamma[(\psi\tau)_M])' &= (M, \psi^\circ\tau_M)' = (M, \tau_M)' = (E, \tau)'|_M \\ &= (E, \psi^\circ\tau)'|_M = (M, (\psi^\circ\tau)_M)' = (M, [\Gamma(\psi\tau)]_M)'.\end{aligned}$$

Conversely, assume the equality holds for a subspace M of E. By Theorem 2.1, $\Gamma(\psi\tau) = \psi^\circ(\tau)$, and by Proposition 2.4, $(M, \Gamma[(\psi\tau)_M])' = (M, (\psi\tau)_M)'$. But Corollary 3.3.1 yields $(M, (\psi\tau)_M)' = (M, \psi(\tau_M))' = (M, \tau_M)'$. But the hypothesis of this proposition, then, $(M, \tau_M)' = (M, (\psi^\circ\tau)_M)'$. Hence if $\varphi \in (M, \tau_M)'$, φ has a $\psi^\circ\tau$-continuous linear extension Φ to E according to the Hahn-Banach Theorem for locally convex topological vector spaces. But $(E, \tau)' = (E, \psi^\circ\tau)'$, so Φ is τ-continuous as well. □

A trivial but interesting consequence of this theorem is

COROLLARY 3.4.1. *Let* (E, τ) *be a stable c.v.s. such that* $\psi\tau$ *is locally convex. Then* (E, τ) *has the H.B.P.*

DEFINITION 3.3. Let (E, τ) be a c.v.s. We will say (E, τ) has the *Geometrical Hahn-Banach Property* (G.H.B.P.) if the τ-closed, convex sets and the $\sigma(E, E')$-closed convex sets of E coincide.

PROPOSITION 3.1. *If* (E, τ) *is a c.v.s. having the G.H.B.P., then every nearly closed subspace* M *of* (E, τ) *has the H.B.P.*

PROOF. Let M be a nearly closed subspace of E. By Theorem 3.1 and Theorem 3.2 we may assume without loss of generality that M is τ-closed in E. Suppose $\varphi \in (M, \tau_M)'$. By Corollary 3.3.1, φ is in $(M, (\bar{\omega}\tau)_M)'$. $\varphi^{-1}(0)$ therefore is $(\bar{\omega}\tau)_M$-closed, hence τ_M-closed in M. But, since M is τ-closed, $\varphi^{-1}(0)$ is τ-closed and convex. Hence $\varphi^{-1}(0)$ is weakly closed. But this implies φ is continuous on M under the topology induced by $\sigma(E, E')$. Hence φ may be extended to a $\sigma(E, E')$-continuous functional $\hat{\varphi}$ on E. So $\hat{\varphi} \in (E, \tau)'$, and the proposition is proved. □

THEOREM 3.5. *If* (E, τ) *is a stable, Hausdorff c.v.s. and* $\psi\tau$ *is locally convex, then* (E, τ) *has the G.H.B.P., and* $(E, \tau)'$ *separates points of* E.

PROOF. Let $0 \neq x \in E$. Since (E, τ) is Hausdorff, for each $\mathcal{F} \in \tau(0)$, $\exists F_{\mathcal{F}} \in \mathcal{F}$ such that $x \notin F_{\mathcal{F}}$. Let $A = \bigcup_{\mathcal{F} \in \tau(0)} F_{\mathcal{F}}$. Then $A \in \mathcal{A}(0)$ and $x \notin A$. Hence there is an absolutely convex $A' \subset A$ such that $\frac{1}{2}A' \in \mathcal{A}(0)$, since $\psi\tau = \Gamma\psi\tau = \psi^\circ\tau$ is a locally convex topology. But $x \notin A' = \frac{1}{2}A' + \frac{1}{2}A'$. Hence $(x + \frac{1}{2}A') \cap \frac{1}{2}A' = \varnothing$ and therefore $\psi\tau$ is Hausdorff. Since $\psi\tau$ then is a Hausdorff locally convex topology, and $(E, \psi\tau)' = (E, \tau)'$, the theorem follows.

4. **Examples.** In this section, we will (1) exhibit a large class of convergence vector spaces (E, τ) which are not topological vector spaces (t.v.s.), but for which $\psi\tau$ is locally convex and which therefore have the H.B.P. and the G.H.B.P.; (2) show that if (E, τ) is a locally convex c.v.s., it does not necessarily follow that $\psi\tau$ is a locally convex convergence structure for E, and thus the convexity hypothesis on $\psi\tau$ in theorems of §3 could not in general be replaced by convexity of τ; (3) exhibit a locally convex, Hausdorff c.v.s. that does not have the H.B.P., and in doing so provide an example of a closed subspace of an *LF*-space that is not an *LF*-space (see [4]) and a counterexample to Theorem 4 of [11, p. 76]; (4) show that in a wide class of convergence vector spaces (E, τ) a subspace M being nearly closed is equivalent to the extendibility of every τ_M-continuous linear function on M to a $\tau_{\overline{M}}$-continuous linear function on $\overline{M}$; and (5) exhibit conditions on a subspace M of an inductive limit of topological vector spaces to insure that the hypothesis of Theorem 3.4 is satisfied. The consequence is that every nearly closed subspace M for which the τ-adherence satisfies these conditions has the H.B.P. More importantly, we characterize those subspaces of a strict inductive limit of reflexive Banach spaces which enjoy the H.B.P.

EXAMPLE (1). Consider a locally convex t.v.s. $(E, \mathcal{T})$, and define a convergence structure $\mathcal{T}_\beta$ by

$$T_\beta(0) = \{\mathcal{F} \in \mathbf{F}(E)\colon \exists \text{ an absolutely convex bounded set } B \text{ in } E \text{ with } \mathcal{F} \geqslant V \cdot B\}, \quad \text{and} \tag{4.1}$$
$$T_\beta(x) = x + T_\beta(0).$$

One can easily show that (E, T_β) is a stable locally convex convergence vector space. A set A is in the generating filter for ψT_β if and only if it absorbs every bounded subset of (E, T). We have, however, the following result whose proof is obtained by a standard argument (e.g. see [9, p. 222]).

PROPOSITION 4.1. *If (E, T) is a t.v.s. which has a countable fundamental system of neighborhoods for 0, then every bornivore (i.e. a set which absorbs every bounded set) is a neighborhood of 0.*

One may now easily verify

PROPOSITION 4.2. *If (E, T) is a metrizable locally convex topological vector space, then*

(i) *$\psi T_\beta = T$, hence is locally convex,*

(ii) *(E, T_β) has the G.H.B.P. and thus also the H.B.P.*

We will now show that the convergence structures, T_β, are in general not topologies.

PROPOSITION 4.3. *Let (E, T) be a t.v.s. where T is the Mackey topology for E. Then*

(i) *$(E, T)' = (E, T_\beta)'$ if and only if (E, T) is bornological, and*

(ii) *if (E, T) is bornological, then T_β is a topology if and only if (E, T) is normable.*

PROOF. (i) Suppose $(E, T)' = (E, T_\beta)'$. If φ is a linear functional on E and maps bounded sets to bounded sets, then clearly $\varphi \in (E, T_\beta)' = (E, T)'$ and (E, T) is bornological.

Conversely, if (E, T) is bornological, and $\varphi \in (E, T_\beta)'$, then for each bounded set B of E, $|\varphi(\lambda B)| \leqslant 1$ for some $\lambda > 0$. Thus, $\varphi(B)$ is bounded and $\varphi \in (E, T)'$. Hence, $(E, T_\beta)' \subset (E, T)'$. The reverse inclusion follows from $T \leqslant T_\beta$.

(ii) If (E, T) is normable, then $\mathcal{F} \in T(0)$ if and only if $\mathcal{F} \geqslant V \cdot B$, where B is the unit ball of E under some norm. Since B is bounded, $T = T_\beta$.

Conversely, suppose T_β is a topology. Since it is locally convex, $T_\beta = \psi^\circ T_\beta$. But since $(E, T_\beta)' = (E, T)$, we see that $T_\beta = \psi^\circ T_\beta \leqslant T$ because T is the Mackey topology. However, $T \leqslant T_\beta$ and hence $T = T_\beta$. Thus, T has a bounded neighborhood of 0. $\square$

EXAMPLE (2). Let E be a vector space and (E_n, τ_n) be a sequence of convergence vector spaces such that (a) $E_n \subset E_{n+1}$, $n = 1, 2, \cdots$, (b) τ_n is finer than the convergence structure induced on E_n by τ_{n+1}, and, (c) $E = \bigcup_{n=1}^{\infty} E_n$. A convergence structure τ on E may be defined by $\mathcal{F} \in \tau(x)$ if and only if $\exists n \geqslant 1$ such that $x \in E_n$ and $\exists \mathcal{F}_n \in \tau_n(x)$ such that $\mathcal{F}$ is finer than the filter in E generated by $\mathcal{F}_n$. This convergence structure τ is called the convergence inductive limit of the structures τ_n. Fischer [5] introduced this notion and showed that τ is the finest convergence structure inducing on each E_n a convergence structure weaker than τ_n and is compatible with the algebraic structure of E.

In this example, we let each (E_n, τ_n) be a normed linear space with closed unit ball B_n and norm $\|\cdot\|_n$. We assume $B_{n+1} \cap E_n = B_n$ for all $n \geqslant 1$. Then $\|x\|_m = \|x\|_n$ for $n \geqslant m$ and $x \in E_m$. The convergence inductive limit structure τ on $E = \bigcup_{n=1}^{\infty} E_n$ is defined by

$$\tau(0) = \{\mathcal{F} \in F(E)\colon \exists \text{ an integer } n \ni \mathcal{F} \geqslant V \cdot B_n\},$$

where $V \cdot B_n$ denotes the filter generated by the filter-base of sets $\{(-\epsilon, \epsilon)B_n\colon \epsilon > 0\}$. In other words a filter, $\mathcal{F}$, converges to $x \in E$ if and only if for some integer n, $x \in E_n$ and $\mathcal{F}$ is finer than the neighborhood filter of x in E_n. Thus $\psi\tau(0)$ consists of exactly those sets A in E which absorb every B_n, $n = 1, 2, \cdots$. The collection $\mathcal{B}$ of sets $\{\bigcup_{n=1}^{\infty} \lambda_n B_n\colon \lambda_n > 0\}$ is clearly a filter base for $\mathcal{A}(0)$. To see that $\psi\tau$ is *not* locally convex, it suffices to show that there exists no $B \in \mathcal{B}$ such that

$$\Gamma(B) \subset \bigcup_{n=1}^{\infty} \frac{1}{n} B_n = A.$$

Suppose on the contrary that $\exists B = \bigcup_{n=1}^{\infty} \lambda_n B_n$ such that $\Gamma(B) \subset A$. Then clearly $\lambda_n \leqslant 1/n$. Choose m such that $\lambda_1 > 3/m$. Also choose $x \in B_1$ such that $\|x\|_1 = 1$ and choose $y \in B_m \backslash E_{m-1}$ such that $\|y\|_m = 1/m$. Then we see that

$$z = \tfrac{1}{2}\lambda_1 x + \tfrac{1}{2}\lambda_m y \in \Gamma(B).$$

However, we also note that

(a) $z \notin E_{m-1}$ (for if so, $y = (2/\lambda_m)(z - \tfrac{1}{2}\lambda_1 x) \in E_{m-1}$ which contradicts the choice of y), and

(b) for $n \geqslant m$,

$$\begin{aligned}\|z\|_n &= \|\tfrac{1}{2}\lambda_1 x + \tfrac{1}{2}\lambda_m y\|_n \geqslant \tfrac{1}{2}|\lambda_1|\ \|x\|_n - \tfrac{1}{2}|\lambda_m|\ \|y\|_n \\ &= \tfrac{1}{2}|\lambda_1| - \tfrac{1}{2}|\lambda_m| \geqslant \tfrac{1}{2}(\lambda_1 - 1/m) > 1/m\end{aligned}$$

by the way m was chosen. But $1/m \geq 1/n$ and so $z \notin (1/n)B_n$ for $n \geq m$. Consequently $z \notin B$ (contradiction). Thus, $\psi\tau$ is not locally convex, though τ clearly is.

EXAMPLE (3). In Example (2) the notion of a convergence inductive limit of a sequence of c.v.s. was introduced. We say such an inductive limit is *strict* if $E_{n+1} \supset E_n$ and $(\tau_{n+1})_{E_n} = \tau_n$.

In this example, we will exhibit a convergence vector space (E, τ) which is the strict convergence inductive limit of a sequence of locally convex Fréchet spaces (F_n, T_n) but which does not have the H.B.P. For this example (E, τ) is locally convex, but $(E, \psi\tau)$ is not. In the process, we obtain a subspace M of E for which the strict inductive limit *topology* determined by the sequence $\{M \cap F_n\}$ has a larger dual than the space (M, T_M) where T_M is the topology induced by T, the inductive limit *topology* on E.

Let Ω be an open set in Euclidean n-space, R_n. For each compact set $K \subset \Omega$, $\mathcal{D}(K)$ is the Fréchet space of infinitely differentiable real valued functions with support in K (see Schwartz [12]). $\mathcal{D}(\Omega) = \bigcup\{\mathcal{D}(K): K$ is compact in $\Omega\}$ provided with the inductive limit topology and $\mathcal{D}'(\Omega)$ is its topological dual. $\mathcal{E}(\Omega)$ denotes the Fréchet space of all infinitely differentiable functions on Ω with the usual topology, and $\mathcal{E}'(\Omega)$ is its topological dual. Let $S \in \mathcal{E}'(R_n)$. If Ω_1 and Ω_2 are open sets such that

$$\Omega_1 + \operatorname{supp} S \subset \Omega_2, \tag{4.2}$$

then the mapping T defined by

$$T\varphi = S * \varphi \tag{4.3}$$

where $*$ denotes convolution, is a continuous injection from $\mathcal{D}(\Omega_1)$ into $\mathcal{D}(\Omega_2)$ and from $\mathcal{E}'(\Omega_1)$ into $\mathcal{E}'(\Omega_2)$ [7]. Let T^* be the transpose of T. We wish to make use of some results of Hörmander [7] and therefore make the following:

DEFINITION 4.1. The pair (Ω_1, Ω_2) of open sets in R_n is called *S-convex* if (4.2) holds, and given any compact set $K_2 \subset \Omega_2$, there exists a compact $K_1 \subset \Omega_1$ such that $\varphi \in \mathcal{D}(\Omega_1)$ and $\operatorname{supp} S * \varphi \subset K_2$ imply that $\operatorname{supp} \varphi \subset K_1$.

DEFINITION 4.2. The distribution $S \in \mathcal{E}'(R_n)$ is said to be *invertible* if there exist constants A_1, A_2 and A_3 such that for every ξ in R_n, one can find $\eta \in R_n$ such that

$$|\xi - \eta| \leq A_1 \log(2 + |\xi|), \quad \text{and} \quad |\hat{S}(\eta)| \geq (A_2 + |\xi|)^{-A_3}. \tag{4.4}$$

Here $\hat{S}$ denotes the Fourier transform of S (see Schwartz [12]). One can now state the following three theorems due to Hörmander [7].

THEOREM 4.1. *The following are equivalent:*

(a) $T^*(\mathcal{E}(\Omega_2)) = \mathcal{E}(\Omega_1)$;

(b) T^{-1} *is a sequentially continuous mapping from the image,* $M \subset \mathcal{D}(\Omega_2)$, *of* T *onto* $\mathcal{D}(\Omega_1)$; *and*

(c) S *is invertible and* (Ω_1, Ω_2) *is an S-convex pair.*

THEOREM 4.2. *If* (Ω_1, Ω_2) *is an S-convex pair, and* $\varphi \in \mathcal{E}'(\Omega_1)$, *then the distance from* supp φ *to* $R_n \backslash \Omega_1$ *is equal to the distance from* supp $(S * \varphi)$ *to* $R_n \backslash \Omega_2$.

THEOREM 4.3. *If* $T^*(\mathcal{D}'(\Omega_2)) = \mathcal{D}'(\Omega_1)$, *then the distance from* sing supp φ *to* $R_n \backslash \Omega_1$ *is equal to the distance from* sing supp $(S * \varphi)$ *to* $R_n \backslash \Omega_2$ *for every* $\varphi \in \mathcal{E}'(\Omega_1) \backslash \mathcal{D}(\Omega_1)$.

We first wish to show that the image, M, of T is a closed subspace of $\mathcal{D}(\Omega_2)$.

PROPOSITION 4.4. *If* (a), (b), *or* (c) *of Theorem* 4.1 *is satisfied, then* $M = T(\mathcal{D}(\Omega_1))$ *is a closed subspace of* $\mathcal{D}(\Omega_2)$.

PROOF. We make use of the following theorem by Dieudonné and Schwartz [4]: Let E and F be Fréchet spaces with duals E' and F' respectively, and suppose θ is a continuous linear mapping of E into F with transpose θ^*. Then θ is a surjection if and only if θ^* is an injection and the image of F' under θ^* is a $\sigma(E', E)$-closed subspace of E'.

Since the conditions of Theorem 4.2 are equivalent, T^* maps $\mathcal{E}(\Omega_2)$ onto $\mathcal{E}(\Omega_1)$, and thus $L = T[\mathcal{E}'(\Omega_1)]$ is a closed subspace of $\mathcal{E}'(\Omega_2)$ by the result quoted above. Let $\varphi \in \mathcal{D}(\Omega_2) \backslash M$. Then $\varphi \in \mathcal{E}'(\Omega_2) \backslash L$ and thus there exists $u \in \mathcal{E}(\Omega_2)$ which strongly separates φ and L. But $\mathcal{E}(\Omega_2) \subset \mathcal{D}'(\Omega_2)$ and $M \subset L$. Thus there exists u in $\mathcal{D}'(\Omega_2)$ which strongly separates φ and M, i.e. M is a weakly-closed subspace of $\mathcal{D}(\Omega_2)$, hence closed. □

Fix α in $\mathcal{D}(R_n)$ such that $\alpha \equiv 1$ on $\Pi_{i=1}^n [-1, 1]$ and let

$$S = \delta + \alpha, \tag{4.5}$$

where δ is the Dirac measure. Then $S \in \mathcal{E}'(R_n)$ and is invertible. Also

$$\text{sing supp } S \subset \text{interior } [\text{supp } S]. \tag{4.6}$$

Let (Ω_1, Ω_2) be an S-convex pair in R_n such that each of Ω_1, Ω_2 has at least one finite boundary point. (For example, let Ω_2 be the ball of radius twice the diameter of supp S, and Ω_1 the largest open set in R_n such that (4.2) is valid.) From the geometry of the situation we obtain

$$d(\text{sing supp } S * \varphi, R_n \backslash \Omega_2) > d(\text{supp } S * \varphi, R_n \backslash \Omega_2) \tag{4.7}$$

for some $\varphi \in \mathcal{E}'(\Omega_1)$ (namely $\varphi(x) = \delta_a(x) = \delta(x-a)$ for any a in Ω_1). Applying Theorem 4.3, we obtain

$$d(\operatorname{supp} \varphi, R_n \backslash \Omega_1) = d(\operatorname{supp} S * \varphi, R_n \backslash \Omega_2) \tag{4.8}$$

for all $\varphi \in \mathcal{E}'(\Omega_1)$. Combining (4.7) and (4.8), we see that

$$d(\operatorname{sing\,supp} \varphi, R_n \backslash \Omega_1) < d(\operatorname{sing\,supp} S * \varphi, R_n \backslash \Omega_2) \tag{4.9}$$

for some φ in $\mathcal{E}'(\Omega_1)$. Thus, it follows from Theorem 4.4 that

$$T^*(\mathcal{D}'(\Omega_2)) \not\supset \mathcal{D}'(\Omega_1). \tag{4.10}$$

Choose $\varphi \in \mathcal{D}'(\Omega_1) \backslash T^*(\mathcal{D}'(\Omega_2))$ and define ψ on M by

$$\psi = \varphi \circ T^{-1}. \tag{4.11}$$

Applying Theorem 4.2, we see that ψ is sequentially continuous, and thus, for each compact set $K \subset \Omega_2$, $\psi|_{M \cap \mathcal{D}(K)}$ is sequentially continuous. But $\mathcal{D}(\Omega_2)$ induces the original topology on each $\mathcal{D}(K)$, and thus $\psi|_{M \cap \mathcal{D}(K)}$ is continuous for each compact $K \subset \Omega_2$.

Suppose ψ has a continuous extension to $\mathcal{D}(\Omega_2)$, say $\Psi \in \mathcal{D}'(\Omega_2)$ such that $\Psi|_M = \psi$. Then, for each $\eta \in \mathcal{D}(\Omega_1)$ we have

$$\begin{aligned} \langle T^*\Psi, \eta\rangle &= \langle \Psi, T\eta\rangle = \langle \psi, T\eta\rangle \\ &= \langle \varphi \circ T^{-1}, T\eta\rangle = \langle \varphi, \eta\rangle. \end{aligned} \tag{4.12}$$

That is, $T^*\Psi = \varphi$ which contradicts the choice of φ.

One can easily verify that if τ is the strict convergence inductive limit on E, and σ is the strict convergence inductive limit on M determined by the family $\{\mathcal{D}(K) \cap M: K \text{ is compact in } \Omega_2\}$ then $\tau|_M = \sigma$. It is clear that ψ as defined by (4.11) is a continuous linear form on M with respect to the σ convergence structure. But we have shown that ψ has no τ-continuous extension to $\mathcal{D}(\Omega_2)$ since $(\mathcal{D}(\Omega_2), \tau)' = (\mathcal{D}(\Omega_2), \psi^\circ\tau)' = \mathcal{D}'(\Omega_2)$.

REMARK 4.1. We have exhibited a sequentially continuous linear form on a closed subspace M of an LF-space which is not continuous. Thus M with the induced topology is not a bornological space and thus cannot be an LF-space. This provides a solution to a problem posed by Dieudonné and Schwartz in [4].

EXAMPLE (4). In this example, we show the condition that a subspace M of a c.v.s. (E, τ) be nearly closed is both necessary and sufficient for the continuous extension of τ_M-continuous linear functionals to $(\overline{M}, \tau_{\overline{M}})$ in case (E, τ) is of a certain type. We also characterize the class of subspaces whose τ-adherence is τ-closed for a class of c.v.s. of considerable interest.

Throughout this example, (E, τ) is a strict convergence inductive limit of an increasing sequence (E_n, T_n) of linear topological spaces.

PROPOSITION 4.5. *Suppose M is a subspace of (E, τ). M is nearly closed if and only if for each $n \geqslant 1$ there exists an integer $N \geqslant n$ such that $\overline{M} \cap E_n \subseteq \overline{M}_N^N \cap E_n$, where $\overline{M}_j^k = T_k$-closure of $M \cap E_j$ in E_k for all $k \geqslant j$.*

PROOF. Let $\hat{F}_n$ be the neighborhood filter for 0 in E_n and F_n the filter in E generated by $\hat{F}_n$. Then $\{F_n\}$ is a fundamental family for τ. Given $n \geqslant 1$, choose N so that $\overline{M} \cap E_n \subseteq \overline{M}_N^N \cap E_n$. Let F be any element of $\hat{F}_n \subseteq F_n$. If $x \in \overline{M} \cap F$, then $x \in \overline{M} \cap E_n$ and hence $x \in \overline{M}_N^N \cap E_n$. But, then $x + F_N$ leaves a trace on M. Hence M is nearly closed. Now, suppose M is nearly closed. Using the notation of the definition, let n be fixed and find $F_\alpha \leqslant F_n$. Let $F \in F_\alpha$ and F_β be found such that $y \in \overline{M} \cap F$ implies $y + F_\beta$ leaves a trace on M. Find N such that $F_N \leqslant F_\beta$. Let $x \in \overline{M} \cap E_n$. For some $\lambda > 0, \lambda x \in \overline{M} \cap F$. Hence $\lambda x + F_\beta$ leaves a trace on M and so $\lambda x + F_N$ also leaves a trace on M. Thus $\lambda x + \hat{F}_N$ leaves a trace on M and therefore on $M \cap E_N = M_N$. That is, $\lambda x \in \overline{M}_N^N$. Since $\overline{M}_N^N$ is a subspace of E_N, $x \in \overline{M}_N^N$. But $x \in E_n$ and thus $x \in \overline{M}_N^N \cap E_n$. $\square$

THEOREM 4.4. *If each (E_n, T_n) is metrizable, and M is a subspace of E, then every continuous linear functional on (M, τ_M) can be extended to a continuous linear functional on $(\overline{M}, \tau_{\overline{M}})$ if and only if for each $n \geqslant 1$ there exists $N \geqslant n$ so that $\overline{M} \cap E_n \subset \overline{M}_N^N \cap E_n$, that is if and only if M is nearly closed.*

PROOF. Suppose there exists n such that for all $m, \overline{M} \cap E_n \not\subset \overline{M}_m^m \cap E_n$. We may assume without loss of generality (as will shortly be evident) that $n = 1$ and $\overline{M}_k^k \cap E_1 \subsetneqq \overline{M}_{k+1}^{k+1} \cap E_1$ for all $k \geqslant 1$. Let $\{U_k\}$ be a zero neighborhood base for (E_1, T_1) such that $U_{k+1} \subset U_k$, $k = 1, 2, \cdots$. For each k, let

$$x_k \in (U_k \cap \overline{M}_{k+1}^{k+1}) \backslash \overline{M}_k^k.$$

We note that $\overline{M}_k^k = \overline{M \cap E_k}^k = E_k \cap \overline{M \cap E_k}^{k+1} = E_k \cap \overline{M}_k^{k+1}$ since $T_{k+1}|_{E_k} = T_k$. Thus for each k, $x_k \notin \overline{M \cap E_k}^{k+1}$. Let φ be a T_k-continuous linear functional on $\overline{M \cap E_k}^k$. Then φ is T_{k+1}-continuous and thus has a unique continuous extension, again call it φ, to $\overline{M \cap E_k}^{k+1}$. But $x_k \notin \overline{M \cap E_k}^{k+1}$ and thus $\overline{M \cap E_k}^{k+1} \oplus \operatorname{span} x_k$ is a T_{k+1}-topological direct sum in $\overline{M}_{k+1}^{k+1}$. Thus there exists a T_{k+1}-continuous linear functional Ψ on $\overline{M \cap E_k}^{k+1} \oplus \operatorname{span} x_k$ such that Ψ coincides with φ on $\overline{M \cap E_k}^k$ and $\Psi(x_k) = 1$. By the Hahn-Banach Theorem, Ψ has a continuous linear extension to $\overline{M}_{k+1}^{k+1}$. Thus, we may construct a sequence φ_k of linear functionals on $\overline{M}_k^k$, $k = 2, 3, \cdots$ such that φ_k is T_k-continuous, $\varphi_k(x_{k-1}) = 1$ and $\varphi_k|_{\overline{M}_{k-1}^{k-1}} = \varphi_{k-1}$ for $k = 3, 4, 5, \cdots$. The choice of φ_2 is subject only to the restriction $\varphi_2(x_1) = 1$. Define φ on M by $\varphi(x) = \varphi_k(x)$ if $x \in M \cap E_k$. φ is well defined, linear and τ_M-continuous. If φ has a $\tau_{\overline{M}}$-continuous extension $\bar{\varphi}$ to $\overline{M} = \bigcup_{k=1}^{\infty} \overline{M}_k^k$, then since x_k is a sequence in $\overline{M}$ whose associated filter is in $\tau_{\overline{M}}(0)$, it must be that $\bar{\varphi}(x_k) \longrightarrow 0$. On the other hand, since $x_k \in \overline{M \cap E}_{k+1}^{k+1} \subseteq \overline{M} \cap E_{k+1}$, $\bar{\varphi}(x_k) = \lim_{j\to\infty} \varphi_{k+1}(y_j)$ for some sequence y_j in $M \cap E_{k+1}$ converging for T_{k+1} to x_k. But then $1 = \varphi_{k+1}(x_k) = \lim_{j\to\infty}\varphi_{k+1}(y_j) = \bar{\varphi}(x_k)$ since φ_{k+1} is continuous on $\overline{M \cap E}_{k+1}^{k+1}$ for T_{k+1}. This is a contradiction. Hence, M is nearly closed. The converse follows from Theorem 3.1 and Proposition 4.5.

THEOREM 4.5. *Suppose each (E_n, T_n) is a Fréchet space. A subspace M of E has the property that every τ_M-continuous linear functional on M can be extended $\tau_{\overline{M}}$-continuously to $\overline{M}$ if and only if $\overline{M}$ is τ-closed.*

PROOF. Suppose $\overline{M}$ is τ-closed. Then for each $n \geqslant 1$, $E_n \cap \overline{M}$ is closed for T_n. But $E_n \cap \overline{M} = E_n \cap \bigcup_{m=1}^{\infty} \overline{M}_m^m = \bigcup_{m=1}^{\infty} E_n \cap \overline{M}_m^m$. Since $T_m|_{E_n} = T_n$ for each $m \geqslant n$, $E_n \cap \overline{M}_m^m$ is closed for T_n in E_n. But $E_n \cap \overline{M}$ is a Fréchet space, so by the Baire Category Theorem, there exists N such that $E_n \cap \overline{M}_N^N$ contains an open set in $E_n \cap \overline{M}$. Hence $E_n \cap \overline{M} = E_n \cap \overline{M}_N^N$ and by Theorem 4.4, M has the required property. The converse follows from Theorem 4.4 and Theorem 3.2.

COROLLARY 4.5.1. *Let (E, τ) and M be as in the theorem. $\overline{M}$ is τ-closed if and only if M is nearly closed.*

EXAMPLE (5). We will now give a simple proof to show that any subspace M of a strict convergence inductive limit (E, τ) of a sequence of Hilbert spaces (E_n, T_n) has the H.B.P. if and only if M is nearly closed.

PROPOSITION 4.6. *Let $\{E_n, T_n\}_{n=1}^{\infty}$ be a sequence of topological vector spaces such that $E_n \subset E_{n+1}$ and $T_{n+1}|_{E_n} = T_n$ for each n. Let $E = \bigcup_{n=1}^{\infty} E_n$ and T the strict inductive limit topology on E. If, for each subspace M of E either* (1) *For some n, $M \subset E_n$ or* (2) *$\forall n$, $\exists$ a subspace of E_n such that $(E_n \cap M) \oplus N_n = E_n$ topologically and $N_n \subset N_{n+1}$ for $n = 1, 2, \cdots$, then $T|_M$ is the inductive limit topology on M defined by the family $\{M \cap E_n : n = 1, 2, \cdots\}$.*

PROOF. If (1) holds, the result is clear. Suppose (2) is valid. For each n, let U_n be an open neighborhood of 0 in E_n. We must find a sequence of 0-neighborhoods $A_n \subset E_n$ $(n = 1, 2, \cdots)$ such that

$$\Gamma\left(\bigcup_{n=1}^{\infty} A_n\right) \cap M \subset \Gamma\left[\bigcup_{n=1}^{\infty} \left(U_n \cap M\right)\right]. \tag{4.13}$$

By the hypothesis, for each n there exists a subspace N_n in E_n such that, $(M \cap E_n) \oplus N_n \cong E_n$ topologically and $N_n \subset N_{n+1}$ for all n. Choose 0-neighborhoods V_n and W_n on E_n such that

$$(M \cap V_n) \oplus (N_n \cap W_n) \subset U_n. \tag{4.14}$$

Let $A_n = (M \cap V_n) \oplus (N_n \cap W_n)$. Then A_n is a 0-neighborhood in E_n. Also we have

$$\begin{aligned}
\Gamma\Big(\bigcup_{n=1}^{\infty} A_n\Big) \cap M &= \Gamma\Big[\bigcup_{n=1}^{\infty} \big\{(M \cap V_n) \oplus (N_n \cap W_n)\big\}\Big] \cap M \\
&\subset \Gamma\Big[\Big(\bigcup_{n=1}^{\infty} (M \cap V_n)\Big) \oplus \Big(\bigcup_{n=1}^{\infty} (N_n \cap W_n)\Big)\Big] \cap M \\
&\subset \Big\{\Big[\Gamma\Big(\bigcup_{n=1}^{\infty} (M \cap V_n)\Big)\Big] \oplus \Big[\Gamma\Big(\bigcup_{n=1}^{\infty} (N_n \cap W_n)\Big)\Big]\Big\} \cap M \qquad (4.15) \\
&= \Gamma\Big[\bigcup_{n=1}^{\infty} (M \cap V_n)\Big] = \Gamma\Big[\bigcup_{n=1}^{\infty} (A_n \cap M)\Big] \\
&\subset \Gamma\Big[\bigcup_{n=1}^{\infty} (U_n \cap M)\Big]. \quad \square
\end{aligned}$$

DEFINITION 4.3. If M and N are subspaces of a Hilbert space H, then the angle $\alpha(M, N)$ between M and N is defined by

$$\alpha(M, N) = \arccos\Big(\sup_{x \in M,\, y \in N;\ \|x\| = \|y\| = 1} \langle x, y\rangle\Big).$$

One can show (see [14, p. 243]) that in case $\alpha(M, N) > 0$ the $M \oplus N$ is a topological direct sum in H.

COROLLARY 4.6.1. *If M is a subspace of a topological strict inductive limit (E, T) of Hilbert spaces (H_n, T_n), $n = 1, 2, \cdots$, such that for each n, $\alpha(H_{n+1} \cap M, H_n) > 0$ and $M \cap H_n$ is closed in H_n, then M is an LF-space.*

PROOF. We choose the N_n's inductively to satisfy Proposition 4.6 in the following way: Since H_1 is Hilbert there exists N_1 such that $(H_1 \cap M) \oplus N_1 = H_1$ topologically. Assume that $\{0\} = N_0, N_1, \cdots, N_n$ are chosen so that $(H_k \cap M) \oplus N_k = H_k$ topologically and $N_{k-1} \subset N_k$ for $k = 1, 2, \cdots, n$. Since N_n is closed in H_n, it is also closed in H_{n+1}. But $H_n \cap M$ is also closed in H_{n+1}, and $\alpha(H_{n+1} \cap M, N_n) > 0$ since $N_n \subset H_n$. Thus $(H_{n+1} \cap M) \oplus N_n$ is a topological direct sum in H_{n+1} and is therefore closed. Since H_{n+1} is Hilbert, there exists a subspace L_{n+1} of H_{n+1} such that $(H_{n+1} \cap M) \oplus N_n \oplus L_{n+1} = H_{n+1}$ topologically. Let $N_{n+1} = N_n \oplus L_{n+1}$. The result now follows from the previous proposition.

THEOREM 4.6. *Let* (E, τ) *be the strict convergence inductive limit of the Hilbert spaces* (H_n, T_n), $n = 1, 2, \cdots$. *Then a subspace* M *of* (E, τ) *has the H.B.P. if* M *is nearly closed and for each* n, $\alpha(H_n \cap M, H_n) > 0$.

PROOF. It was shown by Fischer [5] that for strict convergence inductive limits (E, τ), $\psi^\circ\tau$ is exactly the inductive limit *topology* for E. Let M be a nearly closed subspace of E. Without loss of generality (Theorem 3.1 and Theorem 3.2), we may assume M is closed. Proposition 2.4 applies to show that $(M, \Gamma(\psi\tau)_M)' = (M, (\psi\tau)_M)'$. By Proposition 4.4 and the fact that every closed subspace of a Hilbert space is a topological direct summand, we have

$$\begin{aligned}(M, \Gamma(\psi\tau)_M)' &= (M, (\psi\tau)_M)' = (M, \psi^\circ(\psi\tau)_M)' \\ &= (M, (\psi^\circ\tau)_M)' \subset (M, [\Gamma(\psi\tau)]_M)'.\end{aligned} \tag{4.16}$$

But, since $\Gamma(\psi\tau)_M \geqslant [\Gamma(\psi\tau)]_M$, we have

$$(M, \Gamma(\psi\tau)_M)' = (M, [\Gamma(\psi\tau)]_M)'. \quad \square \tag{4.17}$$

The following more general result follows from Theorem 4.4, Theorem 3.2 and the extension result given by Hogbe-Nlend in [7, p. 66].

THEOREM 4.7. *Let* E *be a strict convergence inductive limit of reflexive Banach spaces. A subspace* M *of* E *has the H.B.P. if and only if* M *is nearly closed.*

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