TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 201, 1975

PI-ALGEBRAS SATISFYING IDENTITIES OF DEGREE 3

BY

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ABSTRACT. A method of classification of PI-algebras over fields of characteristic 0 is described and applied to algebras satisfying polynomial identities of degree 3. Two algebras satisfying the same identities of degree 3 are considered in the same class. For the degree 3 all the possible classes are obtained. In each case the identities of degree 4 that can be deduced from those of degree 3 have been obtained by means of a computer. These computations have made it possible to obtain—except for three cases—all the identities of higher degrees. It turns out that except for a finite number of cases an algebra satisfying an identity of degree 3 is either nilpotent of order 4, or commutative of order 4, namely the product of 4 elements of the algebra is a symmetric function of its factors.

1. **Introduction.** The aim of this paper is to determine the multilinear identities of a PI-algebra that satisfies a polynomial identity of degree 3. We shall restrict ourselves to PI-algebras over fields of characteristic zero, since in this case it is well known that the polynomial identities are completely determined by the multilinear identities. It is easy to show that if $A \neq \{0\}$ is a PI-algebra that satisfies an identity of degree 2, then the multilinear identities of A of degree 2 are determined by one of the following three polynomials:

$$x_1x_2, \quad x_1x_2 + x_2x_1, \quad x_1x_2 - x_2x_1. \tag{1.1}$$

The first characterizes the class of nonzero trivial algebras, the second characterizes the class of nontrivial anticommutative algebras (these are nilpotent of order 3) and the third characterizes the class of nontrivial commutative algebras.

The coefficients of the monomials in (1.1) are ± 1. It was this observation that led to the actual paper. Dealing with identities of degree 3 the following question arises: If A is an algebra that satisfies an identity of degree 3, can the space of multilinear identities, of degree 3 satisfied by A, be spanned by polynomials whose coefficients are only ± 1? The answer to this question is negative. Moreover, there are cases for which no spanning polynomials exist whose coefficients

Received by the editors January 11, 1974.

AMS (MOS) subject classifications (1970). Primary 16A38; Secondary 16A26.

Key words and phrases. Universal PI-algebra, *T*-ideal, codimension, row-echelon normal form, Sylow subgroup, opposite algebra, Grassmann algebra, Peirce complement.

are rational numbers; for example, the universal PI-algebra [1] satisfying the identity:

$$x_1x_2x_3 + \zeta x_2x_3x_1 + \zeta^2 x_3x_1x_2 \tag{1.2}$$

where $1 \neq \zeta \in F$ and $\zeta^3 = 1$.

Our result seems somewhat peculiar if one considers a result of Mal'cev in [6] which was quoted in [7]. It says that any multilinear identity of degree 3 is a linear combination of identities of the following types:

$$\sum_{\sigma \in S_3} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)} \tag{1.3}$$

$$\sum_{\sigma \in S_3} (-1)^{\text{sign}\,\sigma} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)} \tag{1.4}$$

$$x_1x_2x_3 + x_2x_1x_3 - x_2x_3x_1 - x_3x_2x_1 \tag{1.5}$$

$$x_1x_2x_3 + x_1x_3x_2 - x_3x_1x_2 - x_3x_2x_1. \tag{1.6}$$

The coefficients in (1.3)–(1.6) are all equal ± 1, but this result is trivial. Indeed, by permutation of variables one gets from (1.5) and (1.6) two more identities:

$$x_1x_3x_2 - x_2x_3x_1 + x_3x_1x_2 - x_3x_2x_1 \tag{1.7}$$

$$x_2x_1x_3 + x_2x_3x_1 - x_3x_1x_2 - x_3x_2x_1. \tag{1.8}$$

Now the dimension of the space of multilinear identities of degree 3 is 6 and since the six polynomials (1.3)–(1.8) are linearly independent, it is clear that any other identity is a linear combination of these identities. One could even say that any identity of degree 3 is a linear combination of identities of the type $x_1x_2x_3$, since by permutation of variables one gets the six linearly independent monomials of degree 3.

Our first aim will be to obtain a complete determination of the possible spaces of multilinear identities of degree 3, for PI-algebras satisfying an identity of degree 3. We shall see that there are an infinite number of possible cases. In each of these cases we have computed the spaces of multilinear identities of degree 4 that can be obtained from the corresponding identities of degree 3, and except for three cases we have obtained the identities of any higher degree. It is interesting that we have obtained only a finite number of such spaces of multilinear identities of degree 4. Moreover, almost all the cases lead either to the whole space of multilinear polynomials of degree 4, or to the space of codimension 1 generated by the 23 polynomials

$$x_1x_2x_3x_4 - x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}, \qquad \mathrm{id} \neq \sigma \in S_4. \tag{1.9}$$

If the whole space of polynomials is obtained, then the corresponding algebra is nilpotent of order 4. In the other case the corresponding algebra is commutative of order 4, namely the product of 4 elements of the algebra is a symmetric function of its factors.

It was shown in [7] that an algebra satisfying the identity (1.5) or (1.6) is commutative of order 4. It is not apparent how to apply the same method to other cases. The method that we shall develop will be applied to all the cases and commutativity of order 4 will be obtained in an infinite number of cases.

Our method of determining the spaces of identities of degree 3 can be applied to higher degrees, but the trouble is that, already for the degree 4, the amount of necessary computations is enormous. At this moment it is not clear how a computer can be used to overcome these computations.

2. **The general method.** A multilinear identity of degree k over a field F may be written as follows:

$$f(x_1, \cdots, x_k) = \sum_{\sigma \in S_k} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}, \tag{2.1}$$

S_k is the symmetric group of degree k, and $c_\sigma \in F$ are not all 0. The vector space spanned by $x_{\sigma(1)} \cdots x_{\sigma(k)}$, $\sigma \in S_k$, will be denoted by V_k. Its dimension is $k!$ and it is isomorphic to the vector space of the group algebra $F(S_k)$, the isomorphism given by $\Sigma c_\sigma \sigma \longleftrightarrow \Sigma c_\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}$. Given an algebra A satisfying an identity of degree k, let $I_k(A)$ be the subspace of V_k of those polynomials that are identities of A. By definition $I_k(A) \neq \{0\}$ and it is invariant under permutation of variables. Let $J_k(A)$ be the subspace of $F(S_k)$ which is the image of $I_k(A)$ under the above-mentioned isomorphism between V_k and $F(S_k)$. Then $J_k(A)$ is invariant under right multiplication by elements of S_k, hence it is a right ideal in $F(S_k)$. On the other hand, given a right ideal J_k of $F(S_k)$, its corresponding subspace I_k in V_k is invariant under permutation of variables, so there exists an algebra A such that $I_k(A) = I_k$. Indeed, A can be taken as the universal PI-algebra whose ideal of identities is the T-ideal generated by I_k [1]. In conclusion, there is a one-to-one correspondence between the possible subspaces of multilinear identities of degree k and the nonzero right ideals of $F(S_k)$.

Now, to obtain all the possible subspaces of multilinear identities, we can compute all the nonzero right ideals of $F(S_k)$. This computation will be done separately for each of the possible dimensions of the right ideals. First choose a fixed order of the elements of $S_k = \{\sigma_1, \sigma_2, \cdots, \sigma_{k!}\}$. Then, assuming I is an r-dimensional right ideal, choose a basis for I and write the coefficients of the base elements in terms of $\sigma_1, \cdots, \sigma_{k!}$, as the rows of an $r \times k!$ matrix M over F. By passing to another basis, if necessary, one may assume that this matrix is in the row-echelon normal form. So $M = (a_{ij})$ has r columns, numbered say by $j_1 < j_2 < \cdots < j_r$, which are the r unit vectors in their natural order. The entries of the $k! - r$ remaining columns of M will be considered as unknowns, except those which are 0. The unknowns of the ith row of M will be a_{it}, $j_i < t \leqslant k!$ and $t \neq j_{i+1}, \cdots, j_r$. The unknown entries of M satisfy certain equations that can be obtained as follows. Given some row $(a_{i1}, \cdots, a_{ik!})$ of M, multiply $\Sigma_{j=1}^{k!} a_{ij}\sigma_j$ on the right by $\sigma_s \in S_k$, $s = 1, \cdots, k!$. Then write $\Sigma_{j=1}^{k!} a_{ij}\sigma_j\sigma_s$ as $\Sigma_{j=1}^{k!} d_j\sigma_j$. The vector $(d_1, d_2, \cdots, d_{k!})$ is in the row space of M. Hence it is the linear combination of the r rows of M with the uniquely determined coefficients $dj_1, dj_2, \cdots, dj_r$. Using this fact for each of the r rows $k! - 1$ times, one obtains a system of quadratic equations (some of them linear) in the unknown entries of M. The distinct solutions of this system of equations correspond to distinct possible r-dimensional right ideals. To obtain all the r-dimensional right ideals one has to deal separately with each possible type of $r \times k!$ matrix in row-echelon normal form. In the next sections we shall see how our method applies for $k = 3$.

3. The rows obtained from one row when $k = 3$. The order we shall choose for the 6 elements of S_3 will be: $\sigma_1 = \text{id}$, $\sigma_2 = (23)$, $\sigma_3 = (12)$, $\sigma_4 = (123)$, $\sigma_5 = (132)$, $\sigma_6 = (13)$. Given a $r \times 6$, $1 \leqslant r \leqslant 6$, matrix M that corresponds to some r-dimensional right ideal of $F(S_k)$, and if the vector

$$(c_1, c_2, c_3, c_4, c_5, c_6) \tag{3.1}$$

is in the row-space of M, then the rows of the following matrix will be in the row-space of M:

$$P = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_2 & c_1 & c_5 & c_6 & c_3 & c_4 \\ c_3 & c_4 & c_1 & c_2 & c_6 & c_5 \\ c_5 & c_6 & c_2 & c_1 & c_4 & c_3 \\ c_4 & c_3 & c_6 & c_5 & c_1 & c_2 \\ c_6 & c_5 & c_4 & c_3 & c_2 & c_1 \end{pmatrix}. \tag{3.2}$$

Note that if $f = \Sigma_{i=1}^{6} c_i \sigma_i$, then P is the transpose of the matrix that represents f acting from the left on $F(S_3)$ with respect to the basis $\sigma_1, \cdots, \sigma_6$. We shall apply P to rows of M which are clearly $\neq 0$, Hence some row in P starts with a nonzero element and this implies that the first column of M is the unit vector whose first entry is 1. Now what happens if only one of the c_i's is $\neq 0$? In this case it is clear that M will be the 6×6 unit matrix which is the unique possible matrix of degree 6 in row-echelon normal form. The corresponding ideal is $F(S_k)$ itself and the corresponding algebras will be nilpotent of order 3. It remains to compute the r-dimensional right ideals of $F(S_k)$, $r = 1, 2, 3, 4, 5$. We shall do this in three steps. First we shall compute the right ideals of dimensions 1 and 5. These two cases are taken together, since the computations and the results are similar. Then we shall deal with the dimensions 2 and 4. The most complicated will be the computation of the right ideals of dimension 3.

4. **The dimensions 1 and 5.** If $r = 1$, then the corresponding matrix is necessarily $(1\ a\ b\ c\ d\ e)$. Applying the permutations corresponding to the rows of (3.2) we obtain the following system of equations:

$$a^2 = b^2 = e^2 = cd = 1$$

$$a = b = e = ac = ad = bc = bd = ce = de$$

$$c = d = c^2 = d^2 = ab = ae = be.$$

It follows that $c = d = 1$ and for a, b, e there are two possibilities, either all of them are 1 or all of them are -1. So there are two possible 1-dimensional right ideals which in fact are two-sided. Their corresponding matrices are

$$(1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1); \tag{4.1}$$

$$(1\ \ -1\ \ -1\ \ 1\ \ 1\ \ -1). \tag{4.2}$$

In terms of identities our result may be formulated as follows:

THEOREM 1. *There are exactly two types of PI-algebras whose spaces of multilinear identities of degree* 3 *are one-dimensional. One type is characterized by the identity* (1.3) *and the other by* (1.4).

Note that the identity (1.3) is the multilinearization of the identity x^3. By [3] an algebra satisfying (1.3) is nilpotent of order $\leqslant 6$. It will be nilpotent of order 6 if the ideal of identities of the algebra is generated (as a T-ideal) by (1.3). The identity (1.4) is the standard identity of degree 3.

To obtain the 5-dimensional right ideals of $F(S_3)$ we observe that there is only one type of 5×6 matrix in row-echelon normal form that we have to consider, namely:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & d \\ 0 & 0 & 0 & 0 & 1 & e \end{pmatrix}$$

A computation similar to that given in the one-dimensional case leads to $b = c = -1$ and either $a = d = e = 1$ or $a = d = e = -1$. Note that the corresponding right ideals are two-sided and could be obtained as the Peirce complements of the 1-dimensional ideals computed above. One merely has to find for the 1-dimensional ideals idempotents e_1, e_2 that generate them and then $(1 - e_1)F(S_3)$ and $(1 - e_2)F(S_3)$ will be the 5-dimensional (right) ideals of $F(S_3)$. The idempotents are uniquely determined as follows:

$$e_1 = \frac{1}{6}\sum_{i=1}^{6} \sigma_i, \qquad e_2 = \frac{1}{6}\sum_{i=1}^{6} (-1)^{\text{sign } \sigma_i}\sigma_i.$$

So the corresponding ideals have unique complements [2, p. 33]. Moreover, any 5-dimensional right ideal is a complement of a 1-dimensional right ideal and so it is one of the ideals $(1 - e_1)F(S_3)$, $(1 - e_2)F(S_3)$.

In terms of identities our result in the 5-dimensional case is given by

THEOREM 2. *There are exactly two types of PI-algebras whose spaces of multilinear identities of degree* 3 *are* 5-*dimensional. A basis for the multilinear identities of degree* 3 *is for one of those types of PI-algebras given by*

(4.3) $$x_1x_2x_3 - x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}, \qquad \text{id} \neq \sigma \in S_3$$

and for the other type it is given by

(4.4) $$x_1x_2x_3 - (-1)^{\text{sign } \sigma}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}, \qquad \text{id} \neq \sigma \in S_3.$$

Now, an algebra satisfying (4.3) is by definition commutative of order 3 and therefore it will be commutative of any higher order. On the other hand it is natural to say that an algebra satisfying (4.4) is anticommutative of order 3. Such an algebra will be nilpotent of order 4. Indeed, it will satisfy the identities

(4.5) $$(x_1x_2)x_3x_4 + x_3(x_1x_2)x_4; \qquad (x_1x_2x_3)x_4 - (x_3x_1x_2)x_4$$

and hence also $x_1x_2x_3x_4$ (the characteristic being 0). We summarize the previous remarks in the following

COROLLARY. *If A is a PI-algebra whose space of multilinear identities of degree* 3 *is* 5*-dimensional, then A is either nilpotent of order* 4 *or commutative of any order* $\geqslant 3$.

5. **The dimensions 2 and 4.** If $r = 2$, we have to consider two types of matrices in row-echelon normal form. The other types do not correspond to 2-dimensional right ideals.

The first type is:

$$\begin{pmatrix} 1 & a & 0 & b & c & d \\ 0 & 0 & 1 & e & f & g \end{pmatrix}.$$

Applying (3.2) we obtain a system of equations whose solutions are: $b = 0$, $c = g = -1, f = d = -a = -e, a^2 = 1$. So we have two cases corresponding to $a = 1$ and $a = -1$.

The other type is:

$$\begin{pmatrix} 1 & 0 & a & b & c & d \\ 0 & 1 & e & f & g & h \end{pmatrix}.$$

Applying the second permutation of (3.2) to the first row, we obtain that $e = c$, $f = d, g = a$ and $h = b$. Applying the remaining permutations we obtain the following equations for a, b, c, d:

$$a^2 + bc = 1 = bc + d^2; \qquad (a + d)b = 0 = (a + d)c$$

$$(b + c)d = a; \quad ad + c^2 = b; \quad ad + b^2 = c; \quad a(b + c) = d.$$

Now $a + d \neq 0$ is excluded since then we would have $b = c = 0$, so $a = 0$ contradicting $a^2 + bc = 1$. So $d = -a$ and we are left with:

$$a^2 + bc = 1; \quad a(b + c + 1) = 0; \quad b = c^2 - a^2; \quad c = b^2 - a^2.$$

One particular solution is $a = d = 0, b = c = 1$. In addition, we have an infinite number of solutions satisfying $c = -b - 1$ and $a^2 = b^2 + b + 1$. This includes an infinite number of solutions even if the field F is as small as possible, namely the field of rationals. Indeed, there are an infinite number of rational numbers b such that $b^2 + b + 1$ is a square of a rational number. For example take $b = (3n^2 - 4n + 1)/(2n - 1)$, then

$$b^2 + b + 1 = (3n^2 - 3n + 1)^2/(2n - 1)^2,$$

n being an arbitrary integer. In terms of identities the 2-dimensional case is summarized in the following:

THEOREM 3. *If the space of multilinear identities of degree* 3 *of a PI-algebra is 2-dimensional, then it is the invariant subspace (under permutation of variables) generated by one of the following polynomials*:

$$x_1x_2x_3 + x_2x_3x_1 + x_3x_1x_2 \tag{5.1}$$

$$x_2x_1x_3 + x_2x_3x_1 - x_3x_1x_2 - x_3x_2x_1 \tag{5.2}$$

$$x_2x_1x_3 - x_2x_3x_1 + x_3x_1x_2 - x_3x_2x_1 \tag{5.3}$$

$$x_1x_2x_3 + ax_2x_1x_3 + bx_2x_3x_1 - (b+1)x_3x_1x_2 - ax_3x_2x_1, \quad a^2 = b^2 + b + 1. \tag{5.4}$$

There are several interesting remarks concerning the results contained in the previous theorem.

Our first remark is that the invariant subspaces generated by (5.2) and (5.3) may be viewed as members of the family of subspaces corresponding to (5.4), if $a^2 = b^2 + b + 1$ is considered as an equation of a hyperbola. Then the two points of infinity of this hyperbola correspond to (5.2) and (5.3). Returning to $F(S_3)$, the set of 2-dimensional right ideals may be described as follows: To each point on the hyperbola $a_2^2 = a_1^2 + a_1a_0 + a_0^2$, in the projective plane over F, corresponds a 2-dimensional right ideal of $F(S_3)$ which is generated by $a_0(\sigma_1 - \sigma_5) + a_1(\sigma_4 - \sigma_5) + a_2(\sigma_3 - \sigma_6)$, and distinct points corresponds to distinct right ideals, none of them is two-sided. In addition, there is one 2-dimensional right ideal which is two-sided and it is generated by $\sigma_1 + \sigma_4 + \sigma_5$.

Another remark is that the identity (5.2) is the same as (1.8) and together with (1.6) they constitute a basis of the corresponding 2-dimensional invariant subspace.

The identity (1.5) is obtained if we take $a = 1, b = -1$ in (5.4) and together with (1.7) they constitute a basis of the corresponding 2-dimensional invariant subspace.

Now, let us turn to the 4-dimensional right ideals. The computation is similar to that given above for the 2-dimensional right ideals, so we shall write the results in form of 4×6 matrices. One of the possible types of matrices in row-echelon normal form leads to the two solutions given by:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}; \tag{5.5}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{5.6}$$

The other type leads to one particular solution and in addition to an infinite number of solutions given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}; \tag{5.7}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & a & -b \\ 0 & 1 & 0 & 0 & b & 1-a \\ 0 & 0 & 1 & 0 & -b & a \\ 0 & 0 & 0 & 1 & 1-a & b \end{pmatrix}, \tag{5.8}$$

where a, b satisfy $b^2 = a^2 - a + 1$.

Here (5.7) corresponds to a two-sided ideal and none of the others is two-sided. In terms of identities we have

THEOREM 4. *A generator of a 4-dimensional invariant subspace of multilinear identities of degree 3 is given by one of the following identities*:

$$x_1x_2x_3 - x_3x_1x_2 \tag{5.9}$$

$$x_1x_2x_3 + x_2x_3x_1 - x_3x_2x_1 \tag{5.10}$$

$$x_1x_2x_3 + x_2x_3x_1 + x_3x_2x_1 \tag{5.11}$$

$$x_1x_2x_3 + x_1x_3x_2 + (a+b)x_3x_1x_2 + (1-a-b)x_3x_2x_1, \quad b^2 = a^2 - a + 1. \tag{5.12}$$

Note that (5.9), (5.10), (5.11) correspond to the first rows of (5.7), (5.5), (5.6) respectively. However (5.12) corresponds to the sum of the first two rows in (5.8). The polynomial corresponding to the first row of (5.8) will not generate a 4-dimensional subspace if $a = 0$.

6. The 3-dimensional right ideals. We first note that $F(S_3)$ being semi-simple and 6-dimensional it is isomorphic to the direct sum of the 2×2 matrix algebra F_2 and two copies of F. So it has exactly six nonzero proper ideals, two of them are 1-dimensional, two of them are 5-dimensional, one is 2-dimensional and one is 4-dimensional. All of these were mentioned in the two previous sections. Consequently, none of the 3-dimensional right ideals of $F(S_3)$ is two-sided.

Only three types of 3×6 matrices in row-echelon normal form correspond to 3-dimensional right ideals.

The first type is:

$$\begin{pmatrix} 1 & a & 0 & b & 0 & d \\ 0 & 0 & 1 & c & 0 & e \\ 0 & 0 & 0 & 0 & 1 & f \end{pmatrix}$$

and there are two solutions given by:

(6.1)
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix};$$

(6.2)
$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The second type is:

$$\begin{pmatrix} 1 & 0 & a & 0 & c & f \\ 0 & 1 & b & 0 & d & g \\ 0 & 0 & 0 & 1 & e & h \end{pmatrix}$$

and the solutions are:

(6.3)
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix};$$

(6.4)
$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Finally, the third type is:

$$\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & k \end{pmatrix}.$$

Applying (3.2) we obtain 45 equations and one of them is $d^2 = 1$. If $d = 1$ the solution is: $h = 1, c = 1 + a + b, e = f = -b, g = k = -a$ and a, b satisfy $a + b + ab = 0$ so $a \neq -1$ and $b = -a/(a + 1)$. This shows that we have an infinite number of solutions given by:

(6.5)
$$\begin{pmatrix} 1 & 0 & 0 & a & b & 1+a+b \\ 0 & 1 & 0 & 1 & -b & -b \\ 0 & 0 & 1 & -a & 1 & -a \end{pmatrix}$$

where $a \neq -1$ and $b = -a/(a + 1)$.

If $d = -1$ the solution is: $h = -1, c = -1 - a - b, e = -f = b, g = -k = a$ and a, b satisfy as before $a + b + ab = 0$. So again we have an infinite number of solutions given by:

(6.6)
$$\begin{pmatrix} 1 & 0 & 0 & a & b & -1-a-b \\ 0 & 1 & 0 & -1 & b & -b \\ 0 & 0 & 1 & a & -1 & -a \end{pmatrix}$$

where $a \neq -1$ and $b = -a/(a + 1)$.

In terms of identities the 3-dimensional case is given in the following

THEOREM 5. *The space of multilinear identities of degree* 3 *of a PI-algebra is 3-dimensional if and only if it is generated by one of the following polynomials*:

(6.7)
$$x_1x_2x_3 \pm x_1x_3x_2$$

(6.8)
$$x_1x_2x_3 \pm x_2x_1x_3$$

(6.9)
$$x_1x_2x_3 + ax_2x_3x_1 + bx_3x_1x_2 \pm (1 + a + b)x_3x_2x_1, \quad a + b + ab = 0.$$

The result for right ideals can be expressed more concisely as follows:

COROLLARY. *There are two disjoint infinite sets of 3-dimensional right ideals of* $F(S_3)$. *There is a one-to-one correspondence between each of these sets and the points of the hyperbola* $a_0a_1 + a_0a_2 + a_1a_2 = 0$. *The members of one set are generated by* $a_0(\sigma_1 + \sigma_6) + a_1(\sigma_4 + \sigma_6) + a_2(\sigma_5 + \sigma_6)$ *and those of the other set are generated by* $a_0(\sigma_1 - \sigma_6) + a_1(\sigma_4 - \sigma_6) + a_2(\sigma_5 - \sigma_6)$.

7. The identities of degree 4 determined by those of degree 3. Since $F(S_k)$ is semisimple artinian, any right ideal of $F(S_k)$ is generated by one element. Hence to compute the identities of degree 4 determined by those of degree 3, that have been described in the previous sections, it is enough to find a generator in each case, and compute the space of identities of degree 4 that can be derived from this generator [7]. Given a multilinear identity $p(x_1, x_2, x_3)$, the space of multilinear identities of degree 4 that can be derived from it is clearly generated by the 120 polynomials obtained by applying the 24 permutations of S_4 on each of the 5 polynomials:

$$(7.1)\qquad \begin{gathered} p(x_1, x_2, x_3)x_4, \qquad x_1p(x_2, x_3, x_4) \\ p(x_1x_2, x_3, x_4), \quad p(x_1, x_2x_3, x_4), \quad p(x_1, x_2, x_3x_4). \end{gathered}$$

Writing down the coefficients of these polynomials, after an order of the 24 permutations of $x_1x_2x_3x_4$ is chosen, we obtain a 120×24 matrix. By elementary operations on the rows of this matrix, it is possible to obtain its row-echelon normal form, and this can be used to describe the space of identities of degree 4 that can be derived from $p(x_1, x_2, x_3)$. This has been done for all the cases obtained in the previous sections, by the use of the C.D.C. 6600 computer of Tel-Aviv University. I wish to express my thanks to Miss S. Finkelstein for supplying the input of the described matrices to the computer. For the cases depending on a and b, it was also possible to simplify the corresponding matrix. For these cases it was helpful to deal with a 120×72 matrix, each column being replaced by three columns due to the fact that after elementary operations the entries of the matrix are linear combinations of $1, a$ and b.

The following two theorems include all the cases for which the corresponding algebras are either nilpotent of order 4 (codimension 0), or commutative of order 4 (codimension 1).

THEOREM 6. *If the multilinear identities of degree* 3 *of a PI-algebra are determined by one of the following*: (4.4), (5.1), (5.10), (5.11), (5.12), (6.1), (6.3), (6.5)—*except when* $a = 0$, *then the algebra is nilpotent of order* 4.

THEOREM 7. *If the multilinear identities of degree* 3 *of a PI-algebra are determined by one of the following*: (4.3), (5.2), (5.4)—*except when* $b = 0$ *or* $a = b = -1$, (5.9), (6.6), *then the algebra is commutative of order* 4.

In addition to the cases included in these two theroems there are nine more cases which we are now going to discuss.

(1) There is one more case for which the codimension is 1. It is that given by (6.5) when $a = 0$. A basis for the identities of degree 4 in this case is given by:

$$x_1x_2x_3x_4 - (-1)^{\text{sign}\,\sigma}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}, \qquad \text{id} \neq \sigma \in S_4. \tag{7.2}$$

The corresponding algebra will be nilpotent of order 5. This can be deduced in a way similar to that we have used to deduce nilpotency of order 4 from (4.4).

(2) There is one case for which the codimension is 3. It is that given by (5.4)—when $a = 1, b = 0$. To describe the space of multilinear identities of degree 4, let us denote by H the 2-Sylow subgroup of S_4 containing the transposition (13). Then a basis for the identities is given by the seven polynomials:

$$x_1x_2x_3x_4 - x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}, \qquad \text{id} \neq \sigma \in H \tag{7.3}$$

together with the 14 polynomials obtained from (7.3) when the transpositions (23) and (34) are applied. Another way to describe the space of identities is by giving its corresponding right ideal. To do this we introduce the following notation for any group G and a subgroup K:

$$F(G : K) = \text{Span}\,\{x - y \in F(G) \mid x, y \in G,\ xy^{-1} \in K\}. \tag{7.4}$$

Clearly $F(G : K)$ is a right ideal in $F(G)$. Now, the right ideal of $F(S_4)$ that corresponds to the space of identities described before is simply $F(S_4 : H)$. In the present case we can say much more.

THEOREM 8. *Let* A *be an algebra satisfying*

$$x_1x_2x_3 + x_2x_1x_3 - x_3x_1x_2 - x_3x_2x_1; \tag{7.5}$$

then A *is commutative of order* 5.

PROOF. Let $a_1, \cdots, a_5 \in A$, then since H contains (13) and (24) it follows that $a_1a_2a_3a_4a_5 = a_3a_2a_1a_4a_5$ and $a_3a_2a_1(a_4a_5) = a_3(a_4a_5)a_1a_2$. So we have $a_1a_2a_3a_4a_5 = a_3a_4a_5a_1a_2$. This shows that the right ideal that corresponds to the space of identities of degree 5 satisfied by A contains the elements id $-$ (13) and id $-$ (13524). Since S_5 is generated by (13) and (13524) it follows that the right ideal just mentioned will contain id $- \sigma$ for any $\sigma \in S_5$. This shows A is commutative of order 5.

(3) There are four cases for which the codimensions are 4. They are those given by (5.3), (6.2), (5.4)–when $a = b = -1$ and (6.4). For the first two cases we have obtained the same subspaces of identities of degree 4. Hence it is clear that for any higher degree they will have the same subspaces of identities. We have obtained a similar result for the other two cases.

Concerning the first two cases it is enough to see what can be deduced from the identity

$$x_1x_2x_3 - x_1x_3x_2. \tag{7.6}$$

This was treated at the end of [4]. For $n \geqslant 3$, the codimension is n and a basis for the identities of degree n is given by the $n! - n$ polynomials:

$$x_ix_1 \cdots x_{i-1}x_{i+1} \cdots x_n - x_ix_{\sigma(1)} \cdots x_{\sigma(i-1)}x_{\sigma(i+1)} \cdots x_{\sigma(n)} \tag{7.7}$$

where $\mathrm{id} \neq \sigma \in S_n$ satisfies $\sigma(i) = i$ and $i = 1, \cdots, n$. Using our notation (7.4) and if $K_i = \{\sigma \in S_n \mid \sigma(i) = i\}$ we obtain that the right ideal of $F(S_n)$ which corresponds to the subspace of identities of degree n that can be deduced from (7.6) is $F(S_n : K_1)$. This result holds for the identity (5.3) if $n \geqslant 4$.

The situation for the other two cases (5.4)–when $a = b = -1$ and (6.4) is similar to that described above for (5.3) and (6.2). The reason is that an algebra will satisfy (5.4)–when $a = b = -1$, (6.4) if and only if the opposite algebra will satisfy (5.3), (6.2) respectively, So it is clear that a basis for the identities of degree n, $n \geqslant 4$, that can be deduced, either from (5.4)–when $a = b = -1$ or from (6.4), can be obtained from (7.4) by reversing the order of the letters in the monomials. The corresponding right ideals in $F(S_n)$ will be $F(S_n : K_n)$.

Note that for all the cases considered above we have obtained bases for the multilinear identities of any degree n, $n \geqslant 3$. Let us see what can be said concerning the last three cases:

(4) The case (5.4) when $a = -1, b = 0$ is written as $[[x_1, x_2], x_3]$. This is an identity satisfied by the Grassmann algebra. It has been proved in [5] that in this case the codimensions of the subspaces of multilinear identities of degree n are 2^{n-1}. In particular for $n = 4$ the codimension is 8. Hence the dimension of the subspace of identities is 16. The computation done by the computer has given a matrix in row-echelon normal form of degree 16 and corresponding to this matrix we can write a basis for the subspace of identities of degree 4, but we shall not do so.

(5) The identity (1.3) has been obtained in §2 as one of the two 1-dimensional cases. Since the corresponding algebra is nilpotent of order 6, one may ask what are the identities of degrees 4 and 5. Again by the use of the computer, we have obtained a basis for the identities of degree 4 that we shall not write here.

The number of the base elements is 12 and this is also the value of the codimension. So here the unique unknown codimension is that of order 5.

(6) The other 1-dimensional case is that given by (1.4). The identity (1.4) is the so-called "standard identity of degree 3". Here we have obtained a basis of 15 elements for the subspace of multilinear identities of degree 4, which we prefer not to write here. So the codimension of order 4 is 9 and no more codimensions are known.

Concerning codimensions we may say that they are known for each n and for all the cases except the last one for $n > 4$ and the previous one for $n = 5$.

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