IDENTITIES INVOLVING THE COEFFICIENTS OF A CLASS OF DIRICHLET SERIES. VII

BY

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ABSTRACT. Let $a(n)$ be an arithmetical function, and consider the Riesz sum $A_\rho(x) = \Sigma_{n \leqslant x} a(n)(x - n)^\rho$. For $a(n)$ belonging to a certain class of arithmetical functions, $A_\rho(x)$ can be expressed in terms of an infinite series of Bessel functions. K. Chandrasekharan and R. Narasimhan have established this identity for the widest known range of ρ. Their proof depends upon equi-convergence theory of trigonometric series. An alternate proof is given here which uses only the classical theory of Bessel functions.

1. **Introduction.** Let $a(n)$ denote an arithmetical function. In the study of the average order of arithmetical functions, and in other related problems, an expression for

$$(1.1) \qquad \sum_{n \leqslant x} a(n)(x - n)^\rho$$

in terms of an infinite series of analytic functions is often very useful, where ρ assumes certain real values. See, for example, several papers of E. Landau [28], many results of G. H. Hardy [14, part 2], and a paper of K. Chandrasekharan and R. Narasimhan [6]. The case $\rho = 0$ is of particular importance. It is of interest to establish the identity for (1.1) for the widest range of ρ possible. The most significant result in this direction is a theorem of Chandrasekharan and Narasimhan [5, Theorem III]. In this result, $a(n)$ is generated by a Dirichlet series satisfying a functional equation involving a simple gamma factor, $\Gamma(s)$. The representation of (1.1) is then given in terms of an infinite series of ordinary Bessel functions. The method was also used by Chandrasekharan and Narasimhan [6, §7] and the author [3] to establish identities for (1.1) for arithmetical functions generated by Dirichlet series satisfying certain other functional equations involving the gamma function. The method of Chandrasekharan and

Narasimhan is quite deep and depends upon results of A. Zygmund on the equi-convergence of trigonometric integrals.

The purpose of this paper is to give a different proof of the aforementioned principal theorem of Chandrasekharan and Narasimhan [5, Theorem III]. In contrast to their proof, ours uses only well-known, classical results in the theory of Bessel functions. In fact, the method is merely an extension and elaboration of a method of Hardy and Landau ([15], [28, pp. 95–109], [14, pp. 329–343], [22, pp. 189, 225–232]) used in the case when $\rho = 0$ and $a(n) = r_2(n)$, the number of representations of n as the sum of two squares. The method was also used by Hardy ([13], [14, pp. 369–375]) to establish an identity for (1.1) when $\rho = 0$ and $a(n) = \tau(n)$, Ramanujan's arithmetical function. Lastly, the method was used by the author in his thesis [2] to establish identities for (1.1) when $\rho = 0$ and the Dirichlet series is attached to a cusp form. We remark that our hypotheses are slightly different from those of Chandrasekharan and Narasimhan [5]. However, all of the known, particular examples of arithmetical functions generated by Dirichlet series satisfying the functional equation (2.1) below are covered by both theorems.

2. **Definition and preliminary results.** We first indicate those arithmetical functions to be studied.

DEFINITION 1. Let $\{\lambda_n\}$ and $\{\mu_n\}$, $1 \leqslant n < \infty$, be two sequences of positive numbers strictly increasing to ∞, and let $\{a(n)\}$ and $\{b(n)\}$, $1 \leqslant n < \infty$, be two sequences of complex numbers not identically zero. Assume that the Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} b(n)\mu_n^{-s}$$

each converge in some half-plane of the complex s-plane, and let their abscissae of absolute convergence be denoted by σ_a and σ_a^*, respectively. If r is real, we say that ϕ and ψ satisfy the functional equation

(2.1) $\Gamma(s)\phi(s) = \Gamma(r-s)\psi(r-s)$

if there exists in the s-plane a domain D which is the exterior of a compact set S, such that in D a holomorphic function χ exists with the properties:

(i) $\chi(s) = \Gamma(s)\phi(s) \qquad (\sigma > \sigma_a)$

$= \Gamma(r-s)\psi(r-s) \quad (\sigma < r - \sigma_a^*),$

(ii) $\lim_{|t| \to \infty} \chi(\sigma + it) = 0,$

uniformly in every strip $-\infty < \sigma_1 \leqslant \sigma \leqslant \sigma_2 < \infty$. Here, and in the sequel, $\sigma = \text{Re}\,(s)$ and $t = \text{Im}\,(s)$.

We now list some results on Bessel functions to be used in the sequel. The ordinary Bessel function of order ν, $J_\nu(x)$, may be defined by [29, p. 40]

$$(2.2) \qquad J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n! \Gamma(\nu + n + 1)} \qquad (|x| < \infty).$$

We shall frequently use the property [29, p. 45]

$$(2.3) \qquad \frac{d}{dx}\{x^{-\nu/2} J_\nu(cx^{1/2})\} = -\tfrac{1}{2} cx^{-(\nu+1)/2} J_{\nu+1}(cx^{1/2}).$$

As x tends to $+\infty$, we have the asymptotic formula [29, p. 199]

$$(2.4) \qquad J_\nu(x) = (2/\pi x)^{1/2} \cos(x - \pi\nu/2 - \pi/4) + O(x^{-3/2}).$$

Lastly, we need two results on integrals of Bessel functions. Firstly, for $0 < 2\sigma < \nu + 3/2$ [9, p. 22],

$$(2.5) \qquad \int_0^\infty x^{2s-\nu-1} J_\nu(x)dx = \frac{2^{2s-\nu-1}\Gamma(s)}{\Gamma(\nu + 1 - s)}.$$

(In [9], the conditions on σ are given by $0 < 2\sigma < \nu + 1$. However, using (2.4), one can readily show by analytic continuation that the validity of (2.5) can be extended to $0 < 2\sigma < \nu + 3/2$.) Secondly, for $a > 0$ and $\nu > \mu > -1$ [9, p. 48], [29, p. 406],

$$\int_0^\infty x^{\mu-\nu+1} J_\mu(ax) J_\nu(x) dx$$

$$(2.6) \quad
\begin{aligned}
&= \frac{2^{\mu-\nu+1} a^\mu (1 - a^2)^{\nu-\mu-1}}{\Gamma(\nu - \mu)}, & & a < 1, \\
&= 0, & & a > 1, \\
&= 0, & & \nu > \mu + 1, \ a = 1, \\
&= 1/2, & & \nu = \mu + 1, \ a = 1.
\end{aligned}$$

3. **Main theorem.** For real ρ, let

$$A_\rho(x) = \frac{1}{\Gamma(\rho + 1)} \sideset{}{'}\sum_{\lambda_n < x} a(n)(x - \lambda_n)^\rho,$$

where the prime on the summation sign indicates that if $\rho = 0$ and $x = \lambda_m$, $a(m)$ is to be multiplied by $1/2$. We define $A_\rho^*(x)$ by replacing $a(n)$ by $b(n)$ and λ_n by μ_n above. Next, define

$$Q_\rho(x) = \frac{1}{2\pi i} \int_C \frac{\Gamma(s)\phi(s)}{\Gamma(s + \rho + 1)} x^{s+\rho} ds,$$

where C is a cycle with S on its interior. Similarly, put

$$Q_\rho^*(x) = \frac{1}{2\pi i} \int_{C^*} \frac{\Gamma(s)\psi(s)}{\Gamma(s + \rho + 1)} x^{s+\rho} ds,$$

where C^* is a cycle containing all singularities of $\Gamma(s)\psi(s)$ on its interior. Furthermore, let

$$P_\rho(x) = A_\rho(x) - Q_\rho(x) \quad \text{and} \quad P_\rho^*(x) = A_\rho^*(x) - Q_\rho^*(x).$$

Lastly, for $x, y > 0$, ρ real, and an arbitrary nonnegative integer ν, let

(3.1) $$F_{\nu,\rho}(x, y) = P_\nu^*(y)(x/y)^{(r+\rho+\nu)/2}J_{r+\rho+\nu}(2\{xy\}^{1/2}).$$

In our first version of the main theorem, we shall assume that the singularities of $\psi(s)$ lie in the right half-plane $\sigma > 0$. All known examples satisfy this hypothesis. In the second version of the main theorem, we show that this requirement on ψ may be deleted in most cases.

THEOREM 1. *Let ϕ and ψ satisfy Definition 1. Assume that the singularities of $\psi(s)$ lie in $\sigma > 0$. Suppose that for $2\sigma_a^* - r - 3/2 < q \leqslant 2\sigma_a^* - r - 1/2$,*

(3.2) $$\sum_{0 \leqslant j < 2\sigma_a - r - 1/2} F_{j,q}(x, N) \longrightarrow 0,$$

uniformly on each finite interval $0 < x_1 \leqslant x \leqslant x_2$ as N tends to ∞. Then for $\rho > 2\sigma_a^* - r - 3/2$,

(3.3) $$P_\rho(x) = \sum_{n=1}^\infty b(n)(x/\mu_n)^{(r+\rho)/2}J_{r+\rho}(2\{\mu_n x\}^{1/2}).$$

If $\rho > 0$, the infinite series on the right side of (3.3) converges uniformly on $[x_1, x_2]$. If $\rho \leqslant 0$, the series converges uniformly on $[x_1, x_2]$, provided that $[x_1, x_2]$ contains no member of $\{\lambda_n\}$. If $\rho = 0$, the series converges boundedly on $[x_1, x_2]$.

Before proving Theorem 1, we shall discuss the hypothesis (3.2). From (2.4) and (3.1), we see that (3.2) is equivalent to showing that for each integer j, $0 \leqslant j \leqslant 2\sigma_a - r - 1/2$,

(3.4) $$P_j^*(N) = o(N^{(r+q+j)/2+1/4}),$$

as N tends to ∞, uniformly on $[x_1, x_2]$, where $2\sigma_a^* - r - 3/2 < q \leqslant 2\sigma_a^* - r - 1/2$. In most applications, (3.4) is quite easy to show. One does not normally need sophisticated estimates, as the examples at the end of this paper will show. In the case $j = 0$, general theorems have been proven by Landau ([21], [28, pp. 30–64]) and by Chandrasekharan and Narasimhan [6, Theorem 4.1] that give O-estimates for $P_0^*(N)$ when the singularities of ψ are at most

poles. For our purposes, it is unfortunate that these results were not proven for $j > 0$ as well, for Landau's method of finite differences used in [21] and [6] can be easily applied in the more general situation.

Now, for $j > 2\sigma_a - r - 1/2$, we see from (3.3) (with P_j replaced by P_j^*) and (2.4) that

$$(3.5) \qquad P_j^*(N) = O(N^{(r+j)/2 - 1/4}),$$

uniformly on $[x_1, x_2]$. Using the aforementioned results of either Landau or Chandrasekharan and Narasimhan along with (3.5) and a convexity theorem, $P_j^*(N)$ can be estimated for those j (if any) such that $1 \leqslant j \leqslant 2\sigma_a - r - 1/2$. The convexity theorem that one needs is a slight generalization of a convexity theorem of M. Riesz [4, p. 13].

Observe that the corresponding theorem of Chandrasekharan and Narasimhan [5, Theorem III] does not have hypothesis (3.2) but another condition [5, equation (48)] instead. The different hypotheses are probably related.

PROOF OF THEOREM 1.   First, a proof of (3.3) for $\rho > 2\sigma_a^* - r - 1/2$ is not difficult [5, Lemma 5]. (The method used in [5], as noted there, is apparently due to E. Hecke [16]. We remark that the method was subsequently improved and generalized by A. L. Dixon and W. L. Ferrar [7] and T. M. Apostol [1].) Hence, we shall assume for the remainder of the proof that $\rho = q$, where $2\sigma_a^* - r - 3/2 < q \leqslant 2\sigma_a^* - r - 1/2$.

Define for x, $N > 0$ and each nonnegative integer j,

$$I_j(x, N) = \int_0^N P_j^*(u)(x/u)^{(r+q+j+1)/2} J_{r+q+j+1}(2\{ux\}^{1/2}) du.$$

From the definition of $P_j^*(u)$, it is not difficult to see that $P_j^*(u)$ is continuous at $u = 0$ Thus, from (2.2), I_j is well defined. From the definition of $P_j^*(u)$, it is also clear that $\int P_j^*(u) du = P_{j+1}^*(u)$, $j \geqslant 0$. Hence, integrating by parts with the aid of (2.3), we find that

$$I_0(x, N) = F_{1,q}(x, N) - F_{1,q}(x, 0) + I_1(x, N).$$

Choose a positive integer m such that

(A1) $m > 2\sigma_a - r - 1/2$,

(A2) $m + r > -1$,

(A3) all singularities of $\Gamma(s)\psi(s)$ lie in the half-plane $\sigma \geqslant -m$.

After m such integrations by parts, we get

$$(3.6) \qquad I_0(x, N) = \sum_{j=1}^m \{F_{j,q}(x, N) - F_{j,q}(x, 0)\} + I_m(x, N).$$

We have assumed that (3.3) is valid for $\rho > 2\sigma_a^* - r - 1/2$. Hence, by (A1) we have

$$(3.7) \qquad P_m^*(u) = \sum_{n=1}^{\infty} a(n)(u/\lambda_n)^{(r+m)/2} J_{r+m}(2\{\lambda_n u\}^{1/2}).$$

Substituting (3.7) into $I_m(x, N)$ and then inverting the order of summation and integration by absolute convergence, we arrive at

$$I_m(x, N) = x^{(r+q+m+1)/2} \sum_{n=1}^{\infty} a(n)\lambda_n^{-(r+m)/2}$$

$$\times \int_0^N u^{-(q+1)/2} J_{r+m}(2\{\lambda_n u\}^{1/2}) J_{r+q+m+1}(2\{ux\}^{1/2}) du.$$

Observe that the integrals above exist by (2.2) and (A2). Letting $v = 2(ux)^{1/2}$, we get

$$I_m(x, N) = (2x)^q \sum_{n=1}^{\infty} a(n)(x/\lambda_n)^{(r+m)/2}$$

(3.8)

$$\times \left\{\int_0^{\infty} - \int_{2(Nx)^{1/2}}^{\infty}\right\} v^{-q} J_{r+m}(\{\lambda_n/x\}^{1/2} v) J_{r+q+m+1}(v) dv.$$

We must show that the integrals above exist. Suppose that $[x_1, x_2]$ is an interval containing at least one element λ_j with $a(j) \neq 0$. For $\rho > 2\sigma_a^* - r - 1/2$, the right side of (3.3) converges absolutely and uniformly on $[x_1, x_2]$. Thus, the left side of (3.3) is continuous on $[x_1, x_2]$. Since $[x_1, x_2]$ contains λ_j with $a(j) \neq 0$, then necessarily $\rho > 0$, which implies that $2\sigma_a^* - r - 1/2 \geqslant 0$. Thus, in all cases $q > -1$. Hence, using (2.4), we see that the infinite integrals on the right side of (3.8) converge.

We now apply (2.6) with $a = \{\lambda_n/x\}^{1/2}$, $\mu = r + m$, and $\nu = r + q + m + 1$. By (A2) and the fact that $q > -1$, we see that $\nu > \mu > -1$. Hence, from (3.8),

$$(3.9) \qquad I_m(x, N) = \frac{1}{\Gamma(q + 1)} \sum_{\lambda_n < x}' a(n)(x - \lambda_n)^q - K(x, N),$$

where

$$K(x, N) = (2x)^q \sum_{n=1}^{\infty} a(n)(x/\lambda_n)^{(r+m)/2}$$

(3.10)

$$\times \int_{2(Nx)^{1/2}}^{\infty} v^{-q} J_{r+m}(\{\lambda_n/x\}^{1/2} v) J_{r+q+m+1}(v) dv.$$

Putting (3.9) into (3.6), we have thus shown that

$$(3.11) \qquad I_0(x, N) = \sum_{j=1}^{m} \{F_{j,q}(x, N) - F_{j,q}(x, 0)\} + A_q(x) - K(x, N).$$

On the other hand, from (2.3) and an integration by parts, we have

$$I_0(x, N) = -\int_0^N P_0^*(u)\frac{d}{du}\{(x/u)^{(r+q)/2}J_{r+q}(2\{ux\}^{1/2})\}du$$

$$= -F_{0,q}(x, N) + F_{0,q}(x, 0) + \int_0^N (x/u)^{(r+q)/2}J_{r+q}(2\{ux\}^{1/2})dP_0^*(u)$$

(3.12)
$$= -F_{0,q}(x, N) + F_{0,q}(x, 0) + \sum_{\mu_n \leqslant N} b(n)(x/\mu_n)^{(r+q)/2}J_{r+q}(2\{\mu_n x\}^{1/2})$$

$$-\int_0^N (x/u)^{(r+q)/2}J_{r+q}(2\{ux\}^{1/2})\left\{\frac{1}{2\pi i}\int_{C'}\psi(s)u^{s-1}ds\right\}du,$$

where C' is a cycle encircling the singularities of $\psi(s)$, and since the singularities of $\psi(s)$ lie in $\sigma > 0$, C' may be chosen to lie entirely in $\sigma > 0$. Combining (3.11) and (3.12), we have

$$A_q(x) + \sum_{j=0}^m \{F_{j,q}(x, N) - F_{j,q}(x, 0)\}$$

(3.13)
$$+\int_0^N (x/u)^{(r+q)/2}J_{r+q}(2\{ux\}^{1/2})\left\{\frac{1}{2\pi i}\int_{C'}\psi(s)u^{s-1}ds\right\}du$$

$$= \sum_{\mu_n \leqslant N} b(n)(x/\mu_n)^{(r+q)/2}J_{r+q}(2\{\mu_n x\}^{1/2}) + K(x, N).$$

We now let N tend to ∞ in (3.13). We shall examine each expression individually.

Let
$$I = \int_0^\infty (x/u)^{(r+q)/2}J_{r+q}(2\{ux\}^{1/2})\left\{\frac{1}{2\pi i}\int_{C'}\psi(s)u^{s-1}ds\right\}du.$$

Put $v = 2(ux)^{1/2}$ and formally invert the order of integration to obtain

(3.14) $$I = \frac{1}{2\pi i}\int_{C'}\psi(s)x^{r+q-s}2^{r+q+1-2s}ds\int_0^\infty v^{2s-r-q-1}J_{r+q}(v)dv.$$

By (2.4), the inner integral converges "at $\cdot\infty$" provided that $q > 2\sigma - r - 3/2$. Now, by hypothesis, $q > 2\sigma_a^* - r - 3/2$. Since the singularities of $\psi(s)$ have real parts no greater than σ_a^*, C' may be chosen so that, indeed, $q > 2\sigma - r - 3/2$ for all $s \in C'$. The inner integral converges "at 0," because $\sigma > 0$ for all $s \in C'$. By a classical theorem [18, p. 349], the inversion in order of integration is justified.

We now evaluate the inner integral of (3.14) by the use of (2.5). By the discussion immediately above, the requirements on σ and $\nu = r + q$ are fulfilled, and so we obtain

(3.15) $$I = \frac{1}{2\pi i}\int_{C'}\frac{\Gamma(s)\psi(s)}{\Gamma(r+q+1-s)}x^{r+q-s}ds.$$

Next, we calculate $F_{j,q}(x, 0)$ for $0 \leqslant j \leqslant m$. From (3.1) and (2.2), we have

$$\lim_{y \to 0} F_{j,q}(x, y) = -\lim_{y \to 0} Q_j^*(y)(x/y)^{(r+q+j)/2} J_{r+q+j}(2\{xy\}^{1/2})$$

(3.16)

$$= \frac{x^{r+q+j}}{\Gamma(r+q+j+1)} \lim_{y \to 0} \frac{1}{2\pi i} \int_{C^*} \frac{\Gamma(s)\psi(s)}{\Gamma(s+j+1)} y^{s+j} ds$$

$$= -\frac{x^{r+q+j}}{\Gamma(r+q+j+1)} \lim_{y \to 0} \sum_{\mu=0}^{j} \frac{(-1)^\mu \psi(-\mu) y^{j-\mu}}{\mu! \Gamma(-\mu+j+1)}$$

$$= -\frac{x^{r+q+j}(-1)^j \psi(-j)}{\Gamma(r+q+j+1)j!},$$

since the singularities of $\psi(s)$ lie in $\sigma > 0$. Let C'' be the image of C under the transformation $s = r - z$. Then, from (3.15), (3.16), (A3), and (2.1),

$$I - \sum_{j=0}^{m} F_{j,q}(x, 0) = \frac{1}{2\pi i} \int_{C'} \frac{\Gamma(s)\psi(s)}{\Gamma(r+q+1-s)} x^{r+q-s} ds$$

(3.17)

$$+ \sum_{j=0}^{m} \frac{(-1)^j \psi(-j)}{\Gamma(r+q+j+1)j!} x^{r+q+j}$$

$$= \frac{1}{2\pi i} \int_{C''} \frac{\Gamma(s)\psi(s)}{\Gamma(r+q+1-s)} x^{r+q-s} ds$$

$$= -\frac{1}{2\pi i} \int_{C} \frac{\Gamma(z)\phi(z)}{\Gamma(z+q+1)} x^{z+q} dz$$

$$= -Q_q(x).$$

For $j > 2\sigma_a - r - 1/2$, we deduce from (3.5) and (2.4) that

(3.18)

$$F_{j,q}(x, N) = O(N^{(r+j)/2-1/4-(r+q+j)/2-1/4})$$

$$= O(N^{-(q+1)/2}) = o(1),$$

since $q > -1$, uniformly on any interval $0 < x_1 \leqslant x \leqslant x_2 < \infty$, as N tends to ∞.

Letting N tend to ∞ in (3.13) and using (3.2), (3.17), and (3.18), we have so far shown that

$$P_q(x) = \sum_{n=1}^{\infty} b(n)(x/\mu_n)^{(r+q)/2} J_{r+q}(2\{\mu_n x\}^{1/2}) - \lim_{n \to \infty} K(x, N).$$

Lastly, we must show that $K(x, N)$ tends to 0 as N tends to ∞. We must also prove the convergence properties claimed for the infinite series on the right side of (3.3). So far, all expressions which we have shown tend to 0 as

N tends to ∞ tend to 0 uniformly on $[x_1, x_2]$.

From (2.4), we have

$$J_{r+m}(\{\lambda_n/x\}^{1/2}v)J_{r+q+m+1}(v)$$

$$= \frac{2}{\pi v}\left(\frac{x}{\lambda_n}\right)^{1/4}\cos(\{\lambda_n/x\}^{1/2}v + c_1)\cos(v + c_2)$$

(3.19)

$$+ O((x/\lambda_n)^{1/4}v^{-2}) + O((x/\lambda_n)^{3/4}v^{-2}) + O((x/\lambda_n)^{3/4}v^{-3})$$

$$= \frac{2}{\pi v}\left(\frac{x}{\lambda_n}\right)^{1/4}\cos(\{\lambda_n/x\}^{1/2}v + c_1)\cos(v + c_2) + O(\lambda_n^{-1/4}v^{-2}),$$

uniformly for $0 < x_1 \leqslant x \leqslant x_2 < \infty$, where $c_1 = -\pi(r + m)/2 - \pi/4$ and $c_2 = -\pi(r + q + m + 1)/2 - \pi/4$. The contribution of the O-term on the far right side of (3.19) to $K(x, N)$ given by (3.10) is

$$O\left(\sum_{n=1}^{\infty} |a(n)|\lambda_n^{-(r+m)/2-1/4}\int_{2(Nx)^{1/2}}^{\infty} v^{-q-2}dv\right) = O(N^{-(q+1)/2}) = o(1),$$

by (A1) and the fact that $q > -1$, uniformly on $0 < x_1 \leqslant x \leqslant x_2 < \infty$. The contribution of the main term in (3.19) to $K(x, N)$ is $S_1 + S_2$, where for $j = 1, 2$,

$$S_j = \frac{(2x)^q}{\pi}\sum_{n=1}^{\infty} a(n)(x/\lambda_n)^{(r+m)/2+1/4}$$

$$\times \int_{2(Nx)^{1/2}}^{\infty} v^{-q-1}\cos\{(1 + (-1)^{j+1}\{\lambda_n/x\}^{1/2})v + c_{2+j}\}dv,$$

where $c_3 = c_2 + c_1$ and $c_4 = c_2 - c_1$.

We first examine S_1. Integrating by parts, we have

$$\int_{2(Nx)^{1/2}}^{\infty} v^{-q-1}\cos\{(1 + \{\lambda_n/x\}^{1/2})v + c_3\}dv,$$

(3.20)

$$= \frac{v^{-q-1}\sin\{(1 + \{\lambda_n/x\}^{1/2})v + c_3\}}{1 + \{\lambda_n/x\}^{1/2}}\bigg|_{2(Nx)^{1/2}}^{\infty}$$

$$+ \frac{(q+1)}{1 + \{\lambda_n/x\}^{1/2}}\int_{2(Nx)^{1/2}}^{\infty} v^{-q-2}\sin\{(1 + \{\lambda_n/x\}^{1/2})v + c_3\}dv$$

$$= O(N^{-(q+1)/2}) + O(N^{-(q+1)/2}) = O(N^{-(q+1)/2}),$$

uniformly on $0 < x_1 \leqslant x \leqslant x_2 < \infty$. Using (A1), we conclude that $S_1 = O(N^{-(q+1)/2}) = o(1)$, uniformly on $[x_1, x_2]$.

Now examine S_2. First, if $q > 0$, a trivial estimate yields

$$|S_2| \leqslant \frac{(2x)^q}{\pi} \sum_{n=1}^{\infty} |a(n)|(x/\lambda_n)^{(r+m)/2+1/4} \int_{2(Nx)^{1/2}}^{\infty} v^{-q-1}dv$$

$$= O(N^{-q/2}) = o(1),$$

by (A1), uniformly for $0 < x_1 \leqslant x \leqslant x_2 < \infty$. Now assume that $q \leqslant 0$. If no member of $\{\lambda_n\}$, $1 \leqslant n < \infty$, is contained in $[x_1, x_2]$, then an integration by parts along the same lines as that in (3.20) readily gives $S_2 = O(N^{-(q+1)/2})$ $= o(1)$, uniformly on $[x_1, x_2]$.

Lastly, we must examine S_2 in the case when $q = 0$ and $[x_1, x_2]$ contains at least one member of $\{\lambda_n\}$. In this instance, note that $c_4 = -\pi/2$. If $\lambda_n = x$, observe that the corresponding integrand in S_2 is zero. If $\lambda_n \neq x$, set $u = |1 - \{\lambda_n/x\}^{1/2}|v$ and obtain

$$\int_{2(Nx)^{1/2}}^{\infty} v^{-1} \cos \{(1 - \{\lambda_n/x\}^{1/2})v - \pi/2\}dv$$

$$= \operatorname{sgn}(1 - \{\lambda_n/x\}^{1/2}) \int_{2|1-\{\lambda_n/x\}^{1/2}|(Nx)^{1/2}}^{\infty} \frac{\sin u}{u} du.$$

In either case, the integrals in S_2 are uniformly bounded on $[x_1, x_2]$ and tend to 0 as N tends to ∞. By (A1), S_2 converges absolutely and uniformly on $[x_1, x_2]$. Hence, in conclusion, S_2 tends to 0 boundedly on $[x_1, x_2]$ as N tends to ∞. This completes the proof.

The hypothesis in Theorem 1 on the singularities of ψ is for technical reasons. We can alter the method of proof to dispose of this hypothesis. However, any alteration will impose an upper bound for m. Note that assumptions (A1)–(A3) are lower bounds on m. For the next theorem, then, we must be able to choose a positive integer m satisfying (A1)--(A3) and

(A4) $m < r + q + 1/2$.

Put $C^* = C_L \cup C_R$, where C_L is a cycle encircling the singularities of $\Gamma(s)\psi(s)$ in the half-plane $\sigma \leqslant 0$, and C_R is a cycle in the half-plane $\sigma > 0$ encircling the singularities of $\psi(s)$ in $\sigma > 0$. Let

$$T_\rho^*(x) = A_\rho^*(x) - \frac{1}{2\pi i} \int_{C_R} \frac{\Gamma(s)\psi(s)}{\Gamma(s+\rho+1)} x^{s+\rho}dx.$$

For $x, y > 0$, ρ real, and an arbitrary nonnegative integer ν, put

$$G_{\nu,\rho}(x, y) = T_\nu^*(y)(x/y)^{(r+\rho+\nu)/2}J_{r+\rho+\nu}(2\{xy\}^{1/2}).$$

Since $C_R \subset \{s: \sigma > 0\}$, we see that $T_\nu^*(0) = 0$ for $\nu \geqslant 0$. Thus, from (2.2), for $\nu \geqslant 0$,

(3.21) $\lim_{y \to 0} G_{\nu,\rho}(x, y) = 0.$

THEOREM 2. *Let ϕ and ψ satisfy Definition 1. Suppose that a positive integer m can be chosen to satisfy* (A1)–(A4). *Suppose that for $2\sigma_a^* - r - 3/2 < q \leqslant 2\sigma_a^* - r - 1/2$,*

$$\sum_{0 \leqslant j < 2\sigma_a - r - 1/2} G_{j,q}(x, N) \to 0,$$

uniformly on any finite interval $0 < x_1 \leqslant x \leqslant x_2$ as N tends to ∞. Then for $\rho > 2\sigma_a^* - r - 3/2$, (3.3) is valid as well as the statements immediately following (3.3).

PROOF. Proceed in exactly the same fashion as in the proof of Theorem 1 with $P_j^*(u)$ replaced by $T_j^*(u)$ and $F_{j,q}(x, y)$ replaced by $G_{j,q}(x, y)$. When we reach the equation corresponding to (3.8), however, we will have on the right side the additional expression

$$L(x, N) \equiv \int_0^N (x/u)^{(r+q+m+1)/2} J_{r+q+m+1}(2\{ux\}^{1/2})$$

$$\times \left\{ \frac{1}{2\pi i} \int_{C_L} \frac{\Gamma(s)\psi(s)}{\Gamma(s + m + 1)} u^{s+m} ds \right\} du.$$

Thus, corresponding to (3.13), we have, with the use of (3.21),

$$A_q(x) + \sum_{j=0}^m G_{j,q}(x, N) + L(x, N)$$

$$+ \int_0^N (x/u)^{(r+q)/2} J_{r+q}(2\{ux\}^{1/2}) \left\{ \frac{1}{2\pi i} \int_{C_R} \psi(s) u^{s-1} ds \right\} du$$

$$= \sum_{\mu_n \leqslant N} b(n)(x/\mu_n)^{(r+q)/2} J_{r+q}(2\{\mu_n x\}^{1/2}) + K(x, N).$$

As before, we now let N tend to ∞. The analysis is the same, except that in lieu of calculating $F_{j,q}(x, 0)$, we must examine

$$L \equiv \int_0^\infty (x/u)^{(r+q+m+1)/2} J_{r+q+m+1}(2\{ux\}^{1/2})$$

$$\times \left\{ \frac{1}{2\pi i} \int_{C_L} \frac{\Gamma(s)\psi(s)}{\Gamma(s + m + 1)} u^{s+m} ds \right\} du.$$

Put $v = 2(ux)^{1/2}$ and formally invert the order of integration to obtain

(3.22)
$$L = \frac{1}{2\pi i} \int_{C_L} \frac{\Gamma(s)\psi(s)}{\Gamma(s + m + 1)} x^{r+q-s} 2^{r+q-m-2s} ds$$

$$\times \int_0^\infty v^{-r-q+m+2s} J_{r+q+m+1}(v) dv.$$

The inner integral converges provided that $2m + 2\sigma + 2 > 0$ and $-r - q + m + 2\sigma - 1/2 < 0$. By (A3), we may choose C_L so that the former condition

holds. By (A4), we may choose C_L so that the latter condition is fulfilled. The inversion in order of integration is justified, as before, by a theorem in [18, p. 349]. We now evaluate the inner integral in (3.22) by (2.5). Accordingly, we find that

$$L = \frac{1}{2\pi i} \int_{C_L} \frac{\Gamma(s)\psi(s)}{\Gamma(r+q+1-s)} x^{r+q-s} ds.$$

Thus, corresponding to (3.17), we now have

$$\frac{1}{2\pi i} \int_{C_R} \frac{\Gamma(s)\psi(s)}{\Gamma(r+q+1-s)} x^{r+q-s} ds + \frac{1}{2\pi i} \int_{C_L} \frac{\Gamma(s)\psi(s)}{\Gamma(r+q+1-s)} x^{r+q-s} ds$$

$$= \frac{1}{2\pi i} \int_{C''} \frac{\Gamma(s)\psi(s)}{\Gamma(r+q+1-s)} x^{r+q-s} ds = -Q_q(x).$$

The remainder of the proof is exactly as before, and so the proof of Theorem 2 is finished.

4. Examples. We give just three short examples by way of illustration of Theorem 1.

EXAMPLE 1. Let

$$f(\tau) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n \tau / \lambda} \quad (\operatorname{Im} \tau > 0)$$

be a modular form of dimension $-\kappa$, $\kappa > 0$, and Stufe λ, $\lambda > 0$, which vanishes at all rational cusps of a fundamental region. R. A. Rankin [24] has shown that there exists a positive number α such that

$$\sum_{n \leqslant x} |a(n)|^2 = \alpha x^\kappa + O(x^{\kappa - 2/5})$$

as x tends to ∞. By the Cauchy-Schwarz inequality, it follows that if

$$\phi(s) = (2\pi/\lambda)^{-s} \sum_{n=1}^{\infty} a(n) n^{-s},$$

then $\sigma_a \leqslant (\kappa + 1)/2$. Furthermore, ϕ can be analytically continued to an entire function which satisfies the functional equation (2.1) with $\psi = \phi$ and $r = \kappa$. Now, $2\sigma_a - r - 3/2 \leqslant 2(\kappa + 1)/2 - \kappa - 3/2 = -1/2$. By (3.2) and (3.4), we then need to show that

(4.1) $$P_0^*(N) = A_0(N) = o(N^{(\kappa + q)/2 + 1/4})$$

for $q > -1/2$. Rankin [24] has shown that

$$A_0(N) = O(N^{\kappa/2 - 1/5}),$$

which certainly implies (4.1). (In fact, an earlier, easier result of Hecke's [17],

$$A_0(N) = O(N^{\kappa/2}\log N),$$

suffices to show (4.1).) Replacing x by $2\pi x/\lambda$ in (3.3), we deduce that for $\rho > -1/2$,

$$\frac{1}{\Gamma(\rho + 1)}\sum_{n\leqslant x}{}' a(n)(x - n)^\rho = (\lambda/2\pi)^\rho \sum_{n=1}^{\infty} a(n)(x/n)^{(\kappa+\rho)/2}J_{\kappa+\rho}(4\pi\{nx\}^{1/2}/\lambda).$$

EXAMPLE 2. Let $Q(x, y)$ be a positive definite quadratic form with determinant D. The Epstein zeta-function $\zeta(s, Q)$ is defined for $\sigma > 1$ by

$$\zeta(s, Q) = \sum_{m,n=-\infty}^{\infty}{}' \{Q(m, n)\}^{-s},$$

where the prime on the summation sign indicates that the term corresponding to $m = n = 0$ is omitted from the summation. P. Epstein [8] has shown that $\zeta(s, Q)$ has an analytic continuation into the entire complex plane that is holomorphic save for a simple pole at $s = 1$ with residue $\pi D^{-1/2}$. Furthermore, $\zeta(s, Q)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(s, Q) = D^{-1/2}\pi^{s-1}\Gamma(1 - s)\zeta(1 - s, Q^{-1}),$$

where Q^{-1} is the inverse of Q. Also, $\zeta(0, Q) = -1$. Let $r(\mu, Q)$ denote the number of distinct pairs (m, n) such that $Q(m, n) = \mu$. Now $2\sigma_a - r - 3/2 = -1/2$. Thus, by (3.2) and (3.4), we must show that

(4.2) $$P_0(N) = o(N^{(1+q)/2+1/4})$$

for $q > -1/2$. By a classical theorem of Gauss [10], $P_0(N) = O(N^{1/2})$, which implies (4.2). Hence, replacing x by πx in (3.3), we deduce that for $\rho > -1/2$,

(4.3) $$\frac{1}{\Gamma(\rho + 1)}\sum_{\mu<x}{}' r(\mu, Q)(x - \mu)^\rho = \frac{\pi D^{-1/2}x^{\rho+1}}{\Gamma(\rho + 2)} - \frac{x^\rho}{\Gamma(\rho + 1)}$$

$$+ D^{-1/2}\pi^{-\rho}\sum_{\mu>0} r(\mu, Q^{-1})(x/\mu)^{(\rho+1)/2}J_{\rho+1}(2\pi\{\mu x\}^{1/2}),$$

where the values for μ are arranged in increasing order of magnitude.

For $\rho = 0$, (4.3) was first given by Voronoï [25]. Landau ([19], [20], [28, pp. 11–29]) was the first to prove corresponding identities for a much wider class of Epstein zeta-functions associated with positive definite quadratic forms of several variables. See also Hardy's paper ([12], [14, pp. 294–315]) and Walfisz's thesis [26]. Finally, the best possible result, with reference to ρ, was established by Chandrasekharan and Narasimhan [5]. This result is also covered by our Theorem 1, but we have chosen to work out the details for only a simple case.

EXAMPLE 3. Let $K = Q(\sqrt{-D})$ be an imaginary quadratic field with

discriminant D. Then the Dedekind zeta-function $\zeta_K(s)$ is defined for $\sigma > 1$ by $\zeta_K(s) = \Sigma_{n=1}^{\infty} F(n)n^{-s}$, where $F(n)$ is the number of nonzero, integral ideals of norm n in K. $\zeta_K(s)$ can be analytically continued into the entire complex plane where $\zeta_K(s)$ is holomorphic except for a simple pole of residue λ at $s = 1$. Furthermore, $\zeta_K(s)$ satisfies the functional equation [23, p. 66]

$$(\sqrt{-D}/2\pi)^s \Gamma(s)\zeta_K(s) = (\sqrt{-D}/2\pi)^{1-s}\Gamma(1-s)\zeta_K(1-s).$$

Again, $2\sigma_a - r - 3/2 = -1/2$. Since $P_0(N) = O(N^{1/3})$ [23, p. 131], we see from (3.4) that (3.2) holds. Hence replacing x by $2\pi x/\sqrt{-D}$ in (3.3), we deduce that for $\rho > -1/2$,

(4.4)
$$\frac{1}{\Gamma(\rho+1)} \sum_{n \leqslant x}' F(n)(x-n)^\rho = \frac{\lambda x^{\rho+1}}{\Gamma(\rho+2)} + \frac{\zeta_K(0)x^\rho}{\Gamma(\rho+1)}$$
$$+ (\sqrt{-D}/2\pi)^\rho \sum_{n=1}^{\infty} F(n)(x/n)^{(\rho+1)/2} J_{\rho+1}(4\pi\{-nx/D\}^{1/2}).$$

For $\rho \geqslant 0$, (4.4) was first proven by Walfisz ([26], [27]).

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