A HOMOTOPY THEORY OF PRO-SPACES

BY

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ABSTRACT. The category of towers of spaces, $\ldots \to X_{s+1} \to X_s \to \ldots \to X_0$, viewed as pro-spaces, appears to be useful in the study of the relation between homology and homotopy of nonsimply connected spaces. We show that this category admits the structure of a closed model category, in the sense of Quillen; notions of fibration, cofibration, and weak equivalence are defined and shown to satisfy fundamental properties that the corresponding notions satisfy in the category of spaces. This enables one to develop a "homotopy theory" for pro-spaces.

1. **Introduction.** Recent work on the relation between homology and homotopy of nonsimply connected spaces (e.g. [2], [3], and unpublished results of E. Dror and W. Dwyer) involves towers of spaces, $\cdots \to X_{s+1} \to X_s \to \cdots \to X_0$, viewed as pro-spaces. It would be helpful to be able to treat the category of such towers, which we denote by tow-S, like the category of spaces. In this paper we show that notions of *fibration, cofibration,* and *weak equivalence* can be defined for tow-S which give the category the structure of a *closed model category,* in the sense of Quillen [5]; thus one can "do homotopy theory" in tow-S.

In §2 we define the category tow-S, which is just a subcategory of the category of pro-spaces, and elucidate the maps in the category. Next (§3) we recall the definition of a closed model category, listing the axioms that fibrations, cofibrations, and weak equivalences must satisfy; topological spaces and simplicial sets are familiar examples of closed model categories. The definitions of fibration, cofibration, and weak equivalence for tow-S are given in §4, and the proof that tow-S is a closed model category with these definitions occupies §§5–8.

We work over the category S of simplicial sets [4], but the definitions and proofs are not combinatorial.

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research and writing of this paper, and William Dwyer for many helpful conversations.

2. **Pro-spaces.** In this section we introduce the category tow-S.

2.1. DEFINITION. The category tow-S is the category whose objects are *towers* in S (the category of simplicial sets),

$$\cdots \longrightarrow X_{s+1} \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_0,$$

written $\{X_s\}$, and whose maps are given by

$$\mathrm{Hom}_{\mathrm{tow}\text{-}S}(\{X_s\}, \{Y_s\}) = \varprojlim_j \varinjlim_i \mathrm{Hom}_S(X_i, Y_j).$$

The reader familiar with Artin-Mazur [1] will recognize that tow-S is the *full subcategory of* pro-S *with the index restricted to the nonnegative integers.*

A map $\{X_s\} \longrightarrow \{Y_s\}$ in tow-S is best thought of as a *compatible system of maps* $\{X_t \longrightarrow Y_s\}_s$, modulo the relation that $X_t \longrightarrow Y_s$ and $X_u \longrightarrow Y_s$ are equivalent if for some large w the maps $X_w \longrightarrow X_t \longrightarrow Y_s$ and $X_w \longrightarrow X_u \longrightarrow Y_s$ are the same. In particular any *level map*, i.e. tower of maps

$$\begin{array}{ccccccc}
\cdots \longrightarrow & X_{s+1} & \longrightarrow & X_s & \longrightarrow \cdots \longrightarrow & X_0 \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & Y_{s+1} & \longrightarrow & Y_s & \longrightarrow \cdots \longrightarrow & Y_0
\end{array}$$

in S, represents a map $\{X_s\} \longrightarrow \{Y_s\}$ in tow-S. Not every map $\{X_s\} \longrightarrow \{Y_s\}$ can be so presented, however. Instead we have

2.2. DEFINITION. Let $\psi \colon \{X_s\} \longrightarrow \{Y_s\}$ be a map in tow-S. We say that the level map $\{f_s \colon X'_s \longrightarrow Y'_s\}$ is a level representative of ψ if there are equivalences in tow-S, $\{X_s\} \approx \{X'_s\}$ and $\{Y_s\} \approx \{Y'_s\}$ such that the following diagram in tow-S commutes:

$$\begin{array}{ccc}
\{X_s\} & \approx & \{X'_s\} \\
\psi \downarrow & & \downarrow \{f_s\} \\
\{Y_s\} & \approx & \{Y'_s\}
\end{array}$$

Thus for example the identity level map $\{\mathrm{id} \colon X_s \longrightarrow X_s\}$ is a level representative of any equivalence $\{X_s\} \approx \{Y_s\}$. Clearly any map in tow-S has a level representative, obtained simply by taking a subtower of the domain. We shall see in §4, however, that the full generality of Definition 2.2 is useful.

3. **Model categories.** In this section we recall Quillen's definition of closed model category and observe that simplicial sets and topological spaces form closed model categories.
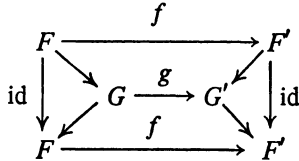
3.1. DEFINITION [6]. A *closed model category* is a category C to-

gether with three classes of maps in C, called *fibrations, cofibrations,* and *weak equivalences,* satisfying the following axioms:

CM1. C is closed under finite direct and inverse limits.

CM2. If f and g are maps in C such that gf is defined, and two of f, g, and gf are weak equivalences, so is the third.

CM3. If a map f is a retract of a map g (i.e. if the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\ f\ } & F' \\
\text{id}\downarrow\ \searrow\ G\xrightarrow{\ g\ }G'\ \swarrow\ \downarrow\text{id} \\
F & \xrightarrow{\ f\ } & F'
\end{array}
$$

commutes), and g is a fibration, cofibration, or weak equivalence, then so is f.

CM4. Given the following solid arrow diagram in C

$$
\begin{array}{ccc}
A & \longrightarrow & E \\
i\downarrow & \nearrow & \downarrow p \\
X & \longrightarrow & B
\end{array}
$$
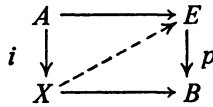
in which i is a cofibration and p is a fibration, the dotted arrow exists if either

(i) p is a weak equivalence, or

(ii) i is a weak equivalence.

CM5. Any map in C can be factored in two ways:

(i) as a cofibration followed by a fibration which is also a weak equivalence;

(ii) as a cofibration which is also a weak equivalence followed by a fibration.

If the dotted arrow exists in the diagram of CM4, we say that i has the *left lifting property* with respect to p, and p has the *right lifting property* with respect to i. A map which is both a fibration [resp. cofibration] and a weak equivalence is called a *trivial fibration* [resp. *trivial cofibration*].

3.2. EXAMPLES. (See [5], [6, p. 259], and [2, Chapter VIII].) We use the fact that the category S of simplicial sets is a closed model category, where fibrations are Kan fibrations (see [4, p. 25]), cofibrations are injective maps, and weak equivalences are weak homotopy equivalences, i.e. maps $X \to Y$ such that $\pi_0 X \to \pi_0 Y$ is an isomorphism of sets and $\pi_n(X, *) \to \pi_n(Y, *)$ is an isomorphism of groups for $n \geqslant 1$ and every choice of basepoint in X. (Here and throughout this paper, we denote the image of a basepoint $*$ also by $*$.) Similarly the category T of topological spaces is a closed model category, with Serre fibrations as fibrations, weak homotopy equivalences as weak equivalences, and retracts of sequences of relative CW complexes [7,

p. 401] as cofibrations. In both of these examples, the associated homotopy category (obtained by localizing at, i.e. formally inverting, weak equivalences) is just the familiar homotopy category of Kan complexes, or, equivalently, of CW complexes.

4. **Statement of results.** Our main result is that fibrations, cofibrations, and weak equivalences can be defined in tow-S to satisfy the axioms CM1–CM5 for a closed model category (3.1).

We begin with cofibration, which has the most natural definition.

4.1. DEFINITION. A map in tow-S is a *cofibration* if it has a level representative $\{f_s \colon X_s \longrightarrow Y_s\}$ such that each f_s is a cofibration in S, i.e. an injective map of simplicial sets.

4.2. REMARK. The need for the generality in the definition of level representative (2.2) can be seen from the following example. Consider the tower $\{I_s\}$ which has a 1-simplex at each level and I_{s+1} projected to the left endpoint of I_s. The map from $\{I_s\}$ to $\{*\}$, a tower of single points, is clearly an equivalence in tow-S, and hence must be [6, p. 234] a cofibration. But no level representative of $\{I_s\} \longrightarrow \{*\}$ using subtowers of the domain and range is a tower of cofibrations.

The definition of weak equivalence is complicated by the fact that the spaces X_s in an object $\{X_s\}$ of tow-S need not be connected or even nonempty.

4.3. DEFINITION. A map in tow-S is a *weak equivalence* if it has a level representative $\{f_s \colon X_s \longrightarrow Y_s\}$ such that

(i) $\{\pi_0 f_s\}$ is an equivalence of pro-sets, i.e. for each $s \geqslant 0$ there is a $t \geqslant s$ and a map $\pi_0 Y_t \longrightarrow \pi_0 X_s$ making the following diagram (of sets) commute:

$$
\begin{array}{ccc}
\pi_0 X_t & \longrightarrow & \pi_0 Y_t \\
\downarrow & \nearrow & \downarrow \\
\pi_0 X_s & \longrightarrow & \pi_0 Y_s
\end{array}
$$

(ii) for each $n \geqslant 1$ and each $s \geqslant 0$ there is a $t \geqslant s$ such that for each choice of basepoint in X_t there is a map $\pi_n(Y_t, *) \longrightarrow \pi_n(X_s, *)$ making the following diagram (of groups) commute:

$$
\begin{array}{ccc}
\pi_n(X_t, *) & \longrightarrow & \pi_n(Y_t, *) \\
\downarrow & \nearrow & \downarrow \\
\pi_n(X_s, *) & \longrightarrow & \pi_n(Y_s, *)
\end{array}
$$

It is routine to check that if one level representative of a map satisfies conditions (i) and (ii), so does any other. In the case that all the spaces X_s and Y_s are connected and $\{X_s\}$ has a compatible basepoint, the definition

becomes simply: *a map* $\{X_s\} \to \{Y_s\}$ *is a weak equivalence if the induced maps* $\{\pi_n X_s\} \to \{\pi_n Y_s\}$ *are pro-isomorphisms for* $n \geqslant 0$. This is the usual definition ([1], [2]).

Before defining fibration, it is convenient to introduce the following (partly standard [7, p. 404]) terminology for maps of simplicial sets.

4.4. DEFINITION. Let N be a nonnegative integer and let $f\colon X \to Y$ be a map in S. Then f is an *N-equivalence* (for X not empty) if for every choice of basepoint in X, $\pi_n(X, *) \to \pi_n(Y, *)$ is an isomorphism for $0 \leqslant n < N$ and an epimorphism for $n = N$. (If X is empty, then $X \to Y$ is an N-equivalence if Y is also empty.) On the other hand, f is a *co-N-equivalence* if for every choice of basepoint in X, $\pi_n(X, *) \to \pi_n(Y, *)$ is an isomorphism for $n > N$ and a monomorphism for $n = N$.

We define fibrations in two stages.

4.5. DEFINITION. A map in tow-S is a *level-fibration* if it has a level representative $\{X_s \to Y_s\}$ such that for each s

(i) there exists an integer $N(s)$ such that $X_s \to Y_s$ is a co-$N(s)$-equivalence;

(ii) $X_s \to Y_s$ is a fibration in S; and

(iii) the induced map $X_{s+1} \to X_s \times_{Y_s} Y_{s+1}$ is a fibration in S.

Induction and diagram chasing show that condition (iii) implies that for each $k \geqslant 1$ the induced map $X_{s+k} \to X_s \times_{Y_s} Y_{s+k}$ is a fibration. In particular a subtower of a level representative satisfying Definition 4.5 also satisfies Definition 4.5.

4.6. DEFINITION. A map in tow-S is a *fibration* if it is a retract (3.1, Axiom CM3) of a level-fibration (4.5).

In particular a level-fibration is a fibration.

4.7. THEOREM . *The category* tow-S, *with fibrations, cofibrations, and weak equivalences defined above, is a closed model category.*

ORGANIZATION OF PROOF. Axiom CM1 is proved by Artin-Mazur [1, Proposition A.4.2]. In §5 we prove CM2 and CM3. In §6 we reduce CM4 and CM5 to more concrete lifting and factoring statements, which are then proved in §8 and §7, respectively.

4.8. REMARK. Everything can just as easily be done for tow-S_*, where S_* is the category of *pointed*, but not necessarily connected, simplicial sets. The fixed basepoint would play no role in the definitions or proofs, except to exclude the case of empty simplicial sets.

5. Proofs of CM2 and CM3.

PROOF OF CM2. The proof consists of setting up the right level diagram.

Suppose $\{X_s\} \longrightarrow \{Y_s\} \longrightarrow \{Z_s\}$ are maps in tow-S. It is easy to see that by taking level representatives (2.2) we can assume that we have a tower $\{X_s \longrightarrow Y_s \longrightarrow Z_s\}$. We shall consider the case in which $\{X_s\} \longrightarrow \{Z_s\}$ and $\{X_s\} \longrightarrow \{Y_s\}$ are weak equivalences; the other two cases are similar but easier. We want to show that $\{Y_s\} \longrightarrow \{Z_s\}$ is a weak equivalence, i.e. (4.3) for each $n \geqslant 1$ and $s \geqslant 0$ there is a $t \geqslant s$ such that for each choice of basepoint $\cdot$ in Y_t there is a map $\pi_n(Z_t, \cdot) \longrightarrow \pi_n(Y_s, \cdot)$ making the following diagram commute

$$
\begin{array}{ccc}
\pi_n(Y_t, \cdot) & \longrightarrow & \pi_n(Z_t, \cdot) \\
\downarrow & \swarrow & \downarrow \\
\pi_n(Y_s, \cdot) & \longleftarrow & \pi_n(Z_s, \cdot)
\end{array}
$$

and a similar statement for π_0. The case of π_0 is easy, so fix $n \geqslant 1, s \geqslant 0$. Since $\{X_s\} \longrightarrow \{Z_s\}$ is a weak equivalence, we can choose $w \geqslant s$ so that the dotted arrow exists in the following diagram for every choice of basepoint in X_w:

$$
\begin{array}{ccccc}
\pi_n(X_w, *) & \longrightarrow & \pi_n(Y_w, *) & \longrightarrow & \pi_n(Z_w, *) \\
\downarrow & & & \swarrow & \downarrow \\
\pi_n(X_s, *) & \longrightarrow & \pi_n(Y_s, *) & \longrightarrow & \pi_n(Z_s, *)
\end{array}
$$

Next, since $\{X_s\} \longrightarrow \{Y_s\}$ is a weak equivalence, we can choose $u \geqslant w$ so that the dotted arrow exists in the following diagram for every choice of basepoint in X_u:

$$
\begin{array}{ccc}
\pi_n(X_u, *) & \longrightarrow & \pi_n(Y_u, *) \\
\downarrow & \swarrow & \downarrow \\
\pi_n(X_w, *) & \longrightarrow & \pi_n(Y_w, *)
\end{array}
$$

Finally we choose $t \geqslant u$ so that the dotted arrow exists in the following diagram:

$$
\begin{array}{ccc}
\pi_0 X_t & \longrightarrow & \pi_0 Y_t \\
\downarrow & \swarrow & \downarrow \\
\pi_0 X_u & \longrightarrow & \pi_0 Y_u
\end{array}
$$

Now let $\cdot$ be any basepoint in Y_t. Because of the last diagram, the image of $\cdot$ in Y_u is in a (path) component which is in the image of $X_u \longrightarrow Y_u$. Choose a basepoint $+$ in X_u such that (the images of) $+$ and $\cdot$ are in the same (path) component in Y_u, and let α be a path between them. Then α induces an isomorphism

$$
\alpha_\# : \pi_n(A, +) \longrightarrow \pi_n(A, \cdot)
$$

for $A = Y_u, Z_u, Y_w, Z_w, Y_s, Z_s$; and $\alpha_\#$ and $\alpha_\#^{-1}$ commute with the appropriate maps. Define the desired map as the composite

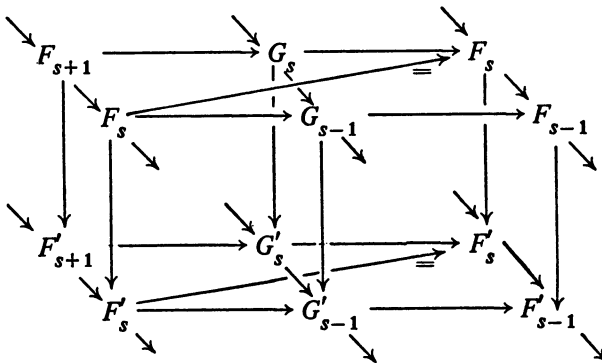$$\pi_n(Z_t, \cdot) \longrightarrow \pi_n(Z_u, \cdot) \xrightarrow{\alpha_\#^{-1}} \pi_n(Z_u, +)$$

$$\longrightarrow \pi_n(Z_w, +) \longrightarrow \pi_n(X_s, +) \longrightarrow \pi_n(Y_s, +) \xrightarrow{\alpha_\#} \pi_n(Y_s, \cdot).$$

A diagram chase gives the desired commutativity relations.

PROOF OF CM3 FOR WEAK EQUIVALENCE. We are given the following commutative diagram in tow-S:

Taking appropriate subtowers we get the following "level" representative of the diagram:

By diagram chasing one can verify that the definition of weak equivalence is satisfied by $\{F_s \to F'_s\}$ if it is satisfied by $\{G_s \to G'_s\}$.

PROOF OF CM3 FOR COFIBRATION. It is clear from the above diagram that the following tower

$$\cdots \longrightarrow F_{s+1} \longrightarrow G_s \longrightarrow F_s \longrightarrow \cdots$$
$$\cdots \longrightarrow F'_{s+1} \longrightarrow G'_s \longrightarrow F'_s \longrightarrow \cdots$$

is a level representative for $\{\overline{F}_s\} \longrightarrow \{\overline{F}'_s\}$. Hence so is the subtower consisting only of the $G_s \to G'_s$. Now if $\{\overline{G}_s\} \longrightarrow \{\overline{G}'_s\}$ is a cofibration, we could have chosen the level representative in such a way (4.1) that each $G_s \to G'_s$ is a cofibration in S; hence $\{\overline{F}_s\} \longrightarrow \{\overline{F}'_s\}$ is also a cofibration.

The *proof of* CM3 *for fibration* is immediate from Definitions 4.5 and 4.6.

6. **Proofs of CM4 and CM5.** In this section we reduce CM5 and CM4 to more concrete factoring and lifting axioms, which are proved in §§7–8.
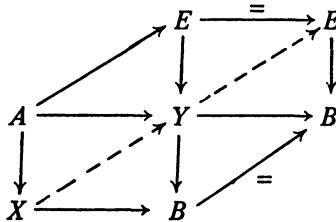
First we show that *it suffices to prove the factoring and lifting axioms for level-fibrations* (4.5), namely

F(i) Any map in tow-S can be factored as a cofibration followed by a trivial level-fibration (i.e. a level-fibration which is also a weak equivalence).

F(ii) Any map in tow-S can be factored as a trivial cofibration followed by a level-fibration.

L(i) Cofibrations have the left lifting property with respect to trivial level-fibrations.

L(ii) Trivial cofibrations have the left lifting property with respect to level-fibrations.

Indeed, CM5 follows from F(i) and F(ii), while CM4(ii) is a straightforward application of Definition 4.6 and L(ii). Then to prove CM4(i), we take the given lifting problem in tow-S

$$
\begin{array}{ccc}
A & \longrightarrow & E \\
\downarrow & \nearrow & \uparrow \\
X & \longrightarrow & B
\end{array}
$$
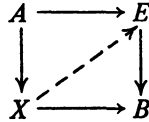
(using Roman letters for objects of tow-S in this proof only), in which $A \longrightarrow X$ is a cofibration and $E \longrightarrow B$ is a trivial fibration, and factor $E \longrightarrow B$ by F(i) into $E \longrightarrow Y \longrightarrow B$, where $Y \longrightarrow B$ is a trivial level-fibration and $E \longrightarrow Y$ is a cofibration, which is trivial by CM2. Then in the following diagram

$$
\begin{array}{ccccc}
& & E & \xrightarrow{\;=\;} & E \\
& \nearrow & \downarrow & \nearrow & \uparrow \\
A & \longrightarrow & Y & \longrightarrow & B \\
\downarrow & \nearrow & \downarrow & \nearrow & \\
X & \longrightarrow & B & &
\end{array}
$$

the dotted arrow $X \longrightarrow Y$ exists by L(i), and the dotted arrow $Y \longrightarrow E$ exists by CM4(ii). Their composite is the desired lifting $X \longrightarrow E$.

Next, to make the proofs easier, we will modify F(ii) and L(ii) slightly.

6.1. DEFINITION. A map in tow-S is a *level-trivial cofibration* if it has a level representative $\{X_s \longrightarrow Y_s\}$ such that for each s, $X_s \longrightarrow Y_s$ is a cofibration and an s-equivalence.

Clearly a level-trivial cofibration is a trivial cofibration

We now replace F(ii) and L(ii) by the following statements

F(ii)' Any map in tow-S can be factored as a level-trivial cofibration followed by a level-fibration

L(ii)' Level-trivial cofibrations have the left lifting property with respect to level-fibrations.

and observe that it suffices to prove F(i), F(ii)$'$, L(i), and L(ii)$'$. For it is clear that F(ii)$'$ implies F(ii); and L(ii) follows from L(ii)$'$, F(ii)$'$, L(i), and CM2 (the proof is similar to the proof of CM4(i), above). The factoring properties F(i) and F(ii)$'$ are proved in §7, while the lifting properties L(i) and L(ii)$'$ are proved in §8.

7. **Proof of factoring properties** F(i) **and** F(ii)$'$. In this section we prove that any map in tow-S, $\{X_s\} \longrightarrow \{Y_s\}$, can be factored

F(i) as $\{X_s\} \longrightarrow \{Z_s\} \longrightarrow \{Y_s\}$ with $\{X_s\} \longrightarrow \{Z_s\}$ a cofibration and $\{Z_s\} \longrightarrow \{Y_s\}$ a trivial level-fibration (4.5); and

F(ii)$'$ as $\{X_s\} \longrightarrow \{Z_s\} \longrightarrow \{Y_s\}$ with $\{X_s\} \longrightarrow \{Z_s\}$ a level-trivial cofibration (6.1) and $\{Z_s\} \longrightarrow \{Y_s\}$ a level-fibration.

The proof consists of factoring in an appropriate way each level of a level representative of $\{X_s\} \longrightarrow \{Y_s\}$ and then modifying the result to fit Definition 4.5.

We shall use the following generalizations of the factoring and lifting properties for simplicial sets.
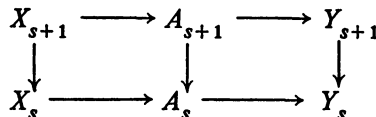
7.1. LEMMA. *Given an integer $N \geqslant 0$ and a map $X \longrightarrow Y$ in S, then there is a factoring $X \longrightarrow Z \longrightarrow Y$ such that $X \longrightarrow Z$ is a cofibration and N-equivalence, and $Z \longrightarrow Y$ is a fibration and co-N-equivalence (4.4).*

7.2. LEMMA. *Given an integer $N \geqslant 0$ and a solid arrow diagram in S*

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & B \end{array}$$
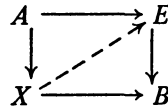
where $A \longrightarrow X$ is a cofibration and N-equivalence, and $E \longrightarrow B$ is a fibration and co-N-equivalence (4.4), then the dotted arrow exists.

We now prove F(ii)$'$; the proof of F(i) is similar, using the ordinary factoring and lifting axioms for S instead of Lemmas 7.1 and 7.2. Let $\{X_s\} \longrightarrow \{Y_s\}$ be the given map in tow-S, which we can assume (2.2) has a level representative $\{X_s \longrightarrow Y_s\}$. By Lemma 7.1 factor each level into $X_s \longrightarrow A_s \longrightarrow Y_s$, with $X_s \longrightarrow A_s$ a cofibration and s-equivalence, and $A_s \longrightarrow Y_s$ a fibration and co-s-equivalence. By Lemma 7.2 the arrow $A_{s+1} \longrightarrow A_s$ exists in the following diagram:

$$\begin{array}{ccccc} X_{s+1} & \longrightarrow & A_{s+1} & \longrightarrow & Y_{s+1} \\ \downarrow & & \downarrow & & \downarrow \\ X_s & \longrightarrow & A_s & \longrightarrow & Y_s \end{array}$$

The problem now is that $\{A_s\} \longrightarrow \{Y_s\}$ need not be a level-fibration because $A_{s+1} \longrightarrow A_s \times_{Y_s} Y_{s+1}$ is not in general a fibration, so we must modify the middle terms. Let $Z_0 = A_0$, and assume by induction that $Z_0, \cdots, Z_{s-1}$

have been defined to replace $A_0, \cdots, A_{s-1}$, so that the pull-back condition is satisfied through level $s-2$. Let $B_s = Z_{s-1} \times_{Y_{s-1}} Y_s$ and by Lemma 7.1 factor $A_s \longrightarrow B_s$ into $A_s \longrightarrow Z_s \longrightarrow B_s$ where $A_s \longrightarrow Z_s$ is an s-equivalence and cofibration, and $Z_s \longrightarrow B_s$ is a co-s-equivalence and fibration:
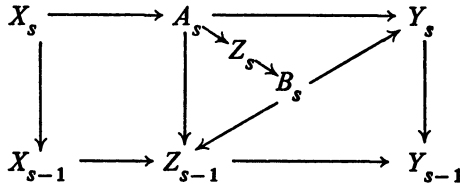
Then $\{Z_s \longrightarrow Y_s\}$ is a level map satisfying Definition 4.5, and $\{X_s \longrightarrow Z_s\}$ is a level map satisfying Definition 6.1, as desired.

PROOF OF 7.1. Essentially we just apply the well-known Moore-Postnikov factorization of a map. In detail, factor $X \longrightarrow Y$ into $X \longrightarrow A \longrightarrow Y$, with $X \longrightarrow A$ a trivial cofibration and $A \longrightarrow Y$ a fibration. Then factor $A \longrightarrow Y$ into $A \longrightarrow B \longrightarrow Y$ where B is the $(N-1)$st Moore-Postnikov "part" of the fibration $A \longrightarrow Y$ (see [4, p. 34], which contains the unneeded hypothesis that A and Y are connected Kan complexes). Thus $A \longrightarrow B$ is a (fibration and) N-equivalence, and $B \longrightarrow Y$ is a fibration and co-N-equivalence. Finally, factor $A \longrightarrow B$ into $A \longrightarrow Z \longrightarrow B$ with $A \longrightarrow Z$ a cofibration and $Z \longrightarrow B$ a trivial fibration. Clearly $X \longrightarrow Z \longrightarrow Y$ has the desired properties.

PROOF OF 7.2. Because $A \longrightarrow X$ is an N-equivalence, there is no obstruction to lifting the N-skeleton, $A \cup X^N$. Because $E \longrightarrow B$ is a co-N-equivalence, there is then no obstruction to lifting the rest of X.

8. **Proof of lifting properties $L(\mathrm{i})$ and $L(\mathrm{ii})'$.** In this section we prove that

$L(\mathrm{i})$ cofibrations have the left lifting property with respect to trivial level-fibrations (4.5),

and

$L(\mathrm{ii})'$ level-trivial cofibrations (6.1) have the left lifting property with respect to level-fibrations.

The proofs are similar, although $L(\mathrm{i})$ is more technical. In both cases the plan is to construct the lifting level by level. Given a level diagram representing the lifting problem
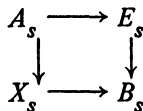
we want to find a monotonic function p and maps $X_{p(s)} \to E_s$ such that $\{X_{p(s)} \to E_s\}$ defines a level map from a subtower of $\{X_s\}$ to $\{E_s\}$, i.e. the following diagram commutes

$$
\begin{array}{ccc}
X_{p(s)} & \longrightarrow & E_s \\
\downarrow & & \downarrow \\
X_{p(s-1)} & \longrightarrow & E_{s-1}
\end{array}
$$

and the map is a lifting, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
A_{p(s)} & \longrightarrow & E_s \\
\downarrow & \nearrow & \downarrow \\
X_{p(s)} & \longrightarrow & B_s
\end{array}
$$

We first do the easier L(ii)$'$. Given the following diagram in tow-S

$$
\begin{array}{ccc}
\{A_s\} & \longrightarrow & \{E_s\} \\
\downarrow & & \downarrow \\
\{X_s\} & \longrightarrow & \{B_s\}
\end{array}
$$

with $\{A_s\} \to \{X_s\}$ a level-trivial cofibration and $\{E_s\} \to \{B_s\}$ a level-fibration, it is not hard to see that we can assume that we have a level diagram

$$
\begin{array}{ccc}
A_s & \longrightarrow & E_s \\
\downarrow & & \downarrow \\
X_s & \longrightarrow & B_s
\end{array}
$$

such that for each s, $A_s \to X_s$ is a cofibration and s-equivalence (6.1), and $E_s \to B_s$ is a fibration and co-$N(s)$-equivalence (4.5) for some integer $N(s)$.

To construct the lifting to E_0, let $p(0) = N(0)$. Then by Lemma 7.2 the dotted arrow exists in the following diagram:

$$
\begin{array}{ccc}
A_{p(0)} & \longrightarrow & E_0 \\
\downarrow & \nearrow & \downarrow \\
X_{p(0)} & \longrightarrow & B_0
\end{array}
$$

For the inductive step, assume that the map $X_{p(s-1)} \to E_{s-1}$ has been defined. Let $p(s) = \max(p(s-1) + 1, N(s-1), N(s))$. Define $E_s' = B_s \times_{B_{s-1}} E_{s-1}$; from Definition 4.5 we know that $E_s \to E_s'$ is a fibration. By universality the map $X_{p(s)} \to E_s'$ exists in the following diagram:
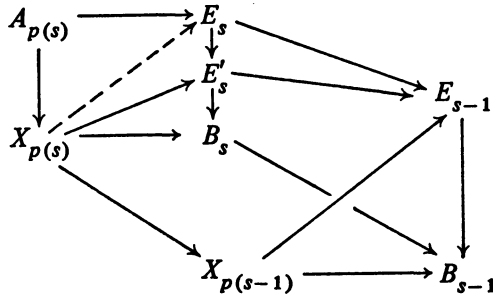
Now since the fibration $E_{s-1} \to B_{s-1}$ is a co-$N(s-1)$-equivalence, so is $E'_s \to B_s$; and hence, since $E_s \to B_s$ is a co-$N(s)$-equivalence, $E_s \to E'_s$ is a co-$p(s)$-equivalence. Thus the dotted arrow in the above diagram exists by Lemma 7.2. Note that it was necessary to lift $X_{p(s)}$ first to E'_s in order that $X_{p(s)} \to E_s$ be compatible with $X_{p(s-1)} \to E_{s-1}$.

For $L(\mathrm{i})$ we must construct the maps $X_{p(s)} \to E_s$ by skeleta $X^k_{p(s)}$, i.e. we will find compatible maps $A_{p(s)} \cup X^0_{p(s)} \to E_{s+m}$, $A_{p(s)} \cup X^1_{p(s)} \to E_{s+m-1}, \cdots, A_{p(s)} \cup X^m_{p(s)} \to E_s$ and finally extend to $X_{p(s)} \to E_s$. To insure, moreover, that $X_{p(s)} \to E_s$ is compatible with $X_{p(s-1)} \to E_{s-1}$ we must, for each skeleton, first lift to an appropriate pull-back as in the proof of $L(\mathrm{ii})'$, above.

We will need two lemmas. The first establishes sufficient conditions for lifting skeleta, while the second shows how to choose a nice enough level representative of a trivial level-fibration.

8.1. LEMMA. *Let N be a nonnegative integer, and consider the following solid arrow diagram in S, where $A \to X$ is a cofibration, and $E \to B$ and $E' \to B'$ are fibrations (we use X^r to denote the r skeleton of X, with X^{-1} the empty space):*
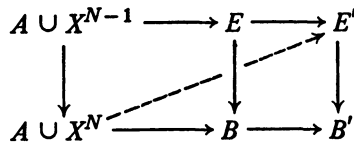
Then the dotted arrow exists if

(i) *$N = 0$ and the image of $\pi_0 B$ in $\pi_0 B'$ is contained in the image of $\pi_0 E'$ in $\pi_0 B'$;*

(ii) *$N \geqslant 1$ and for each choice of basepoint in E, the map $\pi_{N-1}(F, *) \to \pi_{N-1}(F', *)$ is the zero map, where F and F' are the fibers of $E \to B$ and $E' \to B'$ over the basepoint.*

PROOF. For $N = 0$ the construction of the lifting is straightforward. For $N \geqslant 1$ condition (ii) implies that there is no obstruction to lifting to E'.

8.2. LEMMA. *A trivial level-fibration in* tow-S *has a level representative* $\{E_s \to B_s\}$ *which satisfies Definition 4.5 and the following conditions:*

(i) $\pi_n(F_{s+1}, *) \to \pi_n(F_s, *)$ *is the zero map for each* n, $s \geqslant 0$ *and every choice of basepoint in* E_{s+1} *(where* F_i *is the fiber of* $E_i \to B_i$ *over the basepoint), and the image of* $\pi_0 B_{s+1}$ *in* $\pi_0 B_s$ *is contained in the image of* $\pi_0 E_s$ *in* $\pi_0 B_s$.

(ii) *Let* $E'_s = B_s \times_{B_r} E_r$ *and* $E'_{s+1} = B_{s+1} \times_{B_{r+1}} E_{r+1}$, *for* $r \geqslant 0$, $s \geqslant r + 2$. *Choose a basepoint in* E_{s+1} *and let* G_i *be the fiber of* $E_i \to E'_i$. *Then* $\pi_n(G_{s+1}, *) \to \pi_n(G_s, *)$ *is the zero map for each* $n \geqslant 0$, *and the image of* $\pi_0 E'_{s+1}$ *in* $\pi_0 E'_s$ *is contained in the image of* $\pi_0 E_s$ *in* $\pi_0 E'_s$.

The proof is postponed until the end of the section.

We can now prove the lifting property L(i). where we are given a diagram in tow-S

$$
\begin{array}{ccc}
\{A_s\} & \longrightarrow & \{E_s\} \\
\downarrow & & \downarrow \\
\{X_s\} & \longrightarrow & \{B_s\}
\end{array}
$$

in which $\{A_s\} \to \{X_s\}$ is a cofibration and $\{E_s\} \to \{B_s\}$ is a trivial level-fibration. As before, we can assume that in fact we have a level diagram

$$
\begin{array}{ccc}
A_s & \longrightarrow & E_s \\
\downarrow & & \downarrow \\
X_s & \longrightarrow & B_s
\end{array}
$$

in which each $A_s \to X_s$ is a cofibration and $E_s \to B_s$ satisfies Lemma 8.2. To simplify notation, we write $X_s^{(r)}$ for $A_s \cup X_s^r$. For this proof let $N(s)$ be an integer greater than $N(s-1) + 2$, such that $E_s \to B_s$ is a co-$N(s)$-equivalence; $N(-1) = 0$. Let $p(s) = s + N(s) + 1$.

First, for $s = 0$, we construct $X_{p(0)} \to E_0$ in stages $X_{p(0)}^{(k)} \to E_{p(0)-k-1}$ for $k = 0, 1, \cdots, p(0) - 1 = N(0)$. Consider the following diagram:

$$
\begin{array}{ccccccccc}
A_{p(0)} & \longrightarrow & E_{p(0)} & \longrightarrow & E_{p(0)-1} & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & E_0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
X_{p(0)} & \longrightarrow & B_{p(0)} & \longrightarrow & B_{p(0)-1} & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0
\end{array}
$$

By Lemmas 8.1 and 8.2 (i) the desired liftings $X_{p(0)}^{(k)} \to E_{p(0)-k-1}$ exist. Further, by Lemma 7.2 and the choice of $N(0)$, we can extend $X_{p(0)}^{(N(0))} \to E_0$ to $X_{p(0)} \to E_0$.

Now assume by induction on s that we have compatible liftings $X_{p(s-1)}^{(k)} \to E_{p(s-1)-k-1}$ for $k = 0, 1, \cdots, p(s-1) - s = N(s-1)$ and $X_{p(s-1)} \to E_{s-1}$. We wish to construct maps $X_{p(s)}^{(k)} \to E_{p(s)-k-1}$ for

$k = 0, 1, \cdots, p(s) - s - 1 = N(s)$ and $X_{p(s)} \to E_s$, compatible with each other and with the skeleta liftings for $s - 1$. Define $E'_{p(s)-r}$ by the following rules:

$$E'_{p(s)-r} = B_{p(s)-r} \times_{B_{p(s-1)-r-1}} E_{p(s-1)-r-1}, \quad 0 \leqslant r < p(s-1) - s,$$

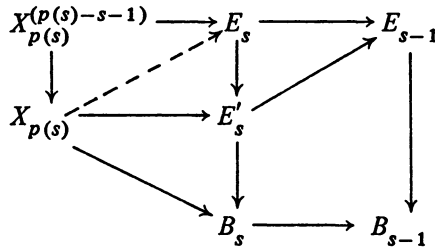$$E'_{p(s)-r} = B_{p(s)-r} \times_{B_{s-1}} E_{s-1}, \quad p(s-1) - s \leqslant r \leqslant p(s) - s.$$

Then there are natural liftings $X^{(r)}_{p(s)} \to E'_{p(s)-r}$ for $0 \leqslant r < p(s-1) - s$ and $X_{p(s)} \to E'_{p(s)-r}$ for $p(s-1) - s \leqslant r \leqslant p(s) - s$, since $X^{(r)}_{p(s)}$ maps to $E_{p(s-1)-r-1}$ through $X^{(r)}_{p(s-1)}$, and $X_{p(s)}$ maps to E_{s-1} through $X_{p(s-1)}$. We wish to construct the dotted arrow in the following diagram
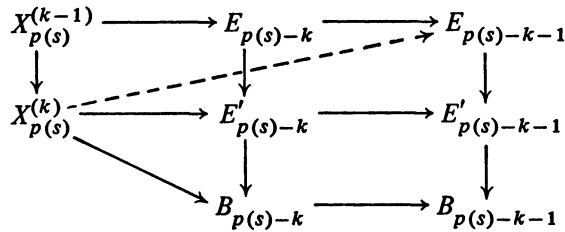

where we are assuming by induction on k that $X^{(k-1)}_{p(s)} \to E_{p(s)-k}$ has already been constructed (there being nothing to prove for $k = -1$). There are two cases. If $0 \leqslant k < p(s-1) - s$, we have the situation of Lemma 8.2 (ii) and the lifting exists by Lemma 8.1. Suppose therefore that $p(s-1) - s \leqslant k < p(s) - s$. In order to apply Lemma 8.1 we want, for every choice of basepoint in $E_{p(s)-k}$, that $\pi_{k-1}(G_{p(s)-k}, *) \to \pi_{k-1}(G_{p(s)-k-1}, *)$ be zero, where G_i is the fiber of $E_i \to E'_i$. First note that the fiber H of $E'_{p(s)-k-1} \to B_{p(s)-k-1}$ is the same as the fiber of $E_{s-1} \to B_{s-1}$ and in particular $\pi_k(H, *) = 0$ since $k \geqslant N(s-1)$. Now since $\pi_{k-1}(F_{p(s)-k}, *) \to \pi_{k-1}(F_{p(s)-k-1}, *)$ is zero by Lemma 8.2, it suffices to show that $\pi_{k-1}(G_{p(s)-k-1}, *) \to \pi_{k-1}(F_{p(s)-k-1}, *)$ is monomorphic. For this we apply the Five Lemma to the following diagram (* and subscripts $p(s) - k - 1$ are suppressed):
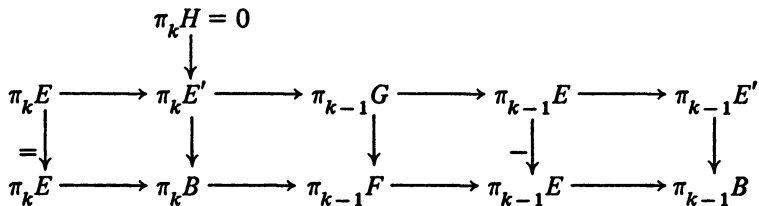

Having constructed the skeleton lifting $X^{(p(s)-s-1)}_{p(s)} \to E_s$, the dotted arrow finally exists in the following diagram by Lemma 7.2 and the choice of $p(s)$:

$$X_{p(s)}^{(p(s)-s-1)} \longrightarrow E_s \longrightarrow E_{s-1}$$

(commutative diagram with vertices $X_{p(s)}^{(p(s)-s-1)}$, E_s, E_{s-1}, $X_{p(s)}$, E_s', B_s, B_{s-1})

PROOF OF 8.2. We start with any level representative $\{E_s \longrightarrow B_s\}$ of the trivial level-fibration which satisfies Definition 4.5. By the definition of weak equivalence (4.3) and level-fibration (4.5) we can, by taking a subtower, assume that for each $s \geqslant 0$ the dotted arrow exists in the following diagram

$$\pi_0 E_{s+1} \longrightarrow \pi_0 B_{s+1}$$
$$\pi_0 E_s \longrightarrow \pi_0 B_s$$
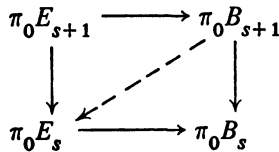
and for each $n \geqslant 1$ and each choice of basepoint in E_{s+1} the dotted arrow exists in the following diagram: •

$$\pi_n(E_{s+1}, *) \longrightarrow \pi_n(B_{s+1}, *)$$
$$\pi_n(E_s, *) \longrightarrow \pi_n(B_s, *)$$
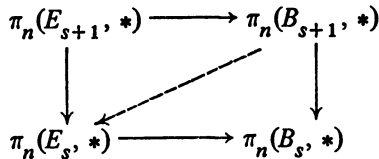
A diagram chase in the long exact sequence of homotopy groups of the fibrations $E_s \longrightarrow B_s$ shows that $\pi_n(F_{s+2}, *) \longrightarrow \pi_n(F_s, *)$ is the zero map for each $s \geqslant 0$, $n \geqslant 0$, and basepoint in E_{s+2}. The subtower $\{E_{2s} \longrightarrow B_{2s}\}$ then satisfies condition (i). A lengthy diagram chase now shows that condition (ii) will be satisfied as well if one takes the subtower $\{E_{4s} \longrightarrow B_{4s}\}$. By the comment following Definition 4.5, the final subtower still satisfies Definition 4.5.

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