

TEICHMÜLLER SPACES AND REPRESENTABILITY OF FUNCTORS⁽¹⁾

BY

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ABSTRACT. The Teichmüller space of compact Riemann surfaces with punctures is exhibited as the object representing a certain functor. This extends the work of Grothendieck, who treated the unpunctured case. The relationship between the two cases is exploited to obtain specific results on the connection between the two Teichmüller spaces.

0. Introduction. Grothendieck [7] has obtained the Teichmüller space of compact Riemann surfaces by representing an appropriate functor. The object of this work is to extend Grothendieck's methods to the case of Riemann surfaces of finite type; that is, compact Riemann surfaces with finitely many points deleted. A similar program has been undertaken by Earle [5] and Hubbard [8] but from a different point of view and using very different methods. Both authors rely heavily on the analytic theory of Teichmüller space. In Grothendieck's work, the only results of the classical theory that intervene are used to show that the object he constructs is indeed Teichmüller space. A fitting extension of his work ought to adhere to the same standard of purity, but we are unable to do so. To complete the extension requires the knowledge that the ordinary Teichmüller space is contractible, but this result is not yet available without analysis (cf. [7, I, Remarques 3.2, 3°]). With this exception, the construction is geometric and depends in large measure on results of Birman [3], [4].

The approach used builds the new Teichmüller space (for surfaces of finite type) out of the old (for compact surfaces) and, in so doing, gives an explicit recipe not only for the Teichmüller space but for its fiber space as well. Part of this recipe has been obtained by Kra [9] using other methods but its full force gives a geometric proof of a theorem of Bers [2] identifying the fiber space for

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n punctures with the Teichmüller space for $n + 1$ punctures.

We are also able to show (Theorem 3.8) that if the deleted points are assigned multiplicities, the Teichmüller space remains independent of the way this is done. This result seems a natural step on the way to a new proof of the Bers-Greenberg isomorphism theorem [10], of which it is also a consequence, but its promise has not been fulfilled. Nonetheless, the result is interesting in its own right so that we deal with the more general concept of Riemann surfaces with signature; i.e., multiplicities.

1. Preliminaries.

DEFINITION 1.1. A Riemann surface of type (g, n) with signature is a compact Riemann surface of genus g together with n distinguished points to each of which is assigned an integer ≥ 2 or the symbol ∞ . The n points are called the punctures and it is customary to order them so that the multiplicities form a nondecreasing sequence. This sequence is called the signature.

It will turn out that the signature is less important than the question of which of the multiplicities are equal. To exploit this fact, we make the following definitions.

DEFINITION 1.2. (i) A signature structure on n points is a partition of the integers $1, \dots, n$ into intervals. Thus a signature structure can be written $\{1, \dots, r_1\}, \{r_1 + 1, \dots, r_2\}, \dots, \{r_{s-1} + 1, \dots, n\}$. If we let $r_0 = 1$ and $r_s = n$ then the above signature structure can be denoted by the $(s + 1)$ -tuple $r = (r_0, \dots, r_s)$.

(ii) If $\nu_1 \leq \nu_2 \leq \dots \leq \nu_n$ is a signature then the associated signature structure is the partition of the indices defined by the requirement that i and j belong to the same element of the partition if and only if $\nu_i = \nu_j$.

(iii) If r is a signature structure, we define $\Sigma(r)$, the stability group of r , to be the subgroup of the full symmetric group on n letters which consists of all permutations which leave the elements of the partition invariant as sets.

EXAMPLES. 1. If $s = 1$, then $r = (0, n)$, so that the partition consists of only one set. This signature structure corresponds to a signature in which all multiplicities are equal. In this case, $\Sigma(r)$ is the full symmetric group.

2. If $s = n$, then $r = (0, 1, \dots, n)$, so that each element of the partition is a singleton. This signature structure is called the complete signature structure and $\Sigma(r) = (1)$.

The following definitions are extensions of those appearing in [3] and [4].

DEFINITIONS 1.3. (i) If M is a manifold, we define $F_n M = \{(p_1, \dots, p_n) \mid p_i \in M, p_i \neq p_j \text{ if } i \neq j\}$. If r is a signature structure on n points then $\Sigma(r)$ acts on $F_n M$ in a natural way and we write $F(r)M = F_n M / \Sigma(r)$. We denote the class of (p_1, \dots, p_n) modulo $\Sigma(r)$ by $[p_1, \dots, p_n]$.

(ii) $\pi_1(F(r), [x_1, \dots, x_n])$ is called the n -string braid group of M with signature structure r . Birman calls this a semipermutated braid group.

(iii) If M is an orientable manifold and x_1, \dots, x_n are distinct points of M , we let $\mathfrak{F}(r)M$ denote the set of orientation-preserving homeomorphisms of M whose action on the set $\{x_1, \dots, x_n\}$ coincides with that of an element of $\Sigma(r)$. We topologize $\mathfrak{F}(r)M$ with the compact-open topology. If $n = 0$, we write \mathfrak{F}_0M .

(iv) The mapping class group of M with signature structure r is the group $\pi_0(\mathfrak{F}(r)M, \text{id})$.

THEOREM 1.4. *Let C be a compact orientable surface of genus $g \geq 2$, x_1, \dots, x_n distinct points of C and r a signature structure on n points. Then there exists an exact sequence*

$$1 \rightarrow \pi_1(F(r)C, [x_1, \dots, x_n]) \rightarrow \pi_0(\mathfrak{F}(r)C, \text{id}) \rightarrow \pi_0(\mathfrak{F}_0C, \text{id}) \rightarrow 1.$$

PROOF. If r is the complete signature structure, this is Theorem 1 and Corollary 1.3 of [4]. The extension to arbitrary r is straightforward.

In the sequel, we will abbreviate $\pi_0(\mathfrak{F}(r)C, \text{id})$ to γ^r and $\pi_0(\mathfrak{F}_0C, \text{id})$ to γ .

It will prove convenient to have the following generalization of $F(r)M$.

DEFINITION 1.5. Let $g: V \rightarrow T$ be a morphism of manifolds. We define

$$F_n(V/T) = \{(p_1, \dots, p_n) \mid p_i \in V; g(p_i) = g(p_j) \\ \text{for all } i \text{ and } j \text{ and } p_i \neq p_j \text{ if } i \neq j\}.$$

If r is a signature structure on n points then $F(r)(V/T) = F_n(V/T)/\Sigma(r)$.

Note that if T is reduced to a point then $F(r)(V/T) = F(r)V$.

The following proposition will be used in the sequel.

PROPOSITION 1.6. *Suppose $g: V \rightarrow T$ is a locally trivial fiber space. Let $t \in T$ and let C be the fiber of g over t . Let x_1, \dots, x_n be n distinct points of C . If $\pi_1(T, t) = \pi_2(T, t) = (1)$ then*

$$\pi_1(F(r)(V/T), [x_1, \dots, x_n]) = \pi_1(F(r)C, [x_1, \dots, x_n]).$$

PROOF. Since V is locally trivial over T it follows that $F(r)(V/T)$ is also locally trivial over T . Furthermore, the fiber of $F(r)(V/T)$ over t is precisely $F(r)C$. By the long exact homotopy sequence for a fiber space, we have:

$$\begin{aligned} \cdots \rightarrow \pi_2(T, t) \rightarrow \pi_1(F(r)C, [x_1, \dots, x_n]) \\ \rightarrow \pi_1(F(r)(V/T), [x_1, \dots, x_n]) \rightarrow \pi_1(T, t). \end{aligned}$$

In view of the hypothesis on the homotopy groups of T , the result follows.

If $r = (r_0, \dots, r_s)$ is a signature structure on n points, then \hat{r} will be the signature structure on $n + 1$ points which is denoted $(r_0, \dots, r_s, n + 1)$; i.e., the partition of the first n integers is given by r and $n + 1$ is in a class by itself.

The important result is

PROPOSITION 1.7. *Let C be a compact surface of genus $g \geq 1$ and r a signature structure on n points. Then we have an exact sequence*

$$\begin{aligned} 1 \rightarrow \pi_1(C - \{x_1, \dots, x_n\}, x_{n+1}) &\rightarrow \pi_1(F(\hat{r})C, [x_1, \dots, x_{n+1}]) \\ &\rightarrow \pi_1(F(r)C, [x_1, \dots, x_n]) \rightarrow 1 \end{aligned}$$

where the homomorphisms are induced as follows:

$$\begin{aligned} C - \{x_1, \dots, x_n\} &\rightarrow F(\hat{r})C \\ y &\mapsto [x_1, \dots, x_n, y] \end{aligned}$$

and

$$\begin{aligned} F(\hat{r})C &\rightarrow F(r)C \\ [y_1, \dots, y_{n+1}] &\mapsto [y_1, \dots, y_n]. \end{aligned}$$

PROOF. This result is due to Fadell and Neuwirth [6] in the case where r is the complete signature structure. The case of arbitrary r is straightforward.

2. The functors. In what follows, all objects are analytic spaces (which need not be reduced) and all morphisms are analytic. We will assume that $g \geq 2$ and $n \geq 0$.

DEFINITION 2.1. (i) Let $f: X \rightarrow S$ be a morphism. X is a curve of genus g over S if and only if f is proper and smooth and the fibers (which are thus compact complex manifolds) are connected, of dimension 1 and genus g .

(ii) An n -punctured curve of genus g over S is a curve of genus g together with n sections e_1, \dots, e_n of f such that $e_i(s) \neq e_j(s)$ if $i \neq j$ for all $s \in S$.

(iii) An n -punctured curve of genus g over S with signature structure is an n -punctured curve of genus g over S together with a signature structure on n points.

(iv) A curve of genus g over S with signature is an n -punctured curve of genus g over S together with a sequence $2 \leq \nu_1 \leq \dots \leq \nu_n \leq \infty$ (abbreviated (ν)) where the ν_i are either integers or the symbol ∞ . Note that each fiber is a Riemann surface with signature.

Where confusion is unlikely, the words “of genus g ” will be omitted. Similarly, since only n -punctured curves can have signatures or signature structures, the puncturing will rarely be mentioned.

DEFINITION 2.2. (i) A morphism of curves over S is a morphism in the (relative) category of analytic spaces over S .

(ii) A morphism of n -punctured curves over S is a morphism of curves over S which commutes with the sections.

(iii) A morphism of curves over S with signature structure r is a morphism $g: X \rightarrow X'$ of curves over S such that $g \circ e_i = e'_{\sigma(i)}$ for all i , where $\sigma \in \text{Hom}(S, \Sigma(r))$. Since $\Sigma(r)$ is discrete, σ may be regarded as a sequence of elements of $\Sigma(r)$ indexed by the connected components of S .

(iv) A morphism of curves over S with signature (ν) is a morphism of curves over S with signature structure r where r is the signature structure associated to (ν) .

With these definitions, we have defined the categories:

- \mathfrak{F}_S , curves over S ,
- $\mathfrak{F}_S(r)$, curves over S with signature structure r ,
- $\mathfrak{F}_S(\nu)$, curves over S with signature (ν) .

If r is the complete signature structure, we write $\mathfrak{F}_S^* = \mathfrak{F}_S(r)$. Note that \mathfrak{F}_S^* is the category of n -punctured curves over S .

If $g: S \rightarrow S'$ then we have a natural functor (inverse image) $\mathfrak{F}_{S'} \rightarrow \mathfrak{F}_S$ given by $X \mapsto X \times_{S'} S$. The categories $\mathfrak{F}_S(r)$ and $\mathfrak{F}_S(\nu)$ are also endowed with this functor.

Let $A(S)$ denote the set of isomorphism classes of objects of any of the above categories. Then A is a contravariant functor from analytic spaces to sets. The moduli problem is the representation of this functor A . This representation of A is known to be impossible and the impossibility is a consequence of the existence of nontrivial automorphisms of curves over S . The approach of Teichmüller involves the elimination of these nontrivial automorphisms by the adjunction of extra “rigidifying” structure.

In the case of ordinary curves of genus g (i.e., over a point), the rigidification is accomplished by a so-called marking. Marking a Riemann surface C means choosing an equivalence class of canonical dissections of C . Two canonical dissections are equivalent if one can be obtained from the other by an orientation-preserving homeomorphism which is homotopic to the identity [1]. Thus markings are in one-to-one correspondence with the elements of a mapping class group.

In our context, if X is a curve over S , we will want to choose an analytic

family of markings for the fibers over S . This will not always be possible, but it will be possible in a sufficiently large class of cases to make our "rigidified" functors representable.

To give a precise meaning to the concept of rigidification, the following lemma is necessary.

LEMMA 2.3. *Let X be a curve of genus g over S . Then X is topologically locally trivial over S .*

PROOF. In fact, if S is a manifold, X is even C^∞ locally trivial over S . Thus it suffices to prove that every point $s \in S$ admits a neighborhood U such that $X|U$ is isomorphic to the pull-back of a curve of genus g over a manifold M via a morphism $U \rightarrow M$. This last may be found in [7, X, Proposition 1.8].

Now let G be the group of homeomorphisms of the fiber C modulo homotopy of continuous maps. If X_s denotes the fiber of X over $s \in S$, we let $I(C, X_s)$ denote the set of homeomorphisms of C onto X_s modulo homotopy.

PROPOSITION 2.4. $\underline{R}(X/S) = \bigcup \{I(C, X_s) \mid s \in S\}$ has the structure of a principal bundle over S with group G .

PROOF. Let $U \subset S$ be an open subset of S over which X is trivial so that $X|U$ is homeomorphic to $U \times C$. To give a homeomorphism of $U \times C$ to $X|U$ over U is to give a map $g_U: U \rightarrow \bigcup \{I(C, X_s) \mid s \in U\}$. We endow $\underline{R}(X/S)$ with the largest topology which makes all of the g_U continuous. Furthermore, $G = I(C, C)$ acts by composition on $I(C, X_s)$ which is thus a homogeneous space for the group G . Finally the maps g_U exhibit the local triviality of $\underline{R}(X/S)$.

In the group G , we distinguish the subgroup γ of classes of orientation-preserving homeomorphisms. The giving of a continuous family of orientations of the fibers of X is equivalent to reducing the structure group of $\underline{R}(X/S)$ to γ . This being done, let $\underline{P}(X/S)$ denote the associated principal bundle.

If X/S is a punctured curve with signature or signature structure, we can still define $\underline{P}(X/S)$; but the additional structure enables us to construct $\underline{P}^r(X/S)$ analogously to $\underline{P}(X/S)$. Here we consider the sets $I^r(C, X_s)$ where the superscript r indicates that we consider only those homeomorphisms whose action on the distinguished points coincides with that of an element of $\Sigma(r)$. Thus $\underline{P}^r(X/S)$ is a principal bundle with group γ^r .

The above definitions are purely topological but, the groups γ being discrete, we can easily put an analytic structure on $\underline{P}(X/S)$. Namely, we require that the structure map $\underline{P}(X/S) \rightarrow S$ be a local isomorphism. If U is an open subset of S over which $\underline{P}(X/S)$ is trivial, we see that $\underline{P}(X/S)|U$ is just a dis-

joint union of copies of U (one for each element of γ). Furthermore, the action of γ on $\underline{P}(X/S)|U$ amounts to nothing more than interchanging these copies of U . In this way, $\underline{P}(X/S)$ becomes an analytic space and γ a group of analytic self-mappings. Similar considerations apply to $\underline{P}^r(X/S)$. Henceforth, any reference to these principal bundles will include the above analytic structure.

PROPOSITION 2.5. $\underline{P}(X/S)$, and $\underline{P}^r(X/S)$ satisfy the following properties:

(i) For fixed S , they are functorial in X/S .

(ii) They are compatible with inverse images.

(iii) Any automorphism of $X \in \text{Ob } \mathfrak{F}_S$ which induces the identity in $\underline{P}(X/S)$ is itself the identity.

(iii)' Any automorphism of $X \in \text{Ob } \mathfrak{F}_S(r)$ or $\text{Ob } \mathfrak{F}_S(\nu)$ which induces the identity in $\underline{P}(X/S)$ is itself the identity.

(iii)'' Any automorphism of $x \in \text{Ob } \mathfrak{F}_S(r)$ or $\text{Ob } \mathfrak{F}_S(\nu)$ which induces the identity in $\underline{P}^r(X/S)$ is itself the identity.

PROOF. (i) and (ii) follow trivially from the construction of the bundles. (iii) is just [7, X, Theorem 3.1] and the remark following I, Lemma 2.4. (iii)' follows from (iii) since any automorphism in $\mathfrak{F}_S(r)$ or $\mathfrak{F}_S(\nu)$ is *a fortiori* an automorphism in \mathfrak{F}_S . Statement (iii)'' follows from (iii)' and the fact that an automorphism inducing the identity on $\underline{P}^r(X/S)$ must induce the identity on $\underline{P}(X/S)$ via the surjection $\gamma^r \rightarrow \gamma$.

Terminology. A functor, possessing any of the properties of the type of (iii), (iii)' or (iii)'', will be called rigidifying.

DEFINITION 2.6. A Teichmüller rigidification on a curve (punctured or not) of genus g over S is a section of $\underline{P}(X/S)$ over S . A Teichmüller r -rigidification of an n -punctured curve of genus g with signature structure r or signature (ν) over S is a section of $\underline{P}^r(X/S)$ over S .

The bundles involved here being principal bundles, the existence of a section means that the bundle is globally trivial. Thus if, for example, a Teichmüller rigidification exists, then $\underline{P}(X/S) = S \times \gamma$. If furthermore, S is nonempty and connected, then the set of Teichmüller rigidifications is in one-to-one correspondence with the elements of γ . Moreover, γ acts freely on the set of Teichmüller rigidifications. Similarly for γ^r . We shall make use of these actions below.

DEFINITION 2.7. A curve over S with a Teichmüller rigidification (resp. Teichmüller r -rigidification) is called a Teichmüller curve over S (resp. Teichmüller r -curve).

Since any S -automorphism of a principal bundle that fixes a section is the identity it follows by (iii), (iii)' and (iii)'' of Proposition 2.5 that any automorphism of a Teichmüller curve or r -curve is the identity. Thus we have accom-

plished our initial goal of eliminating nontrivial automorphisms.

The functors that we now seek to represent are the following:

- $A_P(S)$ = set of isomorphism classes of Teichmüller curves of genus g over S .
- $A_P^r(S)$ = set of isomorphism classes of (n -punctured) Teichmüller curves of genus g with signature structure r over S .
- $A_{pr}^r(S)$ = set of isomorphism classes of (n -punctured) Teichmüller r -curves of genus g with signature structure r over S .
- $A_{pr}^\nu(S)$ = set of isomorphism classes of (n -punctured) Teichmüller r -curves of genus g with signature (ν) over S .

Note that each of these sets is equipped with an action of the corresponding group. For example, if $X \in A_P(S)$, $u \in \gamma$ then $X^u \in A_P(S)$ is the Teichmüller curve whose underlying curve is the same as that of X but whose rigidification is obtained from that of X by right translation by u . (See the remark following Definition 2.6.) Since $\underline{P}(X/S)$ and $\underline{P}^r(X/S)$ are functorial in S , it follows that these actions of γ and γ^r are functorial. Thus if any of the functors above are representable (and we shall see that they all are) then the representing object will in each case admit an action of either γ or γ^r .

The starting point for the representation of these functors is that of A_P . This is proved by Grothendieck [7]. We content ourselves with a precise statement of his result.

THEOREM 2.8 (GROTHENDIECK). *The functor A_P is represented by T_P, V_P, q where T_P is an analytic space, V_P is a curve of genus g over T_P and q is a Teichmüller rigidification of V_P ; i.e., a section of $\underline{P}(V_P/T_P)$. Furthermore, T_P is Hausdorff, nonsingular (thus a manifold) and of dimension $3g - 3$.*

3. The representation. As noted in the remark preceding Definition 1.2, the signature intervenes only as reflected by the associated signature structure. The following proposition makes this precise.

PROPOSITION 3.1. *Let (ν) be a signature and let r be the associated signature structure. If A_{pr}^r is represented by $T_{pr}^r, V_{pr}^r, e_1, \dots, e_n, r, q$, then A_{pr}^ν is represented by $T_{pr}^r, V_{pr}^r, e_1, \dots, e_n, (\nu), q$.*

Recall that to represent a functor F is to give an object T of the domain category together with an element of $F(T)$. In the above case, an element of $A_{pr}^r(T_{pr}^r)$ is an isomorphism class of curves over T_{pr}^r together with n sections, a signature structure and a section of $\underline{P}^r(V_{pr}^r/T_{pr}^r)$.

PROOF. There is only one way to assign a given signature to n points consistent with a given signature structure and there is only one signature structure associated to a given signature.

In the sequel, A_{pr}^r will be written simply as A_{pr} . It will be important to distinguish between A_P^r and A_{pr} . Both functors refer to classes of curves with signature structure but the former are rigidified by \underline{P} (as unpunctured curves) and the latter by \underline{P}^r (as curves with signature structure). We will first represent A_P^r and, using that result, then represent A_{pr} . This last step amounts to a change in the rigidifying group. Such questions are treated fully, in greater generality than we need, in [7, I, §8]. The following is what we need. Let \underline{P}^r be the functor from the category of punctured curves of genus g with signature structure r to the category of principal bundles of group γ^r defined in §2. Let $c: \gamma^r \rightarrow \gamma$ be the surjective homomorphism of Theorem 1.4 and let $\nu = \ker(c) = \pi_1(F(r)C)$. Then for any punctured curve X over S with signature structure r , $\underline{P}(X/S) = \underline{P}^r(X/S)/\nu$. ($\nu \subset \gamma^r$ acts on $\underline{P}^r(X/S)$ on the right.)

PROPOSITION 3.2 [7, I, PROPOSITION 8.1]. *Suppose T_{pr} exists. A necessary and sufficient condition that \underline{P} be rigidifying is that ν act freely on T_{pr} . In the affirmative case, T_P^r exists and is equal to T_{pr}/ν .*

Note. T_{pr} (resp. T_P^r) denotes the object representing A_{pr} (resp. A_P^r). Furthermore, as a subgroup of γ^r , ν acts on T_{pr} as described in the remark preceding Theorem 2.8.

PROPOSITION 3.3 [7, I, PROPOSITION 8.3]. *If T_{pr} exists then so does T_P^r . Furthermore, the latter is isomorphic to the inverse image of the marked section of $\underline{P}(V_P^r/T_P^r)$ via the canonical projection $\underline{P}^r(V_{pr}^r/T_{pr}^r) \rightarrow \underline{P}(V_P^r/T_P^r)$.*

Note. Recall that the existence of T_P^r entails the existence of a curve V_P^r over T_P^r and a section $\underline{P}(V_P^r/T_P^r)$ over T_P^r . It is this that we intend by the words "marked section".

THEOREM 3.4. A_P^r is represented by a manifold T_P^r of dimension $3g - 3 + n$. Furthermore, if T_P, V_P, q represent A_P , then $T_P^r = F(r)(V_P/T_P)$, $V_P^r = T_P^r \times_{T_P} V_P$, the sections $e_i: T_P^r \rightarrow V_P^r$ are given by

$$e_i: [x_1, \dots, x_i, \dots, x_n] \rightarrow ([x_1, \dots, x_i, \dots, x_n], x_i)$$

the signature structure is r and the section of

$$\underline{P}(V_P^r/T_P^r) = \underline{P}(V_P/T_P) \times_{T_P} T_P^r$$

is $q \times_{T_P} \text{id}_{T_P^r}$.

It is sometimes desirable to puncture the curve V_P^r in fact, by removing the images of the sections e_i . If this is done, then we have $\overline{V}_P^r = V_P^r - \bigcup\{e_i(T_P^r) \mid i = 1, \dots, n\} = F(r)(V_P/T_P)$. (Recall the definition preceding Proposition 1.7.)

PROOF. Recall that $F(r)(V_P/T_P) = F_n(V_P/T_P)/\Sigma(r)$. Let S be any analytic space. A morphism $f: S \rightarrow F_n(V_P/T_P)$ gives, upon composition with any of the n projections $F_n(V_P/T_P) \rightarrow T_P$, a morphism $g: S \rightarrow T_P$ which does not depend on the projection chosen. By Theorem 2.8, g gives an isomorphism class of Teichmüller curves over S (obtained by pulling back V_P over g). Furthermore, f also gives n morphisms of S into V_P which never coincide and thus induce n sections of $g^*(V_P)$ over S which are always distinct. Finally we see that two different morphisms $f_1, f_2: S \rightarrow F_n(V_P/T_P)$ give the same morphism $S \rightarrow F(r)(V_P/T_P)$ if and only if $g_1 = g_2$ and the sections associated with f_2 can be obtained from the sections associated with f_1 by the action of an element of $\text{Hom}(S, \Sigma(r))$. In view of the definition of isomorphism for curves with signature structure, the result follows. The fact that T_P^r is a manifold of dimension $3g - 3 + n$ depends on the fact that V_P is smooth over T_P of relative dimension 1 and T_P is itself a manifold of dimension $3g - 3$. The n th fiber power of V_P over T_P is thus smooth over T_P and of relative dimension n . $F_n(V_P/T_P)$ is an open subset of the fiber power and thus a manifold of the same dimension. Finally, the group $\Sigma(r)$ is finite so the quotient is of the same dimension and it acts freely so the quotient is also nonsingular.

PROPOSITION 3.5. $\pi_1(F(r)C) = \pi_1(T_P^r)$ where C is any surface of genus g .

PROOF. $V_P \rightarrow T_P$ is locally trivial since both are manifolds and the structure map is smooth. Furthermore, the first and second homotopy groups of T_P are trivial (since the usual Teichmüller space is contractible). The result is then just Proposition 1.6.

THEOREM 3.6. T_{pr} exists and is the universal covering space of T_P^r .

PROOF. By Proposition 3.3, T_{pr} exists. By Proposition 3.2, we see that $T_P^r = T_{pr}/\nu$, where ν can now be identified as the fundamental group of T_P^r (by Proposition 3.5). Since ν acts freely on T_{pr} (Proposition 3.2), T_{pr} is a covering space of T_P^r and since the covering group, ν , is $\pi_1(T_P^r)$, it follows that T_{pr} is the universal cover.

Note. There are two different fiber spaces over T_{pr} which qualify as universal Teichmüller curves. They are:

- (1) $V_{pr} = V_P^r \times_{T_P} T_{pr}$ (with sections induced from V_P^r);
- (2) $\bar{V}_{pr} = \bar{V}_P^r \times_{T_P} T_{pr}$ whose fibers are literally punctured.

PROPOSITION 3.7. $\bar{V}_{pr} = V_{pr} - \bigcup\{e_i(T_{pr}) \mid i = 1, \dots, n\}$.

PROOF. Immediate.

THEOREM 3.8. *All the T_{pr} are isomorphic (as r ranges over all signature structures on n points).*

PROOF. Let T_{P^*} denote T_{pr} when r is the complete signature structure; i.e., $\Sigma(r) = 1$. Similarly for T_P^* . Then, for any r , $T_P^r = T_P^*/\Sigma(r)$. Since $\Sigma(r)$ is a finite group acting freely, T_P^* is a covering space of T_P^r . Now T_{P^*} is the universal covering space of T_P^* and therefore also of each of the T_P^r . Since T_{pr} is the universal covering space of T_P^r , the result follows.

PROPOSITION 3.9. *If r is the trivial signature structure; i.e., $\Sigma(r) =$ the full symmetric group on n letters, then T_{pr} is the usual Teichmüller space of Riemann surfaces of type (g, n) .*

PROOF. A marked Riemann surface of type (g, n) is an n -punctured curve X of genus g over a reduced one-point space S together with a homotopy class of orientation-preserving homeomorphisms of X which preserve the punctures as a set of points. Such a homotopy class is an element of γ^r . Over a one-point space, an element of γ^r is the same as a section of $\underline{P}^r(X/S)$. Thus a marked Riemann surface of type (g, n) in the usual Teichmüller space theory is the same as an n -punctured Teichmüller r -curve of genus g over a reduced one-point space S in our language. By Theorem 3.6, these last are in one-to-one correspondence with morphisms $S \rightarrow T_{pr}$ which, in turn, are in one-to-one correspondence with the points of T_{pr} . Thus we have defined a set-theoretic bijection $a: T(g, n) \rightarrow T_{pr}$, such that each fiber of V_{pr} over T_{pr} is precisely the corresponding marked Riemann surface. Now, the Teichmüller space $T(g, n)$ carries an analytic fiber space $F(g, n)$ [2] each fiber of which is a disk admitting the action of a common Fuchsian group G . The quotient $F(g, n)/G$ is still an analytic fiber space over $T(g, n)$ and the fiber over each point (which "stands for" a marked Riemann surface) is that marked Riemann surface. Thus $T(g, n)$ carries a Teichmüller curve of genus g , which, by representability, is obtained as the pull-back of \bar{V}_{pr} via a unique morphism $T(g, n) \rightarrow T_{pr}$. In view of the pull-back relation, this map must coincide with a so that a is not only bijective but also analytic. Such a map is an analytic isomorphism.

4. **Complements on the fiber space.** From Propositions 1.6 and 1.7, we obtain the exact sequence (remembering again that T_P is contractible):

$$\begin{aligned}
 (*) \quad 1 \rightarrow \pi_1(C - \{x_1, \dots, x_n\}, x_{n+1}) &\rightarrow \pi_1(F(\hat{r})(V_P/T_P), [x_1, \dots, x_{n+1}]) \\
 &\rightarrow \pi_1(F(r)(V_P/T_P), [x_1, \dots, x_n]) \rightarrow 1.
 \end{aligned}$$

Note that the middle term of this exact sequence can be identified with $\pi_1(\bar{V}'_P)$ and the last term with $\pi_1(T'_P)$ (Theorem 3.4).

PROPOSITION 4.1. \bar{V}_{P^r} is the covering space of $F(\hat{r})(V_P/T_P)$ associated with the normal subgroup $\pi_1(C - \{x_1, \dots, x_n\}, x_{n+1})$ of $\pi_1(F(\hat{r})(V_P/T_P))$.

PROOF. Consider the Cartesian diagram:

$$\begin{array}{ccc}
 \bar{V}_{P^r} & \rightarrow & \bar{V}'_P = F(\hat{r})(V_P/T_P) \\
 \downarrow & & \downarrow \\
 T_{P^r} & \rightarrow & T'_P = F(r)(V_P/T_P)
 \end{array}$$

Both vertical arrows are locally trivial fiber bundles and the fiber in each case is the curve C punctured in n points. Furthermore, since T_{P^r} is a covering space of T'_P , it follows that \bar{V}_{P^r} is a covering space of \bar{V}'_P . To determine the associated fundamental group, we consider the following diagram of long exact sequences, in which \bar{C} will denote $C - \{x_1, \dots, x_n\}$:

$$\begin{array}{cccccccccccc}
 \dots & \rightarrow & \pi_2(\bar{C}) & \rightarrow & \pi_2(\bar{V}_{P^r}) & \rightarrow & \pi_2(T_{P^r}) & \rightarrow & \pi_1(\bar{C}) & \rightarrow & \pi_1(V_{P^r}) & \rightarrow & \pi_1(T_{P^r}) & \rightarrow & 1 \\
 & & \text{id} \wr \downarrow & & \wr \downarrow & & \wr \downarrow & & \text{id} \wr \downarrow & & \downarrow & & \downarrow & & \\
 \dots & \rightarrow & \pi_2(\bar{C}) & \rightarrow & \pi_2(\bar{V}'_P) & \rightarrow & \pi_2(T'_P) & \rightarrow & \pi_1(\bar{C}) & \rightarrow & \pi_1(\bar{V}'_P) & \rightarrow & \pi_1(T'_P) & \rightarrow & 1.
 \end{array}$$

The nonidentity vertical arrows are isomorphisms or injections because of the covering space relationship. Furthermore, since the universal covering space of \bar{C} is the disk, $\pi_2(\bar{C}) = 1$. We also know that T_{P^r} is a universal cover and thus simply connected. The diagram becomes:

$$\begin{array}{ccccccccccc}
 1 & \rightarrow & \pi_2(\bar{V}_{P^r}) & \xrightarrow{a'} & \pi_2(T_{P^r}) & \xrightarrow{b'} & \pi_1(\bar{C}) & \xrightarrow{c'} & \pi_1(\bar{V}_{P^r}) & \rightarrow & 1 \\
 & & \wr \downarrow & & \wr \downarrow & & \text{id} \wr \downarrow & & \downarrow & & \\
 1 & \rightarrow & \pi_2(\bar{V}'_P) & \xrightarrow{a} & \pi_2(T'_P) & \xrightarrow{b} & \pi_1(\bar{C}) & \xrightarrow{c} & \pi_1(\bar{V}'_P) & \rightarrow & \pi_1(T'_P) & \rightarrow & 1.
 \end{array}$$

By comparison with (*), we see that c is injective whence $\text{im } b = 1$. Thus a is an isomorphism so that a' is too. This implies that c' is an isomorphism which identifies the defining subgroup $\pi_1(\bar{V}_{P^r})$ with $\pi_1(\bar{C})$ in (*).

COROLLARY 4.2. $\pi_n(\bar{V}_{P^r}) \rightarrow \pi_n(T_{P^r})$ and $\pi_n(\bar{V}'_P) \rightarrow \pi_n(T'_P)$ are isomorphisms for $n \geq 2$.

PROOF. For $n = 2$, the result is proved above. For $n > 2$, the result follows from the long exact sequences and the triviality of the higher homotopy of \bar{C} .

REMARKS. (1) This corollary says much less than it appears to. In fact, all the groups involved are trivial. This fact, however, follows from the contractibility of T_{pr} which we have not assumed.

(2) If we are willing to use the fact that T_{pr} is contractible then we immediately get $\pi_2(T_{pr}) = \pi_2(T_p^r) = 1$ and we obtain a new proof of Proposition 1.7 and a much easier proof of Proposition 4.1.

COROLLARY 4.3. $T_{pr\hat{r}}$ is the universal covering space of \bar{V}_{pr} .

PROOF. We have already seen that $T_{pr\hat{r}}$ is the universal covering space of $F(\hat{r})(V_p/T_p)$ and Proposition 4.1 shows that \bar{V}_{pr} is a covering space of the same object.

PROPOSITION 4.4. Let $F(g, n)$ be the analytic fiber space of [2] over $T(g, n)$. Then $F(g, n)$ is the universal covering space of \bar{V}_p^r .

PROOF. We know that $F(g, n)$ admits the action of a Fuchsian group G which is identified with $\pi_1(\bar{C})$. Furthermore, $F(g, n)/G = \bar{V}_{pr}$. Thus $F(g, n)$ is a covering space of \bar{V}_{pr} and the map c' of the proof of Proposition 4.1 identifies G with the fundamental group of \bar{V}_{pr} . Thus $F(g, n)$ is the universal space.

COROLLARY 4.5 (BERS). $F(g, n) = T(g, n + 1)$.

PROOF. Corollary 4.3 and Proposition 4.4 identify $F(g, n)$ with $T_{pr\hat{r}}$. Since \hat{r} is a signature structure on $n + 1$ points the result follows from Theorem 3.8 and Proposition 3.9.

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