TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY

COMPARISON THEOREMS FOR BOUNDED SOLUTIONS OF $\Delta u = Pu$

BY

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ABSTRACT. Let P and Q be C^1 densities on a hyperbolic Riemann surface R. A characterization of isomorphisms between the spaces of bounded solutions of $\Delta u = Pu$ and $\Delta u = Qu$ on R in terms of the Wiener harmonic boundary is given.

In 1959, H. Royden [6] proved the following comparison theorem: *If P and Q are nonnegative C^1 densities on a Riemann surface R such that there is a constant c with $c^{-1}P \leqslant Q \leqslant cP$ outside some compact subset of R, then the spaces of bounded solutions of $\Delta u = Pu$ and $\Delta u = Qu$ on R are isomorphic.* In response to a question posed by Royden, M. Nakai [4] in 1960 showed that the same conclusion follows under the assumption that $\int_R |P - Q| < +\infty$. The conditions $\int_R |P - Q| < +\infty$ and $c^{-1}P \leqslant Q \leqslant cP$ outside a compact subset are independent and neither is a necessary condition for the conclusion. Recently A. Lahtinen [2] gave a necessary and sufficient condition for the bounded solutions of $\Delta u = Pu$ on R to be isomorphic to the harmonic bounded functions. This result actually is implicit in the paper of Loeb and Walsh [3] in the axiomatic setting.

In this paper a necessary and sufficient condition is given for the existence of an isomorphism S between the spaces of bounded solutions of $\Delta u = Pu$ and $\Delta u = Qu$ on R with $|u - Su|$ bounded by a potential. This result contains the ones mentioned above as special cases.

1. Let R be a Riemann surface and P a nonnegative C^1 density on R. To avoid trivial considerations we assume R is hyperbolic. We denote by $P(U)$ the space of solutions of $\Delta u = Pu$ on an open subset U of R. The subspace of bounded solutions of $P(U)$ will be denoted by $PB(U)$. The superscript c in the notation $P^c(U)$ and $P^cB(U)$ denotes the subspaces with continuous extensions to ∂U. It is conventional to use the symbol H in case $P \equiv 0$.

Received by the editors January 18, 1974.

AMS (MOS) subject classifications (1970). Primary 31A35; Secondary 30A50, 35J05.

Key words and phrases. Solution of $\Delta u = Pu$, Riemann surfaces, Green's function, maximum principle, Wiener harmonic boundary.

The Wiener ideal boundary will play a fundamental role and therefore we briefly mention some of its properties here (for more details, cf. [7]). The Wiener algebra N associated with R is the set of bounded continuous harmonizable functions on R. The bounded continuous superharmonic functions on R, for example, are contained in N. It can be seen that N is also a vector lattice. The potential subalgebra N_δ consists of the functions in N with 0 harmonizations.

The Wiener compactification R^* of R is a compact Hausdorff space which contains R as an open dense subset and such that the functions in N extend continuously to R^* and separate points there. The space R^* is unique up to a homeomorphism fixing R pointwise. In general we shall use the same symbol for a function in N and its continuous extension to R^*. We use $\overline{A}$ to denote the closure of A in R^* and ∂A for the boundary of A with respect to R. If we use A^* to denote a subset of R^*, then by A we shall mean $A^* \cap R$.

The Wiener harmonic boundary δ is the set of common zeros of functions in N_δ and hence is a compact subset of $R^*\backslash R$. The main properties of δ that we shall need are the following: If U is an open subset of R with ∂U piecewise analytic and $f \in N$, then there is a function $h \in N$ such that $h \in H^c(U)$ and $h - f\,|(R\backslash U) \cup (\overline{U} \cap \delta) = 0$. The function h is called the harmonic projection of f with respect to U. If s is bounded and subharmonic on U with continuous extension to ∂U then $s \leqslant \max_{\partial U \cup (\overline{U} \cap \delta)} s$. Therefore a superharmonic function on U with continuous extension to ∂U is a potential if and only if it vanishes on $\partial U \cup (\overline{U} \cap \delta)$.

We shall use the notation $\|\varphi\|_A$ for the supremum of the function $|\varphi|$ on A.

2. We consider solutions of $\Delta u = Pu$ on subregions G with ∂G piecewise analytic (∂G may be $\varnothing$). The fact that nonnegative solutions are subharmonic gives the following (cf. [3]).

LEMMA. *If* $u \in P^cB(G)$, *then* $\|u\|_G \leqslant \|u\|_{\partial G \cup (\overline{G} \cap \delta)}$. *Moreover, if* $u|\partial G \cup (\overline{G} \cap \delta) \geqslant 0$, *then* $u \geqslant 0$.

We now describe the integral operator T_G which is the basic tool here. Let Ω be a relatively compact region in R with $\partial\Omega$ piecewise analytic. Define $\tau_{G\cap\Omega}$ on the bounded C^1 functions on Ω by setting $\tau_{G\cap\Omega}\varphi = \int_\Omega g_{G\cap\Omega}(\cdot\,, z)\varphi P$, where $g_{G\cap\Omega}(\cdot\,, z)$ denotes the Green's function of $G \cap \Omega$ with pole at z. Then $\tau_{G\cap\Omega}\varphi$ is a C^2 function on $G \cap \Omega$ vanishing continuously on $\partial(G \cap \Omega)$ and satisfies $\Delta\tau_{G\cap\Omega}\varphi = -\varphi P$.

Therefore setting $T_{G\cap\Omega}u = u + \tau_{G\cap\Omega}u$ gives an operator $T_{G\cap\Omega}$:

$P^c(G \cap \Omega) \longrightarrow H^c(G \cap \Omega)$ such that $u - T_{G\cap\Omega} u \,|\, \partial(G \cap \Omega) = 0$. For $u \in P^cB(G)$ with $u \geqslant 0$, $T_G u = \lim_{\Omega\to R} T_{G\cap\Omega}\, u$ exists and is given by $T_G u = u + \tau_G u$, where $\tau_G u = \int_G g_G(\cdot\,, z)uP$. If $u \in P^cB(G)$ is arbitrary, then $T_G u$ is defined to be $T_G u_1 - T_G u_2$ where $u = u_1 - u_2$, $u_i \in P^cB(G)$, $u_i \geqslant 0$. We collect here some of the known properties of T_G that will be needed in later arguments (cf. [4], [5]).

THEOREM. *The operator* T_G: $P^cB(G) \longrightarrow H^cB(G)$ *gives an isometric isomorphism such that* $u - T_G u \,|\, \partial G \cup (\bar{G} \cap \delta) = 0$. *If* $\int_G g_G(\cdot\,, z)P < +\infty$, *for every* $z \in G$, *then* T_G *is onto.*

3. We introduce the set $\delta^P = \{p \in \delta \mid p$ has a nbd U^* in R^* with $\int_U g_U(\cdot\,, z)P < +\infty$ for each $z \in U\}$. Here $g_U(\cdot\,, z)$ for an arbitrary open set U is defined as follows: Let the component of U containing z be denoted by U_z. Then $g_U(\cdot\,, z)$ is the Green's function of U_z on U_z and zero on $U \backslash U_z$.

Clearly δ^P is an open subset of δ. The significance of δ^P stems from the following

THEOREM. *The functions in* $PB(R)$ *restricted to* δ *vanish on* $\delta \backslash \delta^P$.

It is sufficient to prove the assertion for $u \in PB(R)$ with $u \geqslant 0$. By Theorem 2, $\tau_R u(z) < +\infty$ for each $z \in R$. Suppose $p \in \delta$ and $u(p) \neq 0$. By the continuity of u on R^* there is a neighborhood U^* of p and an $\epsilon > 0$ such that $u \,|\, U^* \geqslant \epsilon$. Then

$$+\infty > \tau_R u(z) > \int_U g_R(\cdot\,, z)uP \geqslant \epsilon \int_U g_R(\cdot\,, z)P \geqslant \epsilon \int_U g_U(\cdot\,, z)P,$$

for each $z \in U$. Thus $p \in \delta^P$.

Combining this with Lemma 2 gives the

COROLLARY. *If* $u \in PB(R)$, *then* $\|u\|_R \leqslant \|u\|_{\delta^P}$. *If in addition* $u \,|\, \delta^P \geqslant 0$, *then* $u \geqslant 0$.

4. Denote by $C_C(A)$ the continuous functions with compact support in A and by $C_0(A)$ the closure of $C_C(A)$ with respect to $\|\cdot\|_A$.

THEOREM. *The spaces* $PB(R)$ *and* $C_0(\delta^P)$ *are isomorphic vector lattices. The isomorphism is obtained by restriction to* δ^P, *i.e.* $PB(R) \,|\, \delta^P = C_0(\delta^P)$.

We begin the proof by observing that for any $f \in N$ with $K = \operatorname{supp}(f \,|\, \delta)$ a compact subset of δ^P, there exists a $u \in PB(R)$ with $u - f \,|\, \delta = 0$. In fact we may assume $f \geqslant 0$. Cover K by a finite number of sets U_i^*, $i = 1, \cdots, m$ such that $\int_{U_i} g_{U_i}(\cdot\,, z)P < +\infty$ for each $z \in U_i$. By taking U_i^* slightly smaller if necessary we may assume that ∂U_i is piecewise analytic.

By the Urysohn property for N we can find $\varphi_i \in N$ such that $\operatorname{supp} \varphi_i \subset U_i^*$, $\varphi_i \geqslant 0$ and $\Sigma_1^m \varphi_i | K = 1$. Let $f_i = \varphi_i f \in N$ and note that $f = \Sigma_1^m f_i$ on K. Denote by h_i the harmonic projection of f_i with respect to U_i, i.e. $h_i \in H^cB(U_i)$ and $0 = f_i | \partial U_i = h_i | \partial U_i, f_i | U_i^* \cap \delta = h_i | U_i^* \cap \delta$. Let G be any component of U_i. Then $\int_G g_G(\cdot, z)P < +\infty$ for every $z \in G$. Thus by Theorem 2 there is a function $u \in P^cB(G)$ such that $u | \partial G = 0, u | \overline{G} \cap \delta = h_i | \overline{G} \cap \delta$. Repeating this in each component of U_i we obtain $v_i \in P^cB(U_i)$ with $v_i \geqslant 0, v_i | \partial U_i = 0$ and $v_i - f | U_i^* \cap \delta = 0$. Setting $v_i = 0$ on $R \backslash U_i$ gives a subsolution on R.

Let k_i be the least harmonic majorant of v_i. Take an exhaustion $\{\Omega_n\}$ of R by regular regions. Denote by u_{in} the function in $P^c(\Omega_n)$ such that $u_{in} - v_i | \partial \Omega_n = 0$. Then we have $0 \leqslant v_i \leqslant u_{in} \leqslant u_{i,n+1} \leqslant k_i$. Thus $u_i = \lim_n u_{in}$ is in $PB(R)$ and $0 \leqslant v_i \leqslant u_i \leqslant k_i$. Since $k_i - v_i$ is a potential on R it vanishes on δ and consequently $k_i - u_i$ also vanishes there. That is, $u_i - v_i | \delta = 0$. The function $u = \Sigma_1^m u_i \in PB(R)$ has the property that $u | \delta = f$.

For an arbitrary $f \in C_0(\delta^P)$ there is a sequence $\{f_k\}$ of functions of the sort considered above such that $\|f - f_k\|_{\delta^P} \longrightarrow 0$. Then the corresponding sequence of solutions $\{u_k\}$ is a Cauchy sequence with respect to $\|\cdot\|_R$ by Corollary 3. Thus there is a $u \in PB(R)$ such that $\|u - u_k\|_R \longrightarrow 0$. Let $u | \delta^P = g$ and for a given $\epsilon > 0$ take k such that $\|u - u_k\|_R < \epsilon$ and $\|f - f_k\|_{\delta^P} < \epsilon$. Then the denseness of R in R^* gives $\|g - f_k\|_{\delta^P} < \epsilon$. This means $g = f$ on δ^P.

Thus far we have shown $C_0(\delta^P) \subset PB(R) | \delta^P$. For the proof of the reverse inclusion take $u \in PB(R)$ which without loss of generality can be assumed to be nonnegative. By Theorem 3, $u | \delta \backslash \delta^P = 0$. Set $K_n = \{p \in \delta \,|\, u(p) \geqslant 1/n\}$. K_n is a closed and hence compact subset of δ. Since $K_n \subset \delta^P$ it is compact in δ^P. Now choose $\varphi_n \in N$ such that $\varphi_n K_n = 1, \varphi_n | \delta \backslash K_{n+1} = 0$ and $0 \leqslant \varphi_n \leqslant 1$. Then $\varphi_n u \in C_C(\delta^P)$ and $\|u - \varphi_n u\|_{\delta^P} \leqslant 1/n$.

Corollary 3 implies that the mapping of $PB(R)$ onto $C_0(\delta^P)$ obtained by restriction to δ^P preserves order and sup norm.

A slightly more tractable description of δ^P can be derived from this theorem.

COROLLARY. *$\delta^P = \{p \in \delta \,|\, p$ has a nbd U^* in R^* such that $\int_U g_R(\cdot, z)P < +\infty$ for each $z \in U\}$.*

Since $g_R(\cdot, z) \geqslant g_U(\cdot, z)$ for $z \in U$ we need only show that for each $p \in \delta^P$ there is a neighborhood U^* of p such that $\int_U g_R(\cdot, z)P < +\infty$. But by the theorem there is a function $u \in PB(R)$ such that $u(p) = 2$. Set $U^* = \{q \in R^* \,|\, u(q) > 1\}$. By Theorem 2, $\tau_R u$ exists and, in particular, $+\infty > \tau_R u(z) \geqslant \int_U g_R(\cdot, z)P$, for $z \in U$.

5. The main result is as follows:

THEOREM. *Suppose P and Q are nonnegative C^1 densities on a hyperbolic Riemann surface R. There is an isomorphism S between $PB(R)$ and $QB(R)$ such that $|u - Su|$ is bounded by a potential on R if and only if $\delta^P = \delta^Q$.*

If $\delta^P = \delta^Q$, then both $PB(R)$ and $QB(R)$ are isomorphic to $C_0(\delta^P)$ by restriction to δ^P. Thus define $S: PB(R) \longrightarrow QB(R)$ by $u - Su|\delta^P = 0$. In order to show that $|u - Su|$ is bounded by a potential, express u as $u = u_1 - u_2$ with $u_i \in PB(R)$ and $u_i \geqslant 0$. Note that $u|\delta^P = u_1|\delta^P - u_2|\delta^P$ which implies that $Su = Su_1 - Su_2$, $Su_i \geqslant 0$. Take $h_i \in HB(R)$ with $h_i|\delta = u_i|\delta$, $i = 1, 2$. Since $u_i - h_i$ and $Su_i - h_i$ are bounded subharmonic functions on R which vanish on δ we have $u_i - h_i \leqslant 0, Su_i - h_i \leqslant 0$. Thus $|u - Su|$ is bounded by the potential $(h_1 - u_1) + (h_1 - Su_1) + (h_2 - u_2) + (h_2 - Su_2)$.

Conversely, suppose an isomorphism S as described in the theorem exists. Then $|u - Su||\delta = 0$ for each $u \in PB(R)$. If $p \in \delta^P$, then by Theorem 4 we can find $u \in PB(R)$ with $u(p) = 1$. This implies that $Su(p) = 1$ and in view of Theorem 3 we conclude that $p \in \delta$, i.e. $\delta^P \subset \delta^Q$. By symmetry we obtain $\delta^P = \delta^Q$.

The assumption on $|u - Su|$ implies that S preserves the behavior of functions on δ^P. In view of Theorem 4 this means that S commutes the lattice operations. If the assumption on $|u - Su|$ is replaced by the assumption that S is a vector lattice isomorphism, then by the Kakutani theorem we see that δ^P and δ^Q are homeomorphic.

The results of Royden [6] and Nakai [4] are immediate consequences.

COROLLARY. *If P and Q are C^1 densities on a hyperbolic Riemann surface R such that $c^{-1}P \leqslant Q \leqslant cP$ outside some compact subset and for some constant c, then $PB(R)$ and $QB(R)$ are isomorphic.*

COROLLARY. *If P and Q are C^1 densities on a hyperbolic Riemann surface R and $\int_R |P - Q| < +\infty$, then $PB(R)$ and $QB(R)$ are isomorphic.*

In the first case it is clear that the hypothesis implies that $\delta^P = \delta^Q$. In the second case note that $g_R(\cdot\,, z)|\delta = 0$ and hence there is a neighborhood V^* of δ in R^* with $g_R(\cdot\,, z)|V^* \leqslant 1$. Thus $\int_V g_R(\cdot\,, z)|P - Q| < +\infty$. By the Harnack inequality this is also valid if z is allowed to vary and the conclusion $\delta^P = \delta^Q$ now follows.

Actually the corollaries followed from the slightly weaker hypotheses $c^{-1}P \leqslant Q \leqslant cP$ in V, $\int_V |P - Q| < +\infty$, where V^* is a neighborhood of δ in R^*.

Denote by w the greatest solution of $\Delta u = Pu$ on R which is less than 1 on R. The following result is due to Lahtinen [2] and Loeb and Walsh [3].

COROLLARY. *$HB(R)$ and $PB(R)$ are isomorphic vector lattices if and only if* 1 *is the least harmonic majorant of w on R.*

Let h be the least harmonic majorant of w. Then $h - w$ is a potential and hence vanishes on δ. Therefore, h is the constant 1 if and only if $w|\delta = 1$. This in turn is equivalent to $\delta^P = \delta$ which is equivalent to $PB(R)$ being isomorphic to $HB(R)$.

6. Denote by $PBE(R)$ (resp. $PBD(R)$) the subspace of $PB(R)$ such that $E(u) = \int_R du \wedge *du + u^2 P < +\infty$ (resp. $D(u) = \int_R du \wedge *du < +\infty$). Denote by Δ the Royden harmonic boundary of R, R^* the corresponding compactification and define

$$\Delta^P = \left\{p \in \Delta \,|\, p \text{ has a nbd } U^* \text{ in } R^* \text{ with } \int_U P < +\infty\right\},$$

$$\Delta_P = \left\{p \in \Delta \,|\, p \text{ has a nbd } U^* \text{ in } R^* \text{ with } \iint_{U \times U} g_R(x, y) P(x) P(y) < +\infty\right\}.$$

These definitions lead to criteria for isomorphisms between the closures with respect to the sup norm of the bounded energy finite or bounded Dirichlet finite solutions.

THEOREM. *Suppose P and Q are nonnegative C^1 densities on R. There is an isomorphism S between $\overline{PBE(R)}$ and $\overline{QBE(R)}$ (resp. $\overline{PBD(R)}$ and $\overline{QBD(R)}$) such that $|u - Su|$ is bounded by a potential on R if and only if $\Delta^P = \Delta^Q$ (resp. $\Delta_P = \Delta_Q$).*

The proof is analogous to that of Theorem 5 and therefore we only mention some differences. The operator T_G defined in §2 also maps the spaces $P^c BE(G)$ and $P^c BD(G)$ into $H^c BD(G)$. If $\int_G P < +\infty$, then

$$T_G(P^c BE(G)) = H^c BD(G)$$

(cf. [1]) and if $\iint_{G \times G} g_G(x, y) P(x) P(y) < +\infty$, then

$$T_G(P^c BD(G)) = H^c BD(G)$$

(cf. [5]). This is the motivation for the choice of Δ^P and Δ_P. The proper analogue of Theorem 4 is that the closure of $PBE(R)$ (resp. $PBD(R)$) with respect to the sup norm restricted to Δ^P (resp. Δ_P) is the space $C_0(\Delta^P)$ (resp. $C_0(\Delta_P)$) but this causes no complications. In the proof some complications do occur because of the need to establish the convergence of sequences of functions in the D or E norm.

ADDED IN PROOF. M. Nakai (*Banach spaces of bounded solutions of $\Delta u = Pu$ on hyperbolic Riemann surfaces*, Nagoya Math. J. **53** (1974), 141–155) has simultaneously discovered Theorem 5 and also has given a more detailed analysis of its consequences.

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