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FUCHSIAN MANIFOLDS

BY

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ABSTRACT. Recently Eberlein and O'Neill have investigated Riemannian manifolds of negative sectional curvature. For visibility manifolds, they have obtained a classification into three types: parabolic, axial and fuchsian. Fundamental groups of fuchsian manifolds of finite type will be investigated. The main theorem is that isometry groups of certain (not necessarily compact) fuchsian manifolds are finite. Fundamental groups of fuchsian manifolds of finite type are not amenable. The spectral radius of the random matrix of the fundamental group of a compact Riemannian manifold of negative sectional curvature is less than one.

1. **Introduction.** Recently, Bishop, Eberlein and O'Neill have investigated Riemannian manifolds of negative sectional curvature in [1], [4], [5] and [6]. Let M be a complete Riemannian manifold of sectional curvature $K \leqslant 0$. Then M has a simply connected covering manifold $\widetilde{M}$ which can be given the cone topology so that the boundary $\widetilde{M}(\infty)$ is a sphere and $\widetilde{M} \cup \widetilde{M}(\infty)$ is a closed cell. If any two points in $\widetilde{M}(\infty)$ can be joined by a geodesic, then M is called a visibility manifold. According to the limit set of the fundamental group $\pi_1(M), M$ is called parabolic, axial and fuchsian.

In this paper we require the additional condition for M that there exists a unique geodesic joining any two distinct points in the boundary $\widetilde{M}(\infty)$ of its universal covering manifold $\widetilde{M}$. This condition holds when the sectional curvature K of M satisfies $K \leqslant c < 0$ [4], [6]. Our results seem to be true for visibility manifolds; however we require this additional condition for convenience. This condition has been used by Eberlein in [4]. Under a compactness assumption on the limit set of $\pi_1(M)$ (see §2), M is said to be of finite type. When $\widetilde{M}$ is the hyperbolic plane, M is of finite type if and only if $\pi_1(M)$ is finitely generated [9]. We have not been able to obtain such a characterization for fuchsian manifolds. The purpose of this paper is to extend several results of Greenberg [8], [9], [10] to fuchsian manifolds of finite type and certain symmetric space forms. We also answer a question of Milnor [12] about

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random walks on $\pi_1(M)$. For convenience, we shall from now on omit all notations indicating induced homomorphisms between fundamental groups by projections and liftings. For instance, for a covering space $\Phi\colon \widetilde{X} \longrightarrow X$, we shall ignore the induced monomorphism $\Phi^*\colon \pi_1(\widetilde{X}) \longrightarrow \pi_1(X)$ and identify $\pi_1(\widetilde{X})$ with $\Phi^*\pi_1(\widetilde{X})$.

In §2, preliminaries are stated. Mainly these facts can be found in [1], [2], [4], [5] and [6]. §3 deals with fundamental groups and their subgroups of finite indices. Let M_1 and M_2 be two compact fuchsian manifolds covering the same manifold M. Consider the intersection of images of $\pi_1(M_1)$ and $\pi_1(M_2)$ in $\pi_1(M)$ under natural homomorphisms induced by projections. If the corresponding covering manifold M' of M_1 and M_2 is fuchsian, then M' is of finite type. If M is a covering fuchsian manifold of finite type of a fuchsian manifold M' such that the limit sets $L(\pi_1(M))$ and $L(\pi_1(M'))$ coincide, then M is a finite covering of M'. Let M be a fuchsian manifold of finite type covering a fuchsian manifold M' such that $\pi_1(M)$ is a N-subgroup of $\pi_1(M')$. Then the index $[\pi_1(M') : \pi_1(M)]$ is finite. Consequently, the isometry group of a fuchsian manifold M of finite type is finite, if the limit set of its fundamental group is not contained in the boundaries of its proper totally geodesic submanifolds of the universal covering manifold $\widetilde{M}$. This is shown in §4. These manifolds are not necessarily compact.

In §5, we show that commensurability groups of fundamental groups of certain rank one symmetric space forms are discrete. These space forms are not of finite volume. To determine whether the commensurability group of a lattice (a discrete subgroup whose quotient has finite volume) of a Lie group is discrete or not is an open problem [10]. Of arithmetic subgroups of semisimple Lie groups, the commensurability groups are dense in the whole groups. Finally in §6, we consider random walks on $\pi_1(M)$. Milnor in [12] has raised the following problem. Consider a left-invariant, symmetric random walk on the fundamental group $\pi_1(M)$ of a compact Riemannian manifold M of negative sectional curvature. Is the spectral radius of the matrix associated with the random walk on $\pi_1(M)$ less than 1? The answer is affirmative. Further we obtain another proof of a theorem in [12].

2. **Preliminaries.** Let M be a complete Riemannian manifold of sectional curvature $K \leqslant 0$. Then M has a simply connected covering manifold $\widetilde{M}$ which can be given the cone topology so that the boundary $\widetilde{M}(\infty)$ is a sphere and $\widetilde{M} \cup \widetilde{M}(\infty)$ is a closed n-cell. An isometry ϕ of $\widetilde{M}$ extends naturally to a homeomorphism of $\widetilde{M} \cup \widetilde{M}(\infty)$. Isometries of $\widetilde{M}$ are classified into three classes:

(1) ϕ is elliptic if ϕ has a fixed point in $\widetilde{M}$.

(2) ϕ is axial if ϕ has two fixed points in $\widetilde{M}(\infty)$.

(3) ϕ is parabolic if ϕ has one fixed point in $\widetilde{M}(\infty)$.

If Γ is a properly discontinuous group of isometries of $\widetilde{M}$, one obtains a closed Γ-invariant limit set $L(\Gamma)$ in $\widetilde{M}(\infty)$, i.e. the set of accumulation points of an orbit $\Gamma(p)$ in $\widetilde{M}(\infty)$ of an arbitrary $p \in \widetilde{M}$. It is independent of the choice of $p \in \widetilde{M}$. A visibility manifold M is a complete Riemannian manifold with $K \leqslant 0$ such that for any points $x \neq y$ in $\widetilde{M}(\infty)$, there exists at least one geodesic joining x and y. By investigating limit sets of the fundamental groups $\pi_1(M)$ of visibility manifolds M, Eberlein and O'Neill have divided all visibility manifolds into three types: parabolic, axial or fuchsian. When $L(\pi_1(M))$ is a single point, M is called a parabolic manifold. When $L(\pi_1(M))$ is exactly two points, M is called an axial manifold. Otherwise M is called a fuchsian manifold.

If M is fuchsian, then $\pi_1(M)$ is the disjoint union of its stability groups at boundary points x in $\widetilde{M}(\infty)$. Stability groups are permuted by inner automorphisms, that is $\phi\pi_1(M)_x\phi^{-1} = \pi_1(M)_{\phi x}$. Furthermore, $\pi_1(M)_x = \pi_1(M)_{\phi x}$ if and only if $\phi \in \pi_1(M)_x$.

In this paper we shall only consider visibility manifolds satisfying an additional condition that for any points $x \neq y$ in $\widetilde{M}(\infty)$ there exists at most one geodesic joining x and y. The following results are basic and will be needed in the sequel.

LEMMA 2.1 [2]. *Let S be a closed subset of $\widetilde{M}(\infty)$ which contains more than one point and is invariant under a subgroup G of $I_0(\widetilde{M})$. Then $S \supset L(G)$.*

LEMMA 2.2. *Let G and H be subgroups of $I_0(\widetilde{M})$. Suppose that $hGh^{-1} = G$ for all $h \in H$. Then*

(a) *$L(G)$ is invariant under H* [2], [6].

(b) *If $L(G)$ contains more than one point, then $L(H) \subset L(G)$* [2].

LEMMA 2.3 [6]. *Let α be an axis of an isometry ϕ of $\widetilde{M}$ with end points x and y. Let ψ be an isometry of $\widetilde{M}$ that fixes one of these end points such that ψ and ϕ generate a properly discontinuous group. Then ψ commutes with a power of ϕ and hence fixes the other end point of α.*

Let $A(\Gamma)$ be the set of all fixed points of axial elements of Γ and $P(\Gamma)$ be the set of all fixed points of parabolic elements. Then $A(\Gamma)$, $P(\Gamma)$ and $L(\Gamma)$ are invariant under the normalizer of Γ in the isometry group $I(\widetilde{M})$.

To each properly discontinuous group Γ of isometries of $\widetilde{M}$, we can associate three objects. There is a canonical fundamental domain defined to be the set $\{p \in \widetilde{M} \mid d(p, p_0) \leqslant d(\gamma p, p_0), \gamma(p_0) \neq p_0, \text{ for all } \gamma \in \Gamma\}$. A subset S of $\widetilde{M} \cup \widetilde{M}(\infty)$ is called (geodesically) convex if with every two of its points it contains the geodesic segment between them. We denote by $[S]$ the convex hull of S. It is the intersection of all convex subsets of $\widetilde{M} \cup \widetilde{M}(\infty)$ containing S. The convex figure of Γ is the set $K(\Gamma) = [L(\Gamma)] \cap \widetilde{M}$. This is a convex set which is invariant under Γ. To each point z in $P(\Gamma)$, we associate a sufficiently small horosphere H_z at z such that, if $\phi \in \pi_1(M)$ and $z' = \phi(z)$, then $H_{z'} = H_{\phi(z)} = \phi H_z$. We denote by $K^*(\Gamma)$ the complement in $K(\Gamma)$ of the union of interiors of those horospheres. Then $K^*(\Gamma)$ is neither unique nor convex but it is invariant under Γ. If there exists a ball $B = B(p_0, r) = \{p \in \widetilde{M} \mid d(p, p_0) < r\}$ about a certain point p_0 in $\widetilde{M}$ such that $K^*(\Gamma) \subset \Gamma B = \bigcup_{\gamma \in \Gamma} \gamma B$, then $K^*(\Gamma)$ is said to be compact mod Γ. This is equivalent to the compactness of the quotient obtained from $K^*(\Gamma)$ by identifying points congruent under Γ. Nielsen has proved that a fuchsian group Γ is finitely generated if and only if $K^*(\Gamma)$ is compact mod Γ. The limit set $L(\Gamma)$ is a closed Γ-invariant set in $\widetilde{M}(\infty)$. The intersection of all complete totally geodesic submanifolds of $\widetilde{M}$ containing $L(\Gamma)$ in their boundaries is a totally geodesic submanifold denoted by $\langle L(\Gamma)\rangle$. It is the totally convex hull of $L(\Gamma)$ in $\widetilde{M} \cup \widetilde{M}(\infty)$.

DEFINITION 2.1. A fuchsian manifold M of finite type is a complete Riemannian manifold with $K \leqslant 0$ such that (1) for any $x \neq y$ in the boundary $\widetilde{M}(\infty)$ of the simply connected covering $\widetilde{M}$, there exists a unique geodesic joining them, (2) there exists some $K^*(\pi_1(M))$ which is compact mod $\pi_1(M)$.

3. **Fundamental groups.** In this section we shall obtain several theorems concerning fundamental groups of fuchsian manifolds of finite type.

THEOREM 3.1. *Let M_1 and M_2 be two compact fuchsian manifolds covering the same manifold M. Consider the intersection*(2) *of $\pi_1(M_1)$ and $\pi_1(M_2)$ in $\pi_1(M)$. If the associated covering manifold M' to $\pi_1(M_1) \cap \pi_1(M_2)$ is fuchsian, then M' is of finite type.*

PROOF. Let G_1 and G_2 denote the fundamental groups $\pi_1(M_1)$ and $\pi_1(M_2)$ respectively. Let $G = G_1 \cap G_2$. There exist balls $B_1 = \{p \in \widetilde{M} \mid d(p, p_0) < r_1\}$ and $B_2 = \{p \in \widetilde{M} \mid d(p, p_0) < r_2\}$ for some $p_0 \in \widetilde{M}$ (without loss of generality) such that $K(G_1) \subset G_1 B_1$ and $K(G_2) \subset G_2 B_2$. Let

(2) Please note the convention in our introduction that we are omitting notations of induced homomorphisms between fundamental groups.

$r = \max(r_1, r_2)$ and $B = \{p \in \widetilde{M} \mid d(p, p_0) < r\}$. Then $K(G_1) \subset G_1 B$ and $K(G_2) \subset G_2 B$. Choose coset representatives $\{g_{1i}\}$ and $\{g_{2j}\}$ so that $G_1 = \bigcup_i (G g_{1i})$ and $G_2 = \bigcup_j (G g_{2j})$. Then $K(G_1) \subset G_1 B = G(\bigcup_i g_{1i} B)$ and $K(G_2) \subset G_2 B = G(\bigcup_j g_{2j} B)$. Also $K(G) \subset K(G_1) \cap K(G_2)$ since $L(G) \subset L(G_1) \cap L(G_2)$. For any $g \in G$, $g_{1i} B \cap g g_{2j} B \neq \emptyset$ if and only if $B \cap g_{1i}^{-1} g\, g_{2j} B \neq \emptyset$. Moreover, for any $f \in \pi_1(M)$, $B \cap fB \neq \emptyset$ if and only if $d(p_0, f(p_0)) < 2r$. The discreteness of $\pi_1(M)$ implies that there are only a finite number of elements f with this property. Therefore there are only a finite number of elements $f = g_{1i}^{-1} g g_{2j}$ with $B \cap fB \neq \emptyset$. If $(g_{1i_1})^{-1} g_1 (g_{2j_1}) = g_{1i_2}^{-1} g_2 g_{2j_2}$, then

$$g_{1i_2} g_{1i_1}^{-1} g_1 = g_2 g_{2j_2} g_{2j_1}^{-1} \in G_1 \cap G_2 = G.$$

Therefore $g_{1i_2} g_{1i_1}^{-1}$ and $g_{2j_2} g_{1j_1}^{-1} \in G$, so $g_{1i_1} = g_{1i_2}$, $g_{2j_1} = g_{2j_2}$ and $g_1 = g_2$. It follows that there are only a finite number of the g_{1i}, g_{2j} and g for which $g_{1i} B \cap g g_{2j} B \neq \emptyset$, and therefore only a finite number of the g_{1i} for which $g_{1i} B \cap K(G_2) \neq \emptyset$. Since $K(G) \subset K(G_2)$, there are only a finite number of the g_{1i}, say $g_{1i_1}, \cdots, g_{1i_n}$ so that $g_{1i} B \cap K(G) \neq \emptyset$. Furthermore the elements of G map $K(G)$ and $K(G_1)$ onto themselves and consequently $K(G_1) - K(G)$ onto itself. Thus $g_{1i} B \cap K(G) \neq \emptyset$, if and only if $G(g_{1i} B) \cap K(G) \neq \emptyset$. We have $K(G) \subset G(\bigcup_{k=1}^n g_{1j_k} B)$. Let B' be with center p_0 and radius r' which is large enough to contain $\bigcup_{k=1}^n (g_{1i_k} B)$. Then $K(G) \subset GB'$ or $K(G)$ is compact mod G. Thus M' is of finite type.

THEOREM 3.2. *If M_1 is a covering fuchsian manifold of finite type of a fuchsian manifold M_2 and $L(\pi_1(M_1)) = L(\pi_1(M_2))$. Then M_1 is a finite covering of M_2.*

PROOF. Since M_1 and M_2 are fuchsian, $L(\pi_1(M_1)) = L(\pi_1(M_2))$ contains more than two points. Let G_1 denote $\pi_1(M_1)$ and G_2 denote $\pi_1(M_2)$. Then $K^*(G_1)$ and $K^*(G_2)$ are not empty sets. There is $B = \{p \in \widetilde{M} \mid d(p, p_0) < r\}$ such that $K^*(G_1) \subset G_1 B$. Since G_2 is properly discontinuous, there are only a finite number of elements $g \in G_2$ so that $B \cap gB \neq \emptyset$. Let $g_1, \cdots, g_n$ be those elements. Let $p \in B \cap K^*(G_1)$ (suppose that B is large enough so that $B \cap K^*(G_1)$ is not empty). Let $g \in G_2$. Since $K(G_1) = K(G_2)$, $K^*(G_2) \subset K^*(G_1)$ and $K^*(G_2)$ is invariant under G_2. Thus $g(p) \in K^*(G_1)$. There exists $g_0 \in G_1$ so that $g_0 g(p) \in B$. Thus $B \cap g_0 g B \neq \emptyset$ and $g_0 g = g_k$ for some k. Thus every $g \in G$ is congruent to one of $g_1, \cdots, g_n$ mod G_1 and $[G_2 : G_1]$ is finite.

DEFINITION 3.1. An N-chain of a group G is a sequence of subgroups $G_1, G_2, \cdots, G_n$ such that (a) $G_k \neq \{e\}$ $(k = 1, \cdots, n)$, (b) either G_k is a normal subgroup of G_{k+1} or G_{k+1} is a normal subgroup of G_k.

Two subgroups H and K of G are N-equivalent if there is an N-chain $H = G_1, G_2, \cdots, G_n = K$. A subgroup which is N-equivalent to G is called an N-subgroup. By applying Lemmas 2.1 and 2.2 successively, we obtain

LEMMA 3.1. *Let H and K be N-equivalent subgroups of a properly discontinuous group of isometries of $\widetilde{M}$ such that $L(H)$ contains more than two points. Then $L(H) = L(K)$.*

THEOREM 3.3. *Let M_1 be a fuchsian manifold of finite type covering a fuchsian manifold M_2 such that $\pi_1(M_1)$ is a N-subgroup of $\pi_1(M_2)$. Then $[\pi_1(M_2) : \pi_1(M_1)]$ is finite or M_1 is a finite covering of M_2.*

PROOF. Lemma 3.1 and Theorem 3.2 imply this theorem.

THEOREM 3.4. *Let M_1 be a covering fuchsian manifold of finite type of a fuchsian manifold M_2. Then there exists a subgroup G of $\pi_1(M_2)$ such that* (1) *G is N-equivalent to $\pi_1(M_1)$;* (2) *if $K \subset \pi_1(M_2)$ and K is N-equivalent to $\pi_1(M_1)$, then $K \subset G$;* (3) *$[G : \pi_1(M_1)]$ is finite.*

PROOF. Let G_1 and G_2 denote $\pi_1(M_1)$ and $\pi_1(M_2)$. Let $G = \{g \in G_2 \mid gL(G_1) = L(G_1)\}$. Then $L(G) \supset L(G_1)$. Lemma 2.1 implies that $L(G_1) \supset L(G)$, so that $L(G) = L(G_1)$. Theorem 3.2 implies that $[G : G_1]$ is finite. Thus G_1 has a finite number of conjugate subgroups in G. The intersection of these conjugate subgroups is a normal subgroup F of finite index in G. Since G is infinite, F is nontrivial. Therefore the sequence G, F and G_1 is an N-chain and G is N-equivalent to G_1. If K is N-equivalent to G_1, Lemma 2.2 implies that K leaves $L(G_1)$ invariant, so that $K \subset G$.

THEOREM 3.5. *Let M_1 and M_2 be fuchsian manifolds of finite type covering a fuchsian manifold M. The following statements are equivalent:* (a) *$\pi_1(M_1)$ and $\pi_1(M_2)$ are N-equivalent,* (b) *there is a group J which is simultaneously normal and of finite index in $\pi_1(M_1)$ and $\pi_1(M_2)$,* (c) *$L(\pi_1(M_1)) = L(\pi_1(M_2))$.*

PROOF. Let G, G_1 and G_2 denote $\pi_1(M), \pi_1(M_1)$ and $\pi_1(M_2)$ respectively. If (a) is true, then $\{g \in G \mid gL(G_1) = L(G_1)\} = \{g \in G \mid gL(G_2) = L(G_2)\}$. Since G_1 and G_2 are of finite index in $\{g \in G \mid gL(G_1) = L(G_1)\}$, this is also true of $G_1 \cap G_2$. Therefore $G_1 \cap G_2$ contains a nontrivial subgroup J which is normal and of finite index in $\{g \in G \mid gL(G_1) = L(G_1)\}$. J is also normal and of finite index in G_1 and G_2. Thus (a) implies (b). If (b) is true, then G_1 and G_2 are N-equivalent. Therefore $L(G_1) = L(G_2)$. Hence (b) implies (c). Suppose (c) is true. Then $\{g \in G \mid gL(G_1) = L(G_1)\} = \{g \in G \mid gL(G_2) = L(G_2)\}$ is N-equivalent to G_1 and G_2.

4. **Isometry groups.** The following theorem is shown by Eberlein in [5].

THEOREM 4.1 (EBERLEIN). *Let M be a fuchsian manifold. Then there exists an infinite subset A of $\pi_1(M)$ such that the subgroup G generated by the set A is a free group on the set A.*

It follows that G is a properly discontinuous group of isometries of $\widetilde{M}$. Let F be a finite group generated by $f_1, \cdots, f_n$ and let G be the free group generated by $A = \{a_1, \cdots, a_n\}$. There is a homomorphism $\Phi\colon G \longrightarrow F$ defined by $\Phi(a_i) = f_i$, $i = 1, \cdots, n$. If K is the kernel, then F is isomorphic to G/K. G is the fundamental group of a fuchsian manifold M_1. If we identify points of $\widetilde{M}$ which are congruent under K, we obtain another fuchsian manifold $M_2 = \widetilde{M}/K$. An isometry ϕ of M_2 can be lifted to an isometry $\widetilde{\phi}$ of $\widetilde{M}$. The transformation $\widetilde{\phi}$ maps an orbit Kp, $p \in \widetilde{M}$, onto another orbit. Thus $\widetilde{\phi}K = K\widetilde{\phi}$ or $\widetilde{\phi}$ is in the normalizer $N[K, I(\widetilde{M})]$ of K in $I(\widetilde{M})$. $I(M_2)$ is isomorphic to $N[K, I(\widetilde{M})]/K$ which contains G/K as a subgroup. Thus F is isomorphic to a subgroup of $I(M_2)$. Thus we have

THEOREM 4.2. *Let M_1 be a fuchsian manifold. For each finite group F, one can construct a fuchsian manifold M_2 which covers M such that F is isomorphic to a subgroup of $I(M_2)$.*

REMARK. A stronger result than Theorem 4.2 would be that every finite group is the isometry group of some fuchsian manifold.

LEMMA 4.1. *Let M be a fuchsian manifold such that $L(\pi_1(M))$ is not contained in the boundary of any proper totally geodesic submanifold of $\widetilde{M}$. Then the normalizer $N[\pi_1(M), I(\widetilde{M})]$ of $\pi_1(M)$ in $I(\widetilde{M})$ is discrete.*

PROOF. Suppose that there is a sequence $\{n_k\} \in N[\pi_1(M), I(\widetilde{M})]$ such that $\lim n_k = e$. Then $\lim f n_k f^{-1} n_k^{-1} = e$ where f is any element of $\pi_1(M)$. Since $f n_k f^{-1} n_k^{-1} \in \pi_1(M)$ which is discrete, $f n_k f^{-1} n_k^{-1} = e$ for almost all k. Consider a fixed n_k such that $f n_k f^{-1} n_k^{-1} = e$. Suppose that n_k is elliptic and let the fixed point set of n_k be the proper closed totally geodesic submanifold N. Then each f in $\pi_1(M)$ leaves N invariant. Thus we have a contradiction. Suppose that n_k is parabolic and let the fixed point of n_k be $z \in \widetilde{M}(\infty)$. Then $f(z) = z$ and M must be parabolic or axial by Lemma 2.3. Finally suppose that n_k is axial and let the fixed points of n_k be z_1 and $z_2 \in \widetilde{M}(\infty)$, then either $f(z_1) = z_1, f(z_2) = z_2$ or $f(z_1) = z_2, f(z_2) = z_1$. The second case implies that f leaves the geodesic joining z_1 to z_2 invariant and has a fixed point in $\widetilde{M}$. This is impossible. The first case implies that M is axial.

THEOREM 4.3. *Let M be a fuchsian manifold of finite type such that $L(\pi_1(M))$ is not contained in the boundary of any proper totally geodesic submanifold of $\widetilde{M}$. Then $I(M)$ is finite.*

PROOF. Consider the normalizer $N[\pi_1(M), I(\widetilde{M})]$ of $\pi_1(M)$ in $I(\widetilde{M})$. $I(M)$ is isomorphic to $N[\pi_1(M), I(\widetilde{M})]/\pi_1(M)$. Because every isometry $\phi \in I(M)$ can be lifted to an isometry $\widetilde{\phi}$ of $\widetilde{M}$, the isometry $\widetilde{\phi}$ maps an orbit of $\pi_1(M)p, p \in \widetilde{M}$ onto another orbit. Thus $\widetilde{\phi}\,\pi_1(M) = \pi_1(M)\widetilde{\phi}$ or $\widetilde{\phi}$ is in the normalizer $N[\pi_1(M), I(\widetilde{M})]$. Lemma 4.1 implies that $N[\pi_1(M), I(\widetilde{M})]$ is discrete. Theorem 3.3 implies that the index $[N[\pi_1(M), I(\widetilde{M})] : \pi_1(M)]$ is finite and $I(M)$ is finite.

5. **Commensurability groups.** Let Γ_1 and Γ_2 be subgroups of a group Γ. Γ_1 and Γ_2 are commensurable ($\Gamma_1 \sim \Gamma_2$) if $\Gamma_1 \cap \Gamma_2$ is of finite index in Γ_1 and Γ_2. Let $C(\Gamma) = \{g \in G \mid g\Gamma g^{-1} \sim \Gamma\}$. Then $C(\Gamma)$ is called the commensurability group of Γ. The following result is given in [2] and is needed here.

THEOREM 5.1. *Let x be a noncompact symmetric space of rank one and of dimension n. Let G be a connected Lie subgroup of $I_0(X)$. Then one of the following holds:*

(1) *G has a common fixed point in X and $L(G)$ is empty.*

(2) *G has a common fixed point in $X(\infty)$ and $L(G)$ consists of one point.*

(3) *G modulo a normal subgroup (isomorphic to a subgroup of $O(n-1)$) is the 1-parameter group of axial elements and $L(G)$ consists of two points.*

(4) *G modulo a normal subgroup (isomorphic to a subgroup of $O(n-m)$ $\dim L(G) = m$) is the connected isometry group $I_0(\langle L(G)\rangle)$ of the totally geodesic submanifold $\langle L(G)\rangle$ which is a noncompact symmetric space of rank one.*

(5) $G = I_0(X)$.

THEOREM 5.2. *Let X be a noncompact symmetric space of rank one and of dimension n. Let Γ be a discrete subgroup of $I_0(X)$. If $\dim \langle L(\Gamma)\rangle(\infty) \geqslant n-2, n \geqslant 3$ and $L(\Gamma) \neq$ the boundary $\langle L(\Gamma)\rangle(\infty)$ of $\langle L(\Gamma)\rangle$, then $C(\Gamma)$ is discrete.*

PROOF. Let $G = \{g \in I_0(X) \mid gL(\Gamma) = L(\Gamma)\}$. G is a closed subgroup of $I_0(X)$ which contains $C(\Gamma)$. We shall prove that the identity component G_0 of G is trivial and thus G is discrete. Since $\dim \langle L(\Gamma)\rangle(\infty) \geqslant n-2$, $n \geqslant 3$. $L(\Gamma)$ contains more than two points. Since $G_0 \subset G, G_0$ leaves $L(\Gamma)$ invariant. Lemma 2.1 implies that $L(G_0) \subset L(\Gamma)$. Since $\Gamma \subset G, \Gamma$ normalizes

G_0. Lemma 2.2 implies that Γ leaves $L(G_0)$ invariant, and if $L(G_0)$ contains more than one point, $L(\Gamma) \subset L(G_0)$. Thus $L(\Gamma) = L(G_0)$. We examine the possibilities (1)–(5) in Theorem 5.1.

(1) Suppose that the elements of G_0 have a common fixed point in X. Let $Y = \{x \in X \mid g(x) = x \text{ for all } g \in G_0\}$. If $Y = X$, then $G_0 = \{e\}$, which is what we want to prove. We shall show that all other possibilities lead to contradictions. If $Y \neq X$, then Y is a proper totally geodesic subspace of X. Since G_0 is normal in G, G leaves Y invariant. Let $Y(\infty)$ be the boundary of Y in $X(\infty)$. Then $Y(\infty)$ is invariant under G and $\Gamma \subset G$. Lemma 2.1 implies that $Y(\infty) \supset L(\Gamma)$, and so $Y(\infty) \supset \langle L(G)\rangle(\infty)$. Since $\dim \langle L(G)\rangle(\infty) \geqslant n - 2$, $\dim Y(\infty) \geqslant n - 2$ and $\dim Y \geqslant n - 1$. Since $Y \neq X$, $\dim Y = n - 1$. Thus G_0 which is isomorphic to a subgroup of $O(1)$ is trivial and $Y = X$.

(2) Suppose that the elements of G_0 have a common fixed point $x \in X(\infty)$. Since Γ normalizes G_0, it leaves invariant the set $F = \{x \in X(\infty) \mid g(x) = x, g \in G_0\}$. If F contains more than two points, then G_0 has a fixed point in X. If F contains two points, then Lemma 2.1 implies that $F \supset L(\Gamma)$. Therefore $\dim \langle L(\Gamma)\rangle(\infty) = 0 < n - 2$. Thus F contains a single point. Since Γ leaves that point fixed, Lemma 3.3 implies that $L(\Gamma)$ contains less than or equal to two points. This again contradicts $\dim \langle L(\Gamma)\rangle(\infty) \geqslant n - 2$ and $n \geqslant 3$.

In the remaining cases, $L(G_0)$ contains more than one point. This implies $L(\Gamma) = L(G_0)$.

(3) $L(\Gamma) = L(G_0)$ consists of two points, which violates the conditions $\dim \langle L(\Gamma)\rangle(\infty) \geqslant n - 2, n \geqslant 3$.

(4) and (5). $L(\Gamma) = L(G_0)$ and $\langle L(\Gamma)\rangle(\infty) = \langle L(G_0)\rangle(\infty)$. Thus $L(G_0) = \langle L(G_0)\rangle(\infty)$. This contradicts $L(\Gamma) \neq \langle L(\Gamma)\rangle(\infty)$.

6. **Random walk on $\pi_1(M)$.** Kesten [11] has initiated the study of left-invariant, symmetric random walks on discrete groups. Each such random walk is described by a symmetric matrix m_{gh} whose spectrum has a largest element $\lambda = \limsup (m_{11}^{(s)})^{1/s}$, where $m_{gh}^{(s)}$ denotes the probability of passing from g to h in s steps of the random walk.

This number λ is equal to 1 if the group is solvable, but less than 1 if there exists a free subgroup of at least two generators. In fact, $\lambda < 1$ is equivalent to the group being amenable. An outstanding conjecture of von Neumann is that every nonamenable group contains a free subgroup of at least two generators.

In [12], Milnor has proved the following.

THEOREM 6.1 (MILNOR). *If M is compact Riemannian with sectional curvature $K < 0$, then the growth function of the fundamental group $\pi_1(M)$ is at least exponential.*

THEOREM 6.2 (MILNOR). *If* $\lambda < 1$, *then the group is of exponential growth.*

The following generalization of Theorem 4.1 is given in [3].

THEOREM 6.3. *Let* $\widetilde{M}$ *be a simply connected, complete Riemannian manifold of sectional curvature* $K < 0$ *such that every two points in the boundary* $\widetilde{M}(\infty)$ *can be joined by a unique geodesic. If a subgroup* G *of* $I(\widetilde{M})$ *does not have a common fixed point in* $\widetilde{M}(\infty)$ *and* $L(G)$ *contains more than two points, then* G *contains a free subgroup with an infinite number of generators.*

COROLLARY 6.1. *Let* M *be a complete Riemannian manifold of sectional curvature* $K < 0$ *such that its universal covering manifold* $\widetilde{M}$ *satisfies the assumption in Theorem* 6.3. *The fundamental group* $\pi_1(M)$ *contains a free subgroup with an infinite number of generators. Moreover, the spectral radius* λ *of the random walk on* $\pi_1(M)$ *is less than* 1. *Equivalently* $\pi_1(M)$ *is not amenable.*

COROLLARY 6.2. *If* M *is a compact Riemannian manifold with negative sectional curvature, then* $\lambda < 1$. *Moreover,* $\pi_1(M)$ *is of exponential growth.*

Corollary 6.2 gives another proof of Theorem 6.2 and answers the problem of Milnor in [12].

REFERENCES

1. R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. 145 (1969), 1–49. MR 40 #4891.

2. S. Chen, *On Lobatchewsky manifolds*. Bull. Amer. Math. Soc. 80 (1974), 244–247.

3. ———, *Amenability of isometry groups of Riemannian manifolds*, Chinese J. Math. 2 (1974), 31–38.

4. P. Eberlein, *Geodesic flows on negatively curved manifolds*. I, Ann. of Math. (2) 95 (1972), 492–510. MR 46 #10024.

5. ———, *Some properties of the fundamental group of a fuchsian manifold*, Invent. Math. 19 (1973), 5–13.

6. P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. 46 (1973), 45–109.

7. L. Greenberg, *Discrete subgroups of the Lorentz group*, Math. Scand. 10 (1962), 85–107. MR 25 #5128.

8. ———, *Conformal transformations of Riemann surfaces*, Amer. J. Math. 82 (1960), 749–760. MR 23 #A319.

9. ———, *Discrete groups of motions*, Canad. J. Math. 12 (1960), 415–426. MR 22 #5932.

10. ———, *Commensurable groups of Moebius transformations, discontinuous groups and Riemann surfaces*, Ann. of Math. Studies, no. 79, Princeton Univ. Press, Princeton, N. J., 1974.

11. H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. 92 (1959), 336–354. MR 22 #253.

12. J. Milnor, *A note on curvature and fundamental group*, J. Differential Geometry 2 (1968), 1–7. MR 38 #636.

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