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QUASI-EQUIVALENCE CLASSES OF NORMAL REPRESENTATIONS FOR A SEPARABLE C^*-ALGEBRA(1)

BY

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ABSTRACT. It is shown that the set of quasi-equivalence classes of normal representations of a separable C^*-algebra is a Borel subset of the quasi-dual with the Mackey Borel structure and forms a standard Borel space in the induced Borel structure. It is also shown that the set of factor states which induce normal representations forms a Borel set of the space of factor states with the w^*-topology and that this set has a Borel transversal.

Let A be a separable C^*-algebra. Two representations λ and λ' of A on the Hilbert spaces $H(\lambda)$ and $H(\lambda')$ are said to be *quasi-equivalent* (in symbols: $\lambda \sim \lambda'$) if there is an isomorphism Φ of the von Neumann algebra $\lambda(A)''$ generated by $\lambda(A)$ onto that generated by $\lambda'(A)$ such that $\Phi(\lambda(x)) = \lambda'(x)$ for every $x \in A$. A representation λ of A is a *factor representation* if the center of $\lambda(A)''$ consists of scalar multiples of the identity. The relation of quasi-equivalence partitions the factor representations of A into quasi-equivalence classes. Let $\widetilde{A}$ denote the set of all quasi-equivalence classes of nonzero factor representations of A, and let $[\lambda]$ denote the quasi-equivalence class that contains the representation λ.

For any Hilbert space H, let $\mathrm{Rep}(A, H)$ (resp. $\mathrm{Fac}(A, H)$) denote the space of all representations (resp. factor representations) of A on H taken with the topology of pointwise convergence, i.e., $\lambda_n \longrightarrow \lambda$ if and only if $\lambda_n(x)\zeta \longrightarrow \lambda(x)\zeta$ for all $x \in A$ and $\zeta \in H$. Let H_n $(n = 1, 2, \cdots, \infty)$ be a separable Hilbert space of dimension n, and let $\mathrm{Rep}\, A$ (resp. $\mathrm{Fac}\, A$) be the disjoint union of the spaces $\mathrm{Rep}(A, H_n)$ (resp. $\mathrm{Fac}(A, H_n)$) for $n = 1, 2, \cdots, \infty$. A subset X of $\mathrm{Rep}\, A$ (resp. $\mathrm{Fac}\, A$) is a Borel set if, for each n, the set $X \cap \mathrm{Rep}(A, H_n)$ (resp. $X \cap \mathrm{Fac}(A, H_n)$) is a Borel set in the Borel structure induced by the topology. The Borel space $\mathrm{Rep}\, A$ is *standard* in the sense that it is Borel isomorphic with a Borel subset of a *polonais* (i.e., a complete separable metrizable) space. (Note that a standard Borel space is determined up

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to Borel equivalence by its cardinality and only two infinite cardinals are possible [12, §3].) The set $\operatorname{Fac} A$ is a Borel subset of $\operatorname{Rep} A$ and the Borel structure induced on $\operatorname{Fac} A$ by $\operatorname{Rep} A$ is the structure already assigned to it. The map ψ which assigns to each λ its quasi-equivalence class $[\lambda]$ in $\widetilde{A}$ actually is surjective and induces the so-called *Mackey Borel structure* on $\widetilde{A}$, viz., a set X is Borel in $\widetilde{A}$ if and only if $\psi^{-1}(X)$ is Borel in $\operatorname{Fac} A$ [6, §§5, 7].

A nondegenerate representation λ of A on the Hilbert space $H(\lambda)$ is said to be a *trace representation* of A if the von Neumann algebra $\lambda(A)''$ generated by $\lambda(A)$ is semifinite and if there exists a faithful normal trace t on $\lambda(A)''^{+}$ such that $\lambda(A) \cap N(t)$ generates the von Neumann algebra $\lambda(A)''$. Here $N(t)$ is the *ideal of definition* of t given by the set of all linear combinations of elements in the set

$$N(t)^{+} = \{x \in \lambda(A)''^{+} \mid t(x) < +\infty\}.$$

The set $N(t)$ is an ideal in $\lambda(A)''$ (cf. [6, §6]). (In the sequel a two-sided ideal closed under involution is simply called an *ideal*. Most of the ideals considered in this note will not be closed in the norm topology.)

If λ is a trace representation of A and if $\lambda(A)''$ is a factor von Neumann algebra, then λ is called a *normal representation* of A [9, Definition, p. 13]. Since every trace representation gives a σ-finite von Neumann algebra [6, 6.3.6], the faithful normal trace on $\lambda(A)''^{+}$, where λ is a normal representation, is unique up to a strictly positive scalar multiple [7, I, 6, Theorem 4, Corollary]. Thus, a semifinite factor representation λ of A is normal if and only if $\lambda(A)$ contains a nonzero element of finite trace. Indeed, if the ideal $\lambda(A) \cap N(t)$ is nonzero, then its weak closure is $\lambda(A)''$ [7, I, 3, Theorem 2, Corollary 3].

In this note we show that the set X of quasi-equivalence classes of normal representations of A is a Borel subset of $\widetilde{A}$ and is standard in the induced Borel structure. This answers a question posed by J. Dixmier [6, 7.5.4] (2). We apply this to show that there is a Borel subset of factor states of A (with the Borel structure induced by the w^*-topology) that is Borel isomorphic with X. A. Guichardet [9] proved that the quasi-equivalence classes of finite (resp. type I) normal representations is a Borel subset of $\widetilde{A}$ and is standard in the induced Borel structures. Other structures in this regard have been given by Perdrizet [14]. Although there are separable C^*-algebras with no normal representations [5], there are some (viz., the GCR algebras) for which every factor representation is normal and others (e.g., the reduced group C^*-algebra of a second countable, locally compact unimodular group [9, I, §3, Theorem 1, Corollary]) such that, for every nonzero x in the algebra, there is a normal representation λ with $\lambda(x) \neq 0$.

(2) The rest of the problem has been solved by O. A. Nielsen.

The first lemma is the basis of our analysis of the Borel structure.

LEMMA 1. *Let A be a separable C^*-algebra. There is a countable subset S of A^+ such that, for every normal representation λ of A, the set $\lambda(S)$ contains a nonzero element of finite trace.*

PROOF. Let S_1 be a countable dense subset of $\{x \in A^+ \mid \|x\| = 1\}$. Let α, β be rational numbers with $0 < \alpha < \beta < 1$ and let $f = f_{\alpha,\beta}$ be the continuous real-valued function of a real variable defined by $f(\gamma) = 0$ if $\gamma \leqslant \alpha$, $f(\gamma) = 1$ if $\gamma \geqslant \beta$, and f linear on $[\alpha, \beta]$. Let F be the (countable) family of functions $F = \{f_{\alpha,\beta} | \alpha, \beta$ rational and $0 < \alpha < \beta < 1\}$. Let S be the countable subset of A^+ given by $S = \{f(x) | f \in F, x \in S_1\}$.

Let λ be a normal representation of A and let t be a faithful, normal, semifinite trace on $\lambda(A)''^+$. We show that there is an $x \in S$ such that $0 < t(\lambda(x)) < +\infty$. Let J be the closed ideal of $\lambda(A)''$ generated by the finite projections of $\lambda(A)''$ (cf. [10, §2]). Let y be a nonzero element in $\lambda(A) \cap N(t)$. Since $y^*y \in \lambda(A) \cap N(t)$, we may assume that $y \in \lambda(A)^+$. Let $0 < \alpha < \|y\|$ and let e be the spectral projection of y (in $\lambda(A)''$) corresponding to the interval $[\alpha, \|y\|]$. We have that $e \leqslant \alpha^{-1}y$ and thus that e has finite trace. This proves that $e \in J$. We also have that $\|y - ey\| \leqslant \alpha$. Because $\alpha > 0$ may be arbitrarily small, we get that $y \in J$. This means that the closed ideal $\lambda(A) \cap J$ of $\lambda(A)$ is not zero, and therefore, the canonical homomorphism ϕ of the C^*-algebra $\lambda(A)$ onto the C^*-algebra $\lambda(A)/\lambda(A) \cap J$ is not an isometry. But if

$$\|\lambda(z)\| = \text{glb}\,\{\|\lambda(z) + w\| \mid w \in J \cap \lambda(A)\} = \|\phi(\lambda(z))\|$$

for every $z \in S_1$, then $\|z\| = \|\phi(z)\|$ for every z in the unit sphere of $\lambda(A)$ due to the continuity of the maps $z \longrightarrow \|\lambda(z)\|$ and $z \longrightarrow \|\phi(\lambda(z))\|$ on A. This means that the canonical homomorphism ϕ is an isometry. Therefore, we may find a $z \in S_1$ and an α with $0 < \alpha < 1/2$ such that

$$\|\phi(\lambda(z))\| < (1 - 2\alpha)\|\lambda(z)\| < \|\lambda(z)\| \leqslant 1.$$

Let e' be the spectral projection of $\lambda(z)$ corresponding to the interval $[(1 - \alpha)\|\lambda(z)\|, \|\lambda(z)\|]$. We have that

$$\lambda(z) \geqslant (1 - \alpha)\|\lambda(z)\|e'$$

and thus that

$$\lambda(z) \ (\text{mod } J) \geqslant (1 - \alpha)\|\lambda(z)\|e' (\text{mod } J) \geqslant 0$$

in the C^*-algebra $\lambda(A)''/J$. We get that

$$(1-2\alpha)\|\lambda(z)\| \geqslant \|\phi(\lambda(z))\| = \|\lambda(z)\,(\mathrm{mod}\ J)\| \geqslant (1-\alpha)\|\lambda(z)\|\ \|e'(\mathrm{mod}\ J)\|.$$

We find the norm of the projection $e'(\mathrm{mod}\ J)$ is zero since the only possible choices for its norm are 0 and 1. Hence the projection e' is in J. We recall that all the projections in J are finite projections [10, Proposition 2.1]. Now we may find a function f in F such that $f(\lambda(z)) \neq 0$ and such that $f(\lambda(z)) \leqslant e'$. For example, let $f = f_{\beta,\gamma}$ where β and γ are rational numbers that satisfy $(1-\alpha)\|\lambda(z)\| < \beta < \gamma < \|\lambda(z)\|$. Since $f(\lambda(z)) = \lambda(f(z))$, the element $x = f(z)$ in S is not in the kernel of λ and satisfies the relation $0 \leqslant \lambda(x) \leqslant e'$. This proves that $\lambda(x)$ is a nonzero element of finite trace. Q.E.D.

Let I be an ideal of the C^*-algebra A. A complex-valued function s of the cartesian product $I \times I$ is called a *bitrace* with ideal of definition I if s satisfies the following axioms: (i) s is a positive hermitian form on $I \times I$; (ii) $s(x, y) = s(y^*, x^*)$, for all $x, y \in I$; (iii) $s(zx, y) = s(x, z^*y)$, for all $x, y \in I$ and $z \in A$; (iv) for every $x \in A$, the map $z \longrightarrow xz$ defines a continuous linear operator of I into I in the prehilbert structure induced by s; and (v) the set $I^2 = \{xy | x, y \in I\}$ is dense in I in the prehilbert structure induced by s [4]. Every bitrace s with ideal of definition I induces a trace representation of A in a canonical way. In fact, first assume that s satisfies only the properties (i)–(iii). Let Λ_s be the canonical homomorphism of I into I modulo the ideal $\{x \in I | s(x, x) = 0\}$. For every $x, y \in I$, the relation $(\Lambda_s(x), \Lambda_s(y)) = s(x, y)$ defines an inner product on $\Lambda_s(I)$. Let H_s be the completion of $\Lambda_s(I)$ in this inner product. Now assume s satisfies property (iv). For every $x \in A$ the map $\Delta_s(y) \longrightarrow \Lambda_s(xy)$ (resp. $\Lambda_s(y) \longrightarrow \Lambda_s(yx)$) of $\Lambda_s(I)$ can be extended to a bounded linear operator $\lambda_s(x)$ (resp. $\rho_s(x)$) of the Hilbert space H_s, and the map $x \longrightarrow \lambda_s(x)$ (resp. $x \longrightarrow \rho_s(x)$) defines a representation (resp. antirepresentation) of A on H_s. This means that $\|\lambda_s(x)\| \leqslant \|x\|$ (resp. $\|\rho_s(x)\| \leqslant \|x\|$) since every representation of A is norm decreasing. Suppose now that s satisfies property (v). We then have that $\lambda_s(A)'' = \rho_s(A)'$. Furthermore, there is a faithful normal semifinite trace t on $\lambda_s(A)''^+$ such that $t(\lambda_s(xx^*)) = s(x, x)$ for all $x \in I$ and such that $\lambda(A) \cap N(t)$ generates the von Neumann algebra $\lambda(A)''$ ([4], [9], cf. [6, 6.2]).

Conversely, let λ be a trace representation of A on the Hilbert space H. Let t be a faithful normal semifinite trace on $\lambda(A)''^+$ such that $\lambda(A) \cap N(t)$ generates $\lambda(A)''$. The set $I = \{x \in A | t(\lambda(x)\lambda(x)^*) < +\infty\}$ is an ideal in A and the relation $s(x, y) = t'(\lambda(xy^*))$ defines a bitrace with ideal of definition I. Here t' is the unique extension of t to a linear functional on $N(t)$. The canonical representation λ_s induced by s is quasi-equivalent to λ [6, 6.6.5(ii)].

PROPOSITION 2. *Let* $x_0 \in A^+$, *let* $I = I(x_0)$ *be the ideal of* A *generated by* x_0, *and let* $T = T(x_0)$ *be the family of all bitraces on* I *such that* $s(x_0, x_0) = 1$; *then in the topology of pointwise convergence on* $I \times I$, *the space* T *is polonais.*

PROOF. Let T' be the set of all complex-valued functions s on $I \times I$ satisfying properties (i)–(iv) of the definition of bitraces and the additional property (vi) $s(x_0, x_0) = 1$. Notice that T is a subset of T'.

Let A_e be the C^*-algebra A if A has identity or the C^*-algebra A with identity e_0 adjoined (cf. [6, 1.3.8]) if A has no identity. If A has no identity, the map $(x, \alpha) \longrightarrow x + \alpha e_0$ of the Banach space given by the cartesian product of A with the complexes with norm $\|(x, \alpha)\| = \|x\| + |\alpha|$ onto the C^*-algebra A_e with norm $\|x + \alpha e_0\| = \text{lub}\,\{\|xy + \alpha y\| \mid y \in A, \|y\| \leqslant 1\}$ is continuous and one-one. Therefore, the inverse of the map is continuous, and so there is a constant $\kappa \geqslant 1$ with $\|x\| + |\alpha| \leqslant \kappa\|x + \alpha e_0\|$. Using A_e, we can explicitly express the ideal I as

$$I = \left\{\sum \{x_i x_0 y_i \mid 1 \leqslant i \leqslant m\} \,\middle|\, x_i, y_i \in A_e,\ m = 1, 2, \cdots\right\}.$$

For each $x, y \in I$, we show that the set $\{|s(x, y)| \mid s \in T'\}$ is bounded. Let $s \in T'$; then, for $x, y \in A_e$, we get by direct calculation that

$$s(xx_0y, xx_0y) \leqslant \kappa^4 \|x\|^2 \|y\|^2,$$

and thus, for x_i, y_i $(1 \leqslant i \leqslant m)$, x_i', y_i' $(1 \leqslant i \leqslant n)$ in A_e, we get that

$$\begin{aligned} s\left(\sum x_i x_0 y_i, \sum x_i' x_0 y_i'\right) &\leqslant \sum_{i,j} s(x_i x_0 y_i, x_i x_0 y_i)^{1/2} s(x_j' x_0 y_j', x_j' x_0 y_j')^{1/2} \\ &\leqslant \sum_{i,j} \kappa^4 \|x_i\|\, \|x_j'\|\, \|y_i\|\, \|y_j'\|. \end{aligned} \tag{1}$$

Setting

$$\alpha\left(\sum x_i x_0 y_i, \sum x_i' x_0 y_i'\right) = \sum \kappa^4 \|x_i\|\, \|x_j'\|\, \|y_i\|\, \|y_j'\|,$$

we obtain a positive real-valued function on $I \times I$ that is independent of the choice of s in T'.

It is now possible to define a metric on T'. Let B be a countable dense subset of A_e containing the identity. Let $\{u_i\}$ be an enumeration of the countable dense subset of I

$$C = \left\{\sum \{x_i x_0 y_i \mid 1 \leqslant i \leqslant m\} \,\middle|\, x_i, y_i \in B,\ m = 1, 2, \cdots\right\},$$

and let d be the positive real-valued function of $T' \times T'$ given by

$$d(r, s) = \sum_{i,j} |r(u_i, u_j) - s(u_i, u_j)|/2^{i+j}(\alpha(u_i, u_j) + 1).$$

Due to the bound $|s(u_i, u_j)| \leqslant \alpha(u_i, u_j)$ on s, the function d is finite-valued. To verify that d is a metric it is necessary to verify $d(r, s) = 0$ implies $r = s$; the other properties of a metric are clearly satisfied. Let $d(r, s) = 0$. Let x_i, $x_i' \in A_e$ and let $b_i, b_i' \in B$ for $1 \leqslant i \leqslant m$; then, for every $p \in T'$, the elements $\Lambda_p(\Sigma b_i x_0 b_i')$ tend to $\Lambda_p(\Sigma x_i x_0 x_i')$ in H_p as the b_i and b_i' tend to the x_i and x_i' respectively due to the continuity of the functions $x, y \longrightarrow \Lambda_p(xzy)$ on $A_e \times A_e$, for fixed $z \in I$ (cf. (1)). (Note that this means that $\Lambda_p(C)$ is dense in $\Lambda_p(I)$.) In particular, we get that

$$\left(\Lambda_p\left(\sum b_i x_0 b_i'\right), \Lambda_p\left(\sum b_i x_0 b_i'\right)\right) \longrightarrow \left(\Lambda_p\left(\sum x_i x_0 x_i'\right), \Lambda_p\left(\sum x_i x_0 x_i'\right)\right)$$

as $b_i \longrightarrow x_i$ and $b_i' \longrightarrow x_i'$ for all i. This implies that $r(x, x) = s(x, x)$ for all $x \in I$ and consequently that $r = s$ by polarization.

We now show that the metric topology on T' is the same as the topology of pointwise convergence. In fact, let $\{s_n\}$ be a net on T' that converges to s in T' in the metric or equivalently, pointwise on $C \times C$. But given $x \in I$ and $\epsilon > 0$, there is a $u \in C$ such that $|r(u, u) - r(x, x)| < \epsilon$ for all $r \in T'$. This implies that $\lim s_n(x, x) = s(x, x)$ for all $x \in I$, and thus, that $\{s_n\}$ converges to s pointwise on I.

Now, in the usual way, we can identify T' with a closed subspace of the product of compact subsets of the complex numbers. Let Π be the compact space given by

$$\Pi = \prod \{\{\alpha \text{ complex} \mid |\alpha| \leqslant \alpha(x, y)\} \mid x, y \in I\},$$

and let Φ be the homeomorphism of T' into Π given by $\Phi(s)_{x,y} = s(x, y)$. Let $\{s_n\}$ be a net in T' such that $\{\Phi(s_n)\}$ converges to r in Π. Setting $r_{x,y} = s(x, y)$, we obtain a complex-valued function of $I \times I$ that satisfies properties (i)–(iii) of the definition of a bitrace. For $x \in A, y \in I$, we have that

$$s(xy, xy) = \lim s_n(xy, xy) \leqslant \|x\|^2 \limsup s_n(y, y) = \|x\|^2 s(y, y).$$

This means that s satisfies property (iv). Also we see that $\Phi(s) = r$. Hence, we get that $\Phi(T')$ is closed in Π, and so we have that T' is compact.

We can finish the proof by showing that T is a G_δ in T' since every G_δ in a complete metric space is complete and metrizable in the induced topology

[2, §6, Propositions 2 and 3]. Let $\{u_i'\}$ be an enumeration of the countable set $\{u_i u_j | i, j = 1, 2, \cdots\}$. For every triple i, j, k of integers, let X_{ijk} be the open subset of T' given by

$$X_{ijk} = \{s \in T' | s(u_i' - u_j, u_i' - u_j) < k^{-1}\}.$$

Then T can be written as the G_δ-set $X = \bigcap_{j,k} \bigcup_i X_{ijk}$. In fact, if $s \in X$, then s satisfies (v) since $\Lambda_s(C)$ is dense in I and since $\Lambda_s(C^2) \subset \Lambda_s(I^2)$. The converse relation is known (cf. [9, I, §1, Remark 4]). Q.E.D.

We now can prove the main result.

THEOREM 3. *Let A be a separable C^*-algebra; then the set of quasi-equivalence classes of normal representations of A is a Borel set in the quasi-dual of A with the Mackey Borel structure and is standard in the induced Borel structure.*

PROOF. Let S be a countable subset of A^+ such that, for every normal representation λ of A, the set $\lambda(S)$ contains a nonzero element of finite trace (Lemma 1). For each x in S, let $I(x)$ be the ideal generated by x and let $T(x)$ be the set of all bitraces s on $I(x)$ such that $s(x, x) = 1$. There is a Borel map $\phi = \phi_x$ of the family of all bitraces s on $I(x) \times I(x)$ taken with the Borel structure induced by the topology of pointwise convergence on $I(x) \times I(x)$ into Rep A such that $\phi(s)$ is quasi-equivalent to λ_s [9, Chapter I, §2, Lemma 2]. Because the set $T(x)$ and the inverse image $\phi^{-1}(\text{Fac } A)$ under the Borel map ϕ are certainly Borel sets in the family of all bitraces on $I(x) \times I(x)$ and because the Borel structure induced on $T(x)$ by the family of all bitraces coincides with that already assigned to $T(x)$, the restriction $\theta = \theta_x$ of ϕ to the Borel subset $T = T_x = T(x) \cap \phi^{-1}(\text{Fac } A)$ of $T(x)$ is certainly a one-one Borel map of the Standard Borel space T into Fac A. The fact that T is standard follows from the fact that it is a Borel subset of the polonais space $T(x)$ (Proposition 2). We now verify that θ is one-one. In fact, we show more: If r, s are in T and if $\theta(r)$ and $\theta(s)$ are quasi-equivalent, then $r = s$. Indeed, let $\theta(r) \sim \theta(s)$. Since $\lambda_r \sim \theta(r)$ and $\lambda_s \sim \theta(s)$, there is an isomorphism Φ of $\lambda_s(A)''$ onto $\lambda_r(A)''$ such that $\Phi(\lambda_s(y)) = \Phi(\lambda_r(y))$ for all $y \in A$. Let t_r and t_s be faithful normal semifinite traces on $\lambda_r(A)''$ and $\lambda_s(A)''$ respectively such that

$$t_r(\lambda_r(yy^*)) = r(y, y) \quad \text{and} \quad t_s(\lambda_s(yy^*)) = s(y, y),$$

for all $y \in I(x)$. The function $t_r \cdot \Phi$ defines a faithful, normal, semifinite trace on $\lambda_s(A)''$ because Φ preserves least upper bounds of monotonely increasing nets in $\lambda_s(A)''^+$. Since the trace of $\lambda_s(A)''$ is unique up to a strictly positive scalar multiple, there is an $\alpha > 0$ such that $t_r \cdot \Phi = \alpha t_s$. But we have that

$$\begin{aligned} 1 = r(x, x) &= t_r(\lambda_r(xx^*)) = t_r(\Phi(\lambda_s(xx^*))) \\ &= \alpha t_s(\lambda_s(xx^*)) = \alpha s(x, x) = \alpha. \end{aligned}$$

Hence $\alpha = 1$, and so $t_r \cdot \Phi = t_s$. Thus for all $y \in I(x)$, we have that

$$r(y, y) = t_r(\lambda_r(yy^*)) = t_s(\lambda_s(yy^*)) = s(y, y).$$

This proves that $r = s$. Now θ is a one-one Borel function of the standard Borel space T into the standard Borel space Fac A. Thus, the image $\theta(T)$ of T is a Borel subset of Fac A and the map θ is a Borel isomorphism of T onto $\theta(T)$ [1, Proposition 2.5]. Let ψ be the mapping of Fac A onto $\widetilde{A}$ which associates with each element λ of Fac A its quasi-equivalence class $[\lambda]$. Since the Borel set $\theta(T)$ of Fac A meets each quasi-equivalence class in at most one point, the image $\psi(\theta(T))$ of $\theta(T)$ is a Borel set in $\widetilde{A}$ and ψ is a Borel isomorphism of $\theta(T)$ onto $\psi(\theta(T))$ [6, 7.2.3]. Hence, the set $\psi(\theta(T))$ is a Borel subset of $\widetilde{A}$ and a standard Borel space in the induced Borel structure. Thus we get that $X = \{\psi(\theta_x(T_x)) | x \in S\}$ is a Borel subset of $\widetilde{A}$ since S is countable, and that X is a standard Borel space since X may be written as a disjoint countable union of Borel subsets of the $\psi(\theta_x(T_x))$ and such Borel subsets as well as disjoint countable unions of standard Borel spaces are standard [12, Theorem 3.1 and Theorem 3.2, Corollary 1].

We finish the proof by showing that X contains every quasi-equivalence class of normal representations for A. Let λ be a normal representation of A and let t be a faithful normal semifinite trace of $\lambda(A)''$. There is an element $x \in S$ such that $0 < t(\lambda(x)) < +\infty$ (Lemma 1). Since $\lambda(x) \in \lambda(A)^+$ we have that

$$0 < t(\lambda(x)\lambda(x)) \leqslant \|\lambda(x)\| t(\lambda(x)) < +\infty.$$

There is no loss in generality in the assumption that $t(\lambda(x)\lambda(x)) = 1$. We may define a bitrace r on the ideal

$$I = \{y \in A \,|\, t(\lambda(y)\lambda(y)^*) < +\infty\}$$

by setting $r(y, z) = t'(\lambda(y)\lambda(z)^*)$ for all $y, z \in I$. Here t' is the unique extension of t to a linear functional on its ideal of definition. The canonical representation λ_r induced by r is quasi-equivalent to λ (cf. introductory remarks, Proposition 2). Because $x \in I$, we get that $I(x) \subset I$. Let s be the restriction of r to $I(x) \times I(x)$. It is clear that s satisfies properties (i)–(iv) in the definition of a bitrace on $I(x) \times I(x)$ plus the property (vi) $s(x, x) = 1$. We show that $s \in T(x)$ by showing that $\Lambda_s((I(x))^2)$ is dense in $\Lambda_s(I(x))$. Let $\{x_n\}$ be an increasing approximate identity for $I(x)$ in the positive part of the unit sphere $I(x)$ [6, 1.7.2]. For every $y \in I(x)$, we have from (1) that

$$s((1-x_n)y, (1-x_n)y) \leqslant \kappa^4 \|1-x_n\| t'(\lambda((1-x_n)y)\lambda(y)^*)$$
$$\leqslant \kappa^4 t'(\lambda((1-x_n)y)\lambda(y)^*).$$

Because the function $z \longrightarrow t'(z\lambda(y)^*)$ is continuous on $\lambda(A)''$ [7, I, §6, Proposition 1], we conclude that

$$\lim s((1-x_n)y, (1-x_n)y) = 0.$$

This proves that $\Lambda_s(I(x)^2)$ is dense in $\Lambda_s(I(x))$. Therefore the function s is in the set $T(x)$. We now show that the canonical representation λ_s is unitarily equivalent to λ_r. Because $\lambda_r \sim \lambda$, this would imply on the one hand that λ_s is a factor representation and therefore that $s \in T_x$. On the other hand, this would imply $\theta_x(s) \sim \lambda_s \sim \lambda_r \sim \lambda$, and consequently, we would get $[\lambda] \in \psi(\theta_x(T_x))$. Hence the set X would contain all quasi-equivalence classes of normal representations. We proceed with the proof that λ_s is unitarily equivalent to λ_r. We have that the linear manifold $\Lambda_r(I(x))$ in H_r is invariant under $\lambda_r(A)$ and $\rho_r(A)$. This means that the closure of $\Lambda_r(I(x))$ in H_r corresponds to a projection e in $\lambda_r(A)' \cap \rho_r(A)' = \lambda_r(A)' \cap \lambda_r(A)''$. Because $\lambda_r(A)''$ is a factor von Neumann algebra and because $e\Lambda_r(x) = \Lambda_r(x) \neq 0$, the projection e is equal to the identity, or equivalently, $\Lambda_r(I(x))$ is dense in H_r. This means that the map $\Lambda_s(y) \longrightarrow \Lambda_r(y)$ of $\Lambda_s(I(x))$ onto $\Lambda_r(I(x))$ can be extended to an isometric isomorphism u of H_s onto H_r. For every $y \in A$, $z \in I(x)$, we get that

$$u\lambda_s(y)u^{-1}\Lambda_r(z) = u\Lambda_s(yx) = \Lambda_r(yz) = \lambda_r(y)\Lambda_r(z).$$

Consequently the representations λ_r and λ_s are unitarily equivalent via u. Q.E.D.

A measure μ on a Borel space X is said to be *standard* if there is a μ-null Borel subset M of X such that $X - M$ is standard in the induced Borel structure. Decomposition theorems for traces and trace representations are formulated in terms of standard measures confined almost everywhere to the quasi-equivalence class of normal representations (cf. [3], [6], [9], [13]). This is seen to be unnecessary.

COROLLARY 4. *The set of Borel measures of the quasi-equivalence classes of normal representations of a separable C^*-algebra (with the Mackey Borel structure) coincides with the set of standard Borel measures.*

Let f be a *state* of the C^*-algebra A (i.e., of positive linear functional on A of norm 1); let L_f be the so-called left kernel of f given by

$$L_f = \{x \in A \mid f(x^*x) = 0\}.$$

The set L_f is a closed left ideal. Let $L_f(x)$ denote the image of x in A under the canonical homomorphism of A into A (mod L_f). The relation

$$(L_f(x), L_f(y)) = f(y^*x)$$

defines an inner product on A (mod L_f). Let H_f denote the completion of A (mod L_f). If $x \in A$, the map $L_f(y) \longrightarrow L_f(xy)$ can be extended to a bounded linear operator $\lambda_f(x)$ of the Hilbert space H_f. The map $x \longrightarrow \lambda_f(x)$ is a representation of A called the *canonical representation* induced by f (cf. [6, 2.4ff.]). A state f is called a *factor state* if λ_f is a factor representation of A. Let $F(A)$ be the space of factor states of A with the relativized w^*-topology. The space $F(A)$ is a standard Borel space with the Borel structure induced by the topology ([15, 3.4.5] and [11, Lemma 7]). Two elements f and g of $F(A)$ are said to be quasi-equivalent (in symbols: $f \sim g$) if $\lambda_f \sim \lambda_g$. The relation of quasi-equivalence partitions $F(A)$ into quasi-equivalence classes. The map $\psi_1(f) = [\lambda_f]$ maps $F(A)$ *onto* $\widetilde{A}$. A set X in $F(A)$ is said to be *saturated* for the relation of quasi-equivalence if $g \in X$ whenever $g \sim f$ for some $f \in X$. A subset X_0 of the set X in $F(A)$ is said to be a *transversal* of X if, for each $f \in X$, the set X_0 meets $\psi_1^{-1}([\lambda_f])$ in exactly one point.

THEOREM 5. *Let A be a separable C^*-algebra, let $F(A)$ be the factor states of A, and let X be the set of all factor states whose canonical representations are normal. Then the set X is a saturated Borel set of $F(A)$ with a Borel transversal.*

PROOF. Let ψ_1 be the map of $F(A)$ onto $\widetilde{A}$ given by $\psi_1(f) = [\lambda_f]$. A subset Y of $\widetilde{A}$ is a Borel subset of $\widetilde{A}$ if and only if $\psi_1^{-1}(Y)$ is a Borel subset of $F(A)$ [11, Theorem 8]. Using this fact and Theorem 3, we conclude that the set X is a saturated Borel set of $F(A)$.

Let S be a countable dense subset of A^+ such that, for every normal representation λ of A, the set $\lambda(S)$ contains a nonzero element of finite trace (Lemma 1). For each $x \in S$, let $I(x)$ be the ideal generated by x, and let $T(x)$ be the set of all bitraces s on $I(x) \times I(x)$ such that $s(x, x) = 1$. Let $T = T_x$ be the set of all bitraces s in $T(x)$ such that λ_s is a factor representation. The set T_x is a Borel subset of the polonais space $T(x)$ and thus T_x is standard (Theorem 3 and its proof). For $s \in T$, let $f = f_s$ denote the positive functional on A given by $f(y) = s(yx, x)$ for $y \in A$. We note that

f is a state. Indeed, the property (v) of bitraces shows that the lim $\lambda_f(u_n) = 1$ in the strong topology on H_f where $\{u_n\}$ is an increasing approximate identity in the positive part of the unit sphere of A. (The last statement, incidentally, shows the equivalence of the definition of bitraces used in this note and the definition used by Guichardet [9, I, §1, no. 1].) The canonical representation $\lambda = \lambda_f$ induced by f is quasi-equivalent to λ_s. Because λ_s is a factor representation, it is sufficient to show that λ is equivalent to a subrepresentation of λ_s [6, 5.3.5]. For x_i, y_i $(1 \leqslant i \leqslant m)$ in A, the relation

$$\left(\sum L_f(x_i), \sum L_f(y_i)\right) = \sum_{i,j} (L_f(x_i), L_f(y_j))$$

$$= \sum_{i,j} f(y_j^* x_i) = \left(\sum \Lambda_s(x_i x), \sum \Lambda_s(y_i x)\right)$$

implies the existence of an isometric isomorphism u of H_f onto the closed subspace $H_s' = \operatorname{clos} \lambda_s(A)\Lambda_s(x)$ of H_s. The projection e' of H_s onto H_s' lies in the commutant $\lambda_s(A)'$ of $\lambda_s(A)''$ and satisfies the relation

$$u^{-1}\lambda(y)u = \lambda_s(y)e'$$

for every $y \in A$. This proves that $\lambda \sim e'\lambda_s$, and consequently, that $\lambda \sim \lambda_s$.

Now let $\Phi = \Phi_x$ be the map of T into $F(A)$ given by $\Phi(s) = f_s$. It is clear that Φ is continuous. We have that Φ is one-one; in fact, by the proof of Theorem 3, we have more: If $\Phi(r) \sim \Phi(s)$, then $\lambda_r \sim \lambda_s$ and consequently $r = s$. Hence, we get that Φ is a one-one Borel map of the standard Borel space T into the standard Borel space $F(A)$. This means that $\Phi(T)$ is a Borel subset of $F(A)$ [1, Lemma 2.5]. It is clear that $\Phi(T)$ meets each quasi-equivalence class of $F(A)$ in at most one point.

Let $\{x_i\}$ be an enumeration of S. Let $\Phi_i = \Phi_{x_i}$ and let $T_{x_i} = T_i$. We have that $\psi_1(\Phi_i(T_i)) = Y_i$ is a Borel set in $\widetilde{A}$ [11, Proposition 10]. Since the saturation Z_i of $\Phi_i(T_i)$ can be expressed as $Z_i = \psi_1^{-1}(Y_i)$, we conclude that Z_i is a Borel set in $F(A)$. We define the Borel sets $\{X_i | i = 1, 2, \cdots\}$ in $F(A)$ by

$$X_1 = \Phi_1(T_1) \quad \text{and} \quad X_i = \Phi_i(T_i) - \bigcup_{j<i} Z_j$$

for $i > 1$. Then $X_0 = \bigcup X_i$ is a Borel subset of X and is a transversal for X.

We first verify that two quasi-equivalent factor states f and g in X_0 are equal. Suppose that $f \in X_i$ and $g \in X_j$. Since Z_i and Z_j are saturated and f and g lie in Z_i and Z_j respectively, we must conclude that $i = j$. However, the set X_i is contained in the set $\Phi_i(T_i)$ which meets each quasi-equivalence class in at most one point. Hence we have that $f = g$.

Now we prove that each $f \in X$ is in the saturation of X_0. By Theorem 3, there is an $s \in T_i$ for some positive integer i such that $\psi(\theta_{x_i}(s)) = [\lambda_f]$. (Here we are employing the notation of Theorem 3.) We may assume that this i is the smallest such integer for which such an s exists. Then we have that $\lambda_f \sim \theta_{x_i}(s) \sim \lambda_s \sim \lambda_{f_s}$, and consequently, that $f \sim f_s = \Phi_i(s)$. For every $j < i$, we get that $f_s \notin Z_j$; otherwise, there is an $r \in T_j$ such that $f_r \sim f_s \sim f$ or equivalently such that $\psi(\theta_{x_j}(r)) = [\lambda_f]$. This proves that f is in the saturation of X_0. Q.E.D.

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