

THE BRAUER GROUP OF GRADED AZUMAYA ALGEBRAS. II: GRADED GALOIS EXTENSIONS⁽¹⁾

BY

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ABSTRACT. This paper continues the study of the Brauer group $B_\phi(R, G)$ of G -graded Azumaya R -algebras begun in [5]. A group $\text{Galz}_\phi(R, G)$ of graded Galois extensions is constructed which always contains, and often equals, the cokernel of $B_\phi(R, G)$ modulo the usual Brauer group of R . Sufficient conditions for equality are found. The structure of $\text{Galz}_\phi(R, G)$ is studied, and $\text{Galz}_\phi(R, (Z/p^e Z)^r)$ is computed. These results are applied to give computations of a Brauer group of dimodule algebras constructed by F. W. Long.

In the first paper of this name ([5] = [CGO]) we introduced and studied a Brauer group of graded Azumaya algebras $B_\phi(R, G)$ for G a finite abelian group, R a commutative ring with units group $U(R)$, and $\phi: G \times G \rightarrow U(R)$ a fixed bimultiplicative map. When $G = Z_2$ and ϕ is nontrivial, $B_\phi(R, G)$ is the Brauer-Wall group ([19], [3], [17]), introduced to provide an appropriate image for the Clifford algebra map on quadratic spaces. When ϕ is trivial, $B(R, G)$ was studied in [14].

This paper is a sequel to [CGO] and is motivated by two problems arising in [CGO]. One problem was that of finding an appropriate group of graded Galois extensions into which $B_\phi(R, G)$ is mapped (via a map called π in [CGO]) with kernel $B(R)$, the usual Brauer group. In [CGO] this was solved by an ad hoc method of inducing a product on the image of π from that on $B_\phi(R, G)$, and computing this product explicitly as needed. This sufficed to describe $B_\phi(R, G)$ for G cyclic. In this paper we remedy the ad hoc treatment of the image of π by introducing (§1) a group $\text{Galz}_\phi(R, G)$ of graded Galois extensions, with the group structure intrinsically defined, into which π maps as a homomorphism. It turns out that the group $\text{Galz}_\phi(R, G)$ can be described rather explicitly in many cases, and this leads to progress on the second problem left over from [CGO], namely, describing $B_\phi(R, G)$ when G is noncyclic.

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Suppose, as we shall from §3 on, that R is a connected commutative ring, G is an abelian group of order n and exponent m , and R contains $1/n$ and a primitive m th root of unity. We show in §2 that to compute $\text{Gal}_\phi(R, G)$ it suffices to do it for G a p -group. In §3 we compute $\text{Gal}_\phi(R, G)$ when $G = Z_{p^e} \times \cdots \times Z_{p^e}$ (r copies) in terms of $\text{Comm}(R, G)$, the group of commutative Galois extensions with group G and in terms of two $r \times r$ matrix groups with entries in Z/p^eZ : a group of skew matrices, and an orthogonal group $O^r(M)$. If p is odd and $\phi_s: G \times G \rightarrow U(R)$, defined by $\phi_s(\sigma, \tau) = \phi(\sigma, \tau), \phi(\tau, \sigma)$, is non-degenerate, our description becomes the exact sequence

$$1 \rightarrow \text{Comm}(R, G) \rightarrow \text{Gal}_\phi(R, G) \rightarrow O^r_{Z/p^eZ}(M) \rightarrow 1$$

where M is a nonsingular symmetric matrix representing ϕ_s .

In §4 we show that, with R, G as above, if $\text{Pic}(R)$ is n -torsion free, then the map π maps onto $\text{Gal}_\phi(R, G)$, thereby providing a description of $B_\phi(R, G)$ for a large class of noncyclic groups.

In [12], [13], F. W. Long, motivated by the theory of equivariant Brauer groups of Fröhlich and Wall [9], constructed a Brauer group $BD(R, H)$ of dimodule algebras for any finite commutative, cocommutative Hopf algebra H . We observe (§5) that when $H = RG$ and R, G are as described above, $BD(R, H) \cong B_\phi(R, G \times G)$ for an appropriate ϕ . Thus the results in this paper provide a description of Long's group, whenever R, G and $\text{Pic}(R)$ are as described above and $G = \prod_p G_p$ with $G_p = Z_{p^e} \times \cdots \times Z_{p^e}$ (n times) for any n and e , in terms of $B(R), \text{Comm}(R, G)$ and $O^{2n}_{Z/p^eZ} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Assume throughout the paper that R is a commutative ring, G is a finite abelian group, and $\phi: G \times G \rightarrow U(R)$ is a bilinear map. Recall that if $A = \sum_{\sigma \in G} A_\sigma$, $hA = \bigcup_{\sigma \in G} A_\sigma$. A formula which only makes sense for elements in hA is assumed to be valid as shown only where it makes sense and is extended by linearity on all of A . The ϕ in $B_\phi(R, G)$, etc., will often be omitted. Boldface \otimes denotes the graded tensor product of [CGO]. The product on G will be written multiplicatively, with identity 1.

All unexplained notation is from [CGO].

1. The group of graded Galois extensions. Here is the group of graded Galois extensions which is a candidate for an intrinsic description of the image of the map π of [CGO], and is, in any case, a target for π :

(1.1) DEFINITION. Let $\text{Gal}_\phi(R, G)$ be the set of isomorphism classes (where isomorphisms preserve all structure) of R -algebras S with the following structures:

- (i) S is a G -graded R -algebra.

(ii) S has a right G -action with G acting as grade-preserving R -algebra automorphisms.

(iii) S has a left G -action with G acting as grade-preserving R -algebra automorphisms.

These structures satisfy the following axioms, which are not independent:

(1) S is a graded Azumaya S_1 -algebra ($S_1 =$ trivially graded part of S , as in [CGO]).

(2r) S is a Galois extension of R on the right with group G .

(2l) S is a Galois extension of R on the left with group G .

(3r) For all a, b in hS , $ba^b = \phi(b, a)ab$, where if b is in S_σ , $a^b = a^\sigma$ denotes the right action of σ on a .

(3l) For all a, b in hS , $ab = \phi(a, b)b(^b a)$, where $^b a$ denotes left action.

(4) For all a in S , σ in G , $a^\sigma = \phi(\sigma, a)\phi(a, \sigma)^\sigma a$.

It is easy to see that (3r) + (4) implies (3l), and that (3r) implies the right center of S is S_1 , (3l) implies the left center of S is S_1 , and (2) implies separability of S over S_1 , so (3r) + (3l) + (2) implies (1).

Recall that a graded Azumaya R -algebra A is called fully graded [CGO, p. 309] if, for all σ, τ in G , $A_\sigma A_\tau = A_{\sigma\tau}$. Each class in the Brauer group $B(R, G)$ has a fully graded representative. For a fully graded Azumaya R -algebra A , $\pi(A) = A^{A_1} = \{x \text{ in } A \mid xa = ax \text{ for all } a \text{ in } A_1\}$.

In the rest of this section we prove:

(1.2) THEOREM. *The set $\text{Galz}(R, G)$ is a group, the map π is a homomorphism into $\text{Galz}(R, G)$, and hence (by [CGO, (3.13)]) there is an exact sequence*

$$0 \rightarrow B(R) \rightarrow B(R, G) \rightarrow \text{Galz}(R, G).$$

PROOF. We begin by showing that for a fully graded Azumaya R -algebra A , $\pi(A)$ is an element of $\text{Galz}(R, G)$.

The grading on $\pi(A)$ is that induced from the inclusion $\pi(A) \subseteq A$.

The right action of G on $\pi(A)$ was described in [CGO, (p. 311)]: if b_σ^i in $A_{\sigma^{-1}}$, c_σ^i in A_σ are elements satisfying $\sum_i b_\sigma^i c_\sigma^i = 1$, then defining for a in A^{A_1} , $a^\sigma = \phi(\sigma, a) \sum_i b_\sigma^i a c_\sigma^i$, we showed [CGO, (3.4)] that this action makes $\pi(A)$ into a Galois extension of R with group G . From the definition it is clear that the right action of G preserves the grading by G on $\pi(A)$. We also showed [CGO, (3.3)] that

$$\pi(A) \subseteq \{a \text{ in } A \mid xa^x = \phi(x, a)ax \text{ for all } x \text{ in } hA\};$$

the converse is obvious. Thus $\pi(A)$ satisfies (3r) and (2r) for the right G -structure.

The left G -structure arises from the fact that if A is a graded Azumaya R -algebra, so is $A^\#$, and $\pi(A^\#) = (\pi(A))^\#$ is also a graded R -algebra with a right G -structure. The grading on $A^\#$ is the same as that on A .

Define the left G -structure on πA by $(\sigma a)^\# = (a^\#)^{\sigma^{-1}}$. Then, since $\pi(A)^\#$ satisfies (2r), $\pi(A)$ satisfies (2l). Since $\pi(A)^\# = \{a^\# | x^\# a^\# x = \phi(x, a) a^\# x^\#$ for all x in $hA\}$, therefore (recalling that $x^\# a^\# = \phi(x, a)(ax)^\#$)

$$\pi(A) = \{a | ax = \phi(a, x)x(xa) \text{ for all } x \text{ in } hA\},$$

hence satisfies (3l).

If $d_\sigma^i \in A_\sigma, e_\sigma^i \in A_{\sigma^{-1}}, \sum d_\sigma^{i\#} e_\sigma^i = \sum e_\sigma^i d_\sigma^i \phi(\sigma, \sigma^{-1}) = 1$, then

$$\begin{aligned} (\sigma a)^\# &= (a^\#)^{\sigma^{-1}} = \phi(\sigma^{-1}, a) \sum d_\sigma^{i\#} a^\# e_\sigma^{i\#} \\ &= \phi(\sigma^{-1}, a) \sum \phi(d_\sigma^i, a) \phi(d_\sigma^i a, e_\sigma^i) (e_\sigma^i a d_\sigma^i)^\# \end{aligned}$$

so

$$\sigma a = \phi(\sigma^{-1}, a) \phi(\sigma, a) (\sigma a, \sigma^{-1}) \sum e_\sigma^i a d_\sigma^i,$$

but $a^\sigma = \phi(\sigma, a) \sum e_\sigma^i a (d_\sigma^i \phi(\sigma, \sigma^{-1}))$ so $\sigma a = \phi(\sigma^{-1}, a) \phi(a, \sigma^{-1}) a^\sigma$, verifying (4). Thus $\pi(A)$ satisfies all the axioms.

The product on $\text{Galz}(R, G)$ will be motivated by that induced by π , which we now describe. For A, B in $B(R, G)$,

$$\begin{aligned} \pi(A \otimes B) &= \left\{ \sum a_i \otimes b_i \mid \left(\sum a_i \otimes b_i \right) (x \otimes y) \right. \\ &= \left. \sum (x \otimes y) (a_i \otimes b_i) \text{ for all } x \otimes y \text{ in } A_\sigma \otimes B_{\sigma^{-1}} \right\}. \end{aligned}$$

Since $\pi(A \otimes B) \subseteq \pi(A) \otimes \pi(B)$,

$$\begin{aligned} \left(\sum a_i \otimes b_i \right) (x \otimes y) &= \sum (x \otimes y) (a_i \otimes b_i) = \sum \phi(y, a_i) x a_i \otimes y b_i \\ &= \sum \phi(b_i, x) a_i x \otimes b_i y = \sum \phi(b_i, x) \phi(x, a_i)^{-1} x a_i^\sigma \otimes \phi(b_i, y) y^\sigma b_i. \end{aligned}$$

Since $\phi(\sigma, xy) = 1 = \phi(xy, \sigma)$ for all σ , it follows that $\sum x a_i^\sigma \otimes y^\sigma b_i = \sum x a_i \otimes y b_i$, so $(x \otimes y) (\sum a_i^\sigma \otimes \sigma^{-1} b_i - \sum a_i \otimes b_i) = 0$ for all $x \otimes y$ in $A_\sigma \otimes A_{\sigma^{-1}}$. Choose z_j, w_j in $A_{\sigma^{-1}}, A_\sigma$, with $\sum z_j w_j = 1$, and x_k, y_k in $A_\sigma, A_{\sigma^{-1}}$ with $\sum x_k y_k = 1$; then

$$\begin{aligned} 0 &= \sum_{j,k} (z_j \otimes x_k) (w_j \otimes y_k) \left(\sum a_i^\sigma \otimes \sigma^{-1} b_i - \sum a_i \otimes b_i \right) \\ &= \sum a_i^\sigma \otimes \sigma^{-1} b_i - \sum a_i \otimes b_i. \end{aligned}$$

Thus $\pi(A \otimes B) = \{\sum a_i \otimes b_i \mid \sum a_i^\sigma \otimes \sigma^{-1} b_i = \sum a_i \otimes b_i \text{ for all } \sigma \text{ in } G\}$. So we define $S \cdot T = \{\sum s_i \otimes t_i \text{ in } S \otimes T \mid \sum s_i^\sigma \otimes \sigma^{-1} t_i = \sum s_i \otimes t_i \text{ for all } \sigma \text{ in } G\}$.

The identity of $\text{Galz}(R, G)$ is the image of the identity of $B(R, G)$, namely $GR =$ the set of all functions from G to R with pointwise addition and multiplication and with trivial grading [CGO, (3.7)]. The inverse of S in $\text{Galz}(R, G)$ is $S^\#$, with grading induced from S and left and right actions by G given by $(s^\#)^\sigma = (\sigma^{-1} s)^\#$, $\sigma(s^\#) = (s^{\sigma^{-1}})^\#$ [CGO, p. 314].

We must show that $\text{Galz}(R, G)$ is closed under multiplication and that the axioms for a group hold. Once we do that it will be clear that π is a homomorphism and, given [CGO, §3], the theorem will be proved.

Closure. The right action on $\pi(A \otimes B)$ is induced from the right action on B : let $c_\sigma^i \in B_{\sigma^{-1}}$, $d_\sigma^i \in B_\sigma$ with $\sum c_\sigma^i d_\sigma^i = 1$. Then

$$\begin{aligned} (a \otimes b)^\sigma &= \phi(\sigma, ab) \sum (1 \otimes c_\sigma^i)(a \otimes b)(1 \otimes d_\sigma^i) \\ &= \phi(\sigma, ab) \phi(\sigma^{-1}, a) \sum a \otimes c_\sigma^i b d_\sigma^i \\ &= \sum a \otimes \phi(\sigma, b) c_\sigma^i b d_\sigma^i = a \otimes b^\sigma. \end{aligned}$$

So for S, T in $\text{Galz}_\phi(R, G)$ define the right action on $S \cdot T$ to be that on T . Similarly, the left action on $\pi(A \otimes B)$ is induced from the left action on A , so define the left action of G on $S \cdot T$ to be that on S .

Give $S \cdot T$ the grading induced from the product grading on $S \otimes T$. To check that $S \cdot T$ is in $\text{Galz}(R, G)$ it suffices to show that

2. $S \cdot T$ is a Galois extension of R on the right with group G and on the left with group G , both of which preserve grading;

3. for all a, b in $h(S \cdot T)$, $ba^b = \phi(b, a)ab$;

4. for all a in $h(S \cdot T)$, σ in G , $a^\sigma = \phi(\sigma, a)\phi(a, \sigma)^\sigma a$.

PROOF OF 2. If S, T are in $\text{Galz}(R, G)$ then $S \otimes T$ is a Galois extension of R with group $G \times G$, $(\sigma, \tau)(s \otimes t) = s^\sigma \otimes \tau t$. For if $\{s_i, r_i\}$ are Galois elements for the right action of G on S (i.e., $\sum_i s_i r_i^\sigma = \delta_{1, \sigma}$ for all $\sigma \in G$) and $\{t_j, u_j\}$ are Galois elements for the left action of G on T , then it is quickly checked that

$$\{s_i \otimes t_j, \phi^{-1}(t_j, r_i) r_i \otimes u_j\}$$

is a set of Galois elements for $S \otimes T$; hence condition 1.3b of [4] holds for the action of $G \times G$ on $S \otimes T$. It follows easily by [4, 2.2] that $S \otimes T$ is a Galois extension of $S \cdot T$ with group $D = \{(\sigma, \sigma^{-1}) \in G \times G\}$, and, hence, by [4, 1.6], that there exists an element $c = \sum_k c_k \otimes d_k$ in $S \otimes T$ with $\sum_{\sigma \in G} \sum_k c_k^\sigma \otimes \sigma^{-1} d_k = 1 \otimes 1$, i.e., whose trace is one. By taking homogeneous components we can choose c to be in $S_1 \otimes T_1$.

Then, following [4, p. 23], let $\{v_j, w_j\}$ be Galois elements for the right action of G on T , and $\{s_p, r_i\}$ be as above. Consider $\{x_{ij}, y_{ij}\}$ in $S \cdot T$ as follows:

$$x_{ij} = \sum_{\sigma} \left((s_i^{\sigma} \otimes \sigma^{-1} v_j) \left(\sum_k c_k^{\sigma} \otimes \sigma^{-1} d_k \right) \right),$$

$$y_{ij} = \sum_{\sigma} \phi^{-1}(v_j, r_i) (r_i^{\sigma} \otimes \sigma^{-1} w_j).$$

Each of these is the trace over D of an element in $S \otimes T$ so is in $S \cdot T$. It is easily verified that $\{x_{ij}, y_{ij}\}$ are Galois elements for the right action of G on $S \otimes T$; hence (2r) holds.

A similar argument holds to show (2l).

3. Let $a \otimes b, c \otimes d$ be homogeneous elements of $S \cdot T \subseteq S \otimes T$. Then

$$\begin{aligned} \phi(ab, cd)(c \otimes d)(a \otimes b) &= \phi(ab, cd)\phi(d, a)ca \otimes db \\ &= \phi(ab, cd)\phi(d, a)ac^a\phi(a, c)^{-1} \otimes bd^b\phi(b, d)^{-1} \\ &= \phi(b, c)\phi(a, d)\phi(d, a)ac^a \otimes bd^b \\ &= \phi(a, d)\phi(d, a)(a \otimes b)(c^a \otimes d^b) = \phi(a, d)\phi(d, a)(a \otimes b)(c \otimes a^a d^b) \\ &= (a \otimes b)(c \otimes d^{ab}) = (a \otimes b)(c \otimes d)^{(a \otimes b)} \end{aligned}$$

and similarly for homogeneous elements of $S \cdot T$ which are sums of tensors in $S \otimes T$.

$$\begin{aligned} 4. \quad (a \otimes b)^{\sigma} &= a \otimes b^{\sigma} = \phi(\sigma, b)\phi(b, \sigma)a \otimes b^{\sigma} \\ &= \phi(\sigma, b)\phi(b, \sigma)\phi(a, \sigma)\phi(\sigma, a)^{\sigma}a \otimes b \\ &= \phi(\sigma, ab)\phi(ab, \sigma)^{\sigma}(a \otimes b) \end{aligned}$$

and similarly for sums of tensors.

So $\text{Gal}(R, G)$ is closed under multiplication.

Associativity. We have $S(TU) = (S \otimes (T \otimes U)^D)^D$. Since $T \otimes U$ is a Galois extension of $(T \otimes U)^D$ with group D , every element of $(T \otimes U)^D$ is the trace of some element of $T \otimes U$, etc. Thus a typical element of $S(TU)$ is of the form

$$\sum_i \sum_{\sigma} s_i^{\sigma} \otimes \sigma^{-1} \left(\sum_j \sum_{\tau} t_{ij}^{\tau} \otimes \tau^{-1} u_{ij} \right) = \sum_{ij} \sum_{\tau} \left(\sum_{\sigma} s_i^{\sigma} \otimes \sigma^{-1} t_{ij}^{\tau} \right)^{\tau} \otimes \tau^{-1} u_{ij},$$

a sum of elements in $((S \otimes T)^D \otimes U)^D = (ST)U$.

Identity. The identity of $\text{Galz}(R, G)$ is $GR =$ functions from G to R with trivial grading. On GR the left and right G -structures coincide, and if S is any graded Galois extension $S \otimes GR = S \otimes GR$ since GR is trivially graded. Thus as a graded algebra and right Galois extension $S \cdot GR = S$ by the standard argument [CGO, p. 313]; since the left G -structure on $S \cdot GR$ is the same as that on S , GR is the identity.

Inverse. The inverse of S is $S^\#$, as described above. Consider the map $f: (S \otimes S^\#)^D \rightarrow GR$, defined by linearity and $f(xy^\#)(\tau) = x(\tau y)$. The image of $f(xy^\#)$ is in fact in R since

$$\begin{aligned} (x^\tau y)^\sigma &= x^\sigma \cdot {}^\tau y^\sigma = f(x^\sigma \otimes (y^\sigma)^\#)(\tau) \\ &= f(x^\sigma \otimes \sigma^{-1}(y^\#))(\tau) = f(x \otimes y^\#)(\tau) = x^\tau y; \end{aligned}$$

f is an algebra map since

$$\begin{aligned} f(x \otimes y^\#)(z \otimes w^\#)(\tau) &= f(xz \otimes y^\# w^\# \phi(y, z))(\tau) \\ &= f(xz \otimes \phi(y, w)(wy)^\# \phi(y, z))(\tau) \\ &= f(xz \otimes (wy)^\#)(\tau) \text{ since } zw \text{ is in } R \quad (f(z \otimes w^\#)(1) = zw) \\ &= xz^\tau (wy) = xz^\tau w^\tau y \\ &= (x^\tau y)(z^\tau w) \text{ since } z^\tau w \text{ is in } R \\ &= f(x \otimes y^\#)(\tau) \cdot f(z \otimes w^\#)(\tau). \end{aligned}$$

By [10, Corollary 2, p. 5], it follows that $(S \otimes S^\#)^D = GR$, and $S^\#$ is the inverse of S in $\text{Galz}(R, G)$. This completes the proof.

2. **Reduction to p -groups.** The next two sections are devoted to a study of $\text{Galz}_\phi(R, G)$ for a given fixed bilinear map $\phi: G \times G \rightarrow U(R)$. In this section we prove

(2.1) **THEOREM.** *Let $G = H \times J$ where the orders of H and J are relatively prime. Let ϕ_H, ϕ_J be the restrictions of ϕ to H, J . Then $\text{Galz}_\phi(R, G) = \text{Galz}_{\phi_H}(R, H) \times \text{Galz}_{\phi_J}(R, J)$.*

(2.2) **COROLLARY.** *Let $G = \prod G_p$ be the decomposition of G into its p -primary components, and let ϕ_p be ϕ on $G_p \times G_p$. Then*

$$\text{Galz}_\phi(R, G) = \prod_p \text{Galz}_{\phi_p}(R, G_p).$$

The equality of (2.1) will be obtained by decomposing any S in

$\text{Galz}_\phi(R, G)$ as

$$(2.3) \quad S \cong S^J \otimes_R S^H.$$

To prove (2.1) it suffices to show

(2.4) The decomposition (2.3) is as R -algebras and G -modules.

(2.5) The algebras S^J and S^H are in $\text{Galz}_{\phi_H}(R, H)$, $\text{Galz}_{\phi_J}(R, J)$, respectively.

(2.6) The decomposition (2.3) gets along with the group structures on $\text{Galz}_\phi(R, G)$, etc.

PROOF OF (2.4). Since S is a Galois extension of R with group $G = H \times J$ it is standard that $S \cong S^J \otimes_R S^H$ as R -modules and G -modules. The only nontrivial aspect of (2.4) is in showing that (2.3) is a decomposition as R -algebras, that is, that elements of S^J and S^H commute with each other. To do this we look at the R -module generated by $\{xy - yx \mid x \in S^J, y \in S^H\} \subseteq S$. To show it is zero and (2.4) holds it suffices to show it when we replace R by a faithfully flat extension, namely, the direct sum of the stalks on the Boolean spectrum of R , a faithfully flat extension of R [18, 2.9]. Since these stalks are connected [18, 2.13] it suffices to assume R is connected. In that case, S , being a Galois extension of R with abelian group G , is a central Galois extension of T , the center of S , with group L [11, Proposition 8]. Then $S = \Sigma \bigoplus J_\sigma$ with $J_1 = T$, J_σ rank one projective T modules, for σ in L [11]. Here $J_\sigma = \{a \in S \mid ax^\sigma = xa\}$. The noncommutativity of S is expressed by a skew nondegenerate bilinear form ψ on L : for a in J_σ , b in J_τ , $ba = \psi(\tau, \sigma)ab$ [8], [6]. Since $L \cap H$ and $L \cap J$ have relatively prime orders, $L \cap H$ and $L \cap J$ are orthogonal with respect to ψ , so ψ must be nondegenerate on both $L \cap H$ and $L \cap J$. We have $J_\sigma J_\tau = J_{\sigma\tau}$ [11], so we may write $S = (\Sigma_{\sigma \in (H \cap L)} J_\sigma) \otimes_T (\Sigma_{\sigma \in (J \cap L)} J_\sigma) = S_H \otimes_T S_J$. Clearly $S_J \subseteq S^{S^H}$, the commutator of S_H in S ; in fact, $S_J = S^{S^H}$: for S is an Azumaya T -algebra and S_H and S_J both have center T by the nondegeneracy of ψ on $L \cap H$ and $J \cap H$, and are contained in each other's commutators; [2, (3.3)] applies to yield $S_J = S^{S^H}$, $S_H = S^{S^J}$. Now for x in S , σ in $H \cap L$, a in $J_\sigma \subseteq S_H$, we have $ax^\sigma = xa$. If x is in S^H , $x^\sigma = x$ so $ax = xa$. So $S^H \subseteq S^{S^H} = S_J$. Similarly, $S^J \subseteq S_H$. So elements of S^H and of S^J commute with each other.

PROOF OF (2.5). The difficulty in showing that if S is in $\text{Galz}_\phi(R, G)$ then S^J is in $\text{Galz}_{\phi_H}(R, H)$ lies in showing that the gradings of S^J are in H . So we begin (2.5) with some lemmas on the gradings of S in $\text{Galz}_\phi(R, G)$.

(2.7) LEMMA. *Let R be connected, S in $\text{Galz}_\phi(R, G)$. For each σ in $K = \{\sigma \in G \mid S_\sigma \neq 0\}$:*

- (i) $S_\sigma = \{a \in S \mid ax^\sigma = \phi(a, x)xa\}$ is a rank one projective S_1 -module,
- (ii) $S_\sigma S_\tau = S_{\sigma\tau}$.

PROOF OF (2.7). Since S is in $\text{Galz}_\phi(R, G)$, S is a graded Azumaya S_1 -algebra, by (1.1).

Let ${}^eS = S^\# \otimes_{S_1} S$ [CGO, (1.3)] and view S as an S - S bimodule, $S(1, \sigma)$, by the usual left S -module action of S on S and a right action by $s \cdot x = sx^\sigma$. Then the Morita equivalence [CGO, 2.8] between S_1 -modules and right eS -modules shows that $J_\sigma = \{a \in S \mid ax^\sigma = \phi(ax)xa\} = {}^eS(1, \sigma)$ is a rank one projective S_1 -module. Now $J_\sigma = S_\sigma$. For clearly $S_\sigma \subseteq J_\sigma$; on the other hand, if a is in S_ρ , $ax^\rho = \phi(a, x)xa$ for all $x \in S$, so $a \in J_\sigma \cap S_\rho$ iff $x^\rho = x^\sigma$ for all $x \in S$. But S is a Galois extension of R with group G , so if $x^\rho = x^\sigma$ for all x in S , $\rho = \sigma$. Thus $J_\sigma = S_\sigma$. That $S_\sigma S_\tau = J_\sigma J_\tau = J_{\sigma\tau} = S_{\sigma\tau}$ follows by tracing through the maps in the isomorphism of $J_\sigma \otimes J_\tau$ with $J_{\sigma\tau}$ in [15, Lemma 5] (as was observed in [7, Lemma 1]).

(2.8) LEMMA. Assume R is connected. If $K = \{\sigma \in G \mid S_\sigma \neq 0\}$ then K is a group and S is a Galois extension of S_1 with group K .

PROOF OF (2.8). From (2.7) (ii) it is clear that K is a submonoid of G , so since G is finite, K is a group. Now note that $S_1 \subseteq S^K$. For if σ is in K , a is in S_σ , x is in S_1 , then $ax^\sigma = \phi(a, x)xa$. But S_1 is in the center of S (use the same formula with x arbitrary and a in S_1), so $ax = xa$, so $a(x^\sigma - x) = 0$. Choose a_i in S_σ , b_i in $S_{\sigma^{-1}}$ with $\sum b_i a_i = 1$; then $0 = \sum b_i a_i (x^\sigma - x) = x^\sigma - x$. Thus $S_1 \subseteq S^K$. Now, since S is a Galois extension of S^K with group K , S^K is a direct summand of S , S is a projective S^K -module [4] and $\text{rank}_{S_1} S = [K : 1] = \text{rank}_{S^K} (S)$. Since S^K is S_1 -projective, S_1 is an S_1 -direct summand of S^K . Thus $S^K = S_1$.

PROOF OF (2.5). We check the conditions of (1.1) for $T = S^J$. The right action of H on T is clear. We must show that the set of gradings of T is contained in H .

We first observe that T is a projective T_1 -module. For this it suffices to assume R is local. Now T is a Galois extension of R with group $H = G$ restricted to T , and T_1 is the fixed ring of K restricted to T since $T_1 = T \cap S_1$, by (2.8). So T is a Galois extension of T_1 with group $K \cap H$; hence T is T_1 -projective, and $\text{rank}_{T_1} (T)$ divides the order of H at each prime ideal of T_1 .

To show that the set of gradings of T is contained in H it suffices to assume T_1 is local. We have $T = \sum_{\sigma \in L} T_\sigma$, where $L = \{\sigma \in G \mid T_\sigma \neq 0\}$, so the T_σ , being T_1 -submodules of T , are T_1 -projective. The multiplication map $T \otimes_{T_1} S_1 \rightarrow TS_1$ is an isomorphism since both are Galois extensions of

S_1 with group $H \cap K \cong K/(J \cap K)$ [4, (3.4)]. This map sends $T_\sigma \otimes_{T_1} S_1$ one-one into S_σ . Since S_σ is a rank one projective S_1 -module, T_σ must be a rank one projective T_1 -module. We show $T_\sigma \cdot T_\tau \neq 0$, which will suffice to show that L is a group whose order = $\text{rank}_{T_1}(T)$ divides the order of H ; hence $L \subseteq H$.

To show $T_\sigma \cdot T_\tau \neq 0$ it suffices to show that $T_\sigma S_1 \cdot T_\tau S_1 \neq 0$, since S_1 is in the center of S . But $T_\sigma S_1$ and $T_\tau S_1$ are rank one projective S_1 -submodules of the rank one projectives S_σ and S_τ . Since S_1 is semilocal, we may pick free bases $S_\sigma = S_1 x$, $S_\tau = S_1 y$, $T_\sigma S_1 = S_1 cx$, $T_\tau S_1 = S_1 dy$, $x, y \in S$, c, d in S_1 ; then $S_\sigma S_\tau = S_{\sigma\tau} = S_1 xy$. Since x, y, cx, dy and xy are bases for free S_1 -modules of rank 1, none can be divisors of zero in S_1 . Thus $T_\sigma S \cdot T_\tau S_1 = S_1 cdx y \neq 0$, for as is easily seen, $cdxy$ cannot be a zero divisor. Thus $L \subseteq H$.

To show that T is a left H -module observe that since T is graded by H , for all a in T , σ in J , $a^\sigma = \sigma a$, by (1.1), (4). Thus $S^J = T \subseteq {}^J S$. The left-right symmetry of the argument thus far gives the opposite inclusion. Thus $T = {}^J S$ and has a natural left H -action induced from the left action of G .

It is clear then that (2r) and (2l) of (1.1) hold for T , and that (3r), (3l) and (4) hold by inheritance from S . That proves (2.5).

PROOF OF (2.6). With $G = H \times J$ as above, let S, T be in $\text{Galz}_\phi(R, G)$. Then $S = S^J \otimes S^H, T = T^J \otimes T^H$ with S^J, T^J in $\text{Galz}_\phi(R, H), S^H, T^H$ in $\text{Galz}_{\phi_J}(R, J)$, by (2.4) and (2.5). We have $S \cdot T = S^J \cdot T^J \otimes S^H \cdot T^H$. For H and J are mutually orthogonal with respect to ϕ , so the usual "switch" map

$$S \otimes T = (S^J \otimes S^H) \otimes (T^J \otimes T^H) \cong (S^J \otimes T^J) \otimes (S^H \otimes T^H)$$

is an R -algebra isomorphism. Then if DG , etc., is the kernel of the multiplication map from $G \times G$ to G , acting on $S \otimes T$ by $(\sigma^{-1}, \sigma)(s \otimes t) = s^\sigma \otimes \sigma t$ (cf. proof of (1.2)) it is easy to see that if $(S \otimes T)^{DG} = S \cdot T$ is the fixed ring under the action of DG , then $(S \otimes T)^{DG} \cong (S^J \otimes T^J)^{DH} \otimes (S^H \otimes T^H)^{DJ}$. This completes the proof of (2.6) and hence of (2.1).

3. Matrix computations. In view of Corollary (2.2) in order to describe $\text{Galz}_\phi(R, G)$, it is enough to do it for p -groups. This section will be devoted to a description of $\text{Galz}_\phi(R, G)$ for G a p -group which is a finite product of cyclic groups of equal exponent, i.e., groups which are free modules of finite rank over $Z/p^e Z$. Before restricting ourselves to such G we first obtain a general remark about elements of $\text{Galz}(R, G)$ under the following assumption, which will remain in force throughout this section:

(3.1) Assume G is a finite abelian group of order n and exponent m ,

and R is a connected commutative ring containing $1/n$ and a primitive m th root of unity.

Under the assumption (3.1) $G^* = \text{Hom}(G, U(R)) \cong G$, and we may decompose any abelian Galois extension, just as in classical Kummer Theory of fields, into a direct sum of R -submodules, indexed by G^* , on each of which each element of G acts as multiplication by a root of unity [6].

Let S be in $\text{Galz}(R, G)$. Then S is graded in three ways: by G , and by $G^* = \text{Hom}(G, U(R))$ in two different ways, arising from the left and right actions of G on S , respectively, as follows: Since S is, with respect to each of the G -actions, a Galois extension of R with group G , $S = \sum_{\chi} S_{\chi} = \sum_{\chi} \chi^* S$ with respect to the G^* -gradings, where

$$S_{\chi} = \{s \text{ in } S \mid s^{\sigma} = \chi(\sigma)s\}, \quad \chi^* S = \{s \text{ in } S \mid s^{\sigma} = \chi(\sigma)s \text{ for all } \sigma \text{ in } G\}.$$

(3.2) LEMMA. *Each S_{χ} is a ψS and each is homogeneous with respect to G .*

PROOF. Since the left action by G , the right action by G , and the grading by G all commute with each other, we may view S as an $RG \otimes RG \otimes RG^*$ -module, hence as an $RG^* \otimes RG^* \otimes RG$ -comodule, i.e., as graded by $G^* \times G^* \times G$. Thus

$$S = \sum_{(\psi, \chi, \sigma)} S_{(\psi, \chi, \sigma)}$$

($S_{(\psi, \chi, \sigma)}$ = elements homogeneous of degree (ψ, χ, σ) on $G^* \times G^* \times G$)

and, in particular, each $S_{\chi} = \sum_{(\psi, \sigma)} S_{(\psi, \chi, \sigma)}$; each $\psi S = \sum_{(\chi, \sigma)} S_{(\psi, \chi, \sigma)}$. But each S_{χ} , respectively ψS , is a rank one projective R -module, so that (since R is connected) there can be only one summand. Thus $S_{\chi} = S_{(\psi, \chi, \sigma)}$ for some ψ, σ ; also $\psi S = S_{(\psi, \chi, \sigma)}$ for some χ, σ . Thus if $S_{\chi} = S_{(\psi, \chi, \sigma)}$, $S_{\chi} = \psi S \subseteq S_{\sigma}$.

We now assume that G is a p -group. We wish to compute $\text{Galz}_{\phi}(R, G)$. This is relatively complicated in general, but becomes manageable in case

(3.3) G is a product of cyclic groups all of the same order $m = p^e$.

For in that case we can map $\text{Galz}(R, G)$ into a group of orthogonal matrices over $Z/p^e Z$. We now describe how this is done.

Let $G = \prod_{i=1}^r Z_i$ with $Z_i = \langle \sigma_i \rangle$, cyclic of order p^e , and $G^* = \prod_{i=1}^r Z_i^*$ with $Z_i^* = \langle \chi_i \rangle$, where, if ζ is a fixed m th root of unity, $\chi_i(\sigma_i) = \zeta$. Let S be in $\text{Galz}_{\phi}(R, G)$.

From [6] and (3.1) we can write $S = \sum_{\chi \in G^*} S_{\chi}$ where G acts on s in S_{χ} by $s^{\tau} = \chi(\tau)s$. By (3.2), $S_{\chi} = S_{(\psi, \chi, \sigma)}$, that is, $\tau s = \psi(\tau)s$ and s is

homogeneous of grade σ . By [6] all of these gradings respect the multiplication in S ; hence

$$S_{(\psi_1, x_1, \sigma_1)} \cdot S_{(\psi_2, x_2, \sigma_2)} = S_{(\psi_1 \psi_2, x_1 x_2, \sigma_1 \sigma_2)}.$$

We will associate to S some $r \times r$ matrices with entries in Z/mZ .

Notation. Let $\chi = (\chi_1, \dots, \chi_r)$, $\sigma = (\sigma_1, \dots, \sigma_r)$ and, if β is a vector $(\beta_1, \dots, \beta_r)$ in $(Z/mZ)^r$, write $\chi^\beta = \chi_1^{\beta_1} \chi_2^{\beta_2} \dots \chi_r^{\beta_r}$, and similarly for σ^β .

For each i , $S_{\chi_i} = S_{(\chi^{p_i}, \chi^{\epsilon_i}, \sigma^{\alpha_i})}$ where $p_i, \epsilon_i, \alpha_i$ are (row) vectors in $(Z/mZ)^r$, $\epsilon_i = (0, \dots, 1, \dots, 0)$ (1 in i th entry). Now S is generated as an algebra by the S_{χ_i} , so the left G -action on S is described by the $r \times r$ matrix

$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix}$$

of left G^* -gradings of the S_{χ_i} , $i = 1, \dots, r$, and similarly the G -grading on S is described by the $r \times r$ matrix

$$A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}$$

of G -gradings of the S_{χ_i} . The corresponding matrix of right G^* -gradings is

$$\begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_r \end{pmatrix} = I.$$

Let u_i be a typical element of S_{χ_i} for $i = 1, \dots, r$. From [6] (cf. proof of (2.4) above) we know that $u_i u_j = \delta(\chi_i, \chi_j) u_j u_i$ where $\delta: G^* \times G^* \rightarrow U(R)$ is a skew bilinear map; δ thus describes the way elements of S (fail to) commute with each other. By bilinearity, $\delta(\chi_i, \chi_j) = \zeta^{d_{ij}}$ and $D = (d_{ij})$ is clearly an $r \times r$ skew-symmetric matrix with entries in Z/mZ .

From bilinearity of ϕ we have $\phi(\sigma_i, \sigma_j) = \zeta^{\phi_{ij}}$ where $\phi = (\phi_{ij})$ is an $r \times r$ matrix in Z/mZ . Set $M = \phi + \phi'$ where $'$ means transpose.

The structure of S as an element of $\text{Galz}_\phi(R, G)$ defines some relations among P, A, D, ϕ, M .

The relations:

- (1) $u_i u_j^{u_i} = \phi(u_i, u_j) u_j u_i$,
- (2) $u_i u_j = \delta(\chi_i, \chi_j) u_j u_i$ with δ skew bilinear,
- (3) $u_i^\sigma = \phi(\sigma, u_i) \phi(u_i, \sigma)^\sigma u_i$,

(4) S is a left Galois extension with group G , for all i, j, σ are necessary in order that S be in $\text{Galz}_\phi(R, G)$.

Condition (4) is equivalent to the nonsingularity of P .

Conditions (1) and (2) combine to yield:

$$u_i u_j^{u_i} = \phi(u_i, u_j) \delta(\chi_j, \chi_i) u_i u_j, \quad \text{i.e.,}$$

$$\langle \chi^{\epsilon_j}, \sigma^{\alpha_i} \rangle = \phi(\sigma^{\alpha_i}, \sigma^{\alpha_j}) \delta(\chi_j, \chi_i), \quad \text{i.e.,}$$

$$\zeta^{\alpha_i \epsilon_j'} = \zeta^{\alpha_i \phi \alpha_j'} \zeta^{d_{ji}}, \quad \text{i.e.,}$$

$$d_{ij} + \alpha_i \epsilon_j' = \alpha_i \phi \alpha_j', \quad \text{i.e.,}$$

$$D + AI = A\phi A'.$$

Condition (3) becomes

$$\langle \chi^{\epsilon_i}, \sigma_j \rangle = \phi(\sigma_j, \sigma^{\alpha_i}) \phi(\sigma^{\alpha_i}, \sigma_j) \langle \chi^{\rho_i}, \sigma_j \rangle, \quad \text{i.e.,}$$

$$\epsilon_i \epsilon_j' = \epsilon_j \phi \alpha_i' + \alpha_i \phi \epsilon_j' + \rho_i \epsilon_j', \quad \text{i.e.,}$$

$$I = A\phi' + A\phi + P.$$

Summing up, conditions (1)–(4) yield:

$$D = A\phi A' - A \text{ is skew,}$$

$$P = I - AM \text{ is nonsingular.}$$

(3.4) THEOREM. With \dot{R}, G as in (3.1), (3.3), $\text{Galz}(R, G)$ is described by the following two exact sequences:

$$(3.4.1) \quad 1 \rightarrow \text{Galz}_\phi^0(R, G) \rightarrow \text{Galz}_\phi(R, G) \xrightarrow{\beta} O_{Z/mZ}^r(M),$$

$$(3.4.2) \quad 1 \rightarrow \text{Comm}(R, G) \rightarrow \text{Galz}_\phi^0(R, G) \xrightarrow{\nu} \text{Skew}_{Z/mZ}^r(M),$$

where $O_{Z/mZ}^r(M) = \{P \in GL_r(Z/mZ) \mid P'MP = M\}$ and β , defined by $\beta(S) = P$, is an antihomomorphism, $\text{Skew}_{Z/mZ}^r(M) = \{A \in M_r(Z/mZ) \mid AM = 0 \text{ and } A' = -A\}$ and ν , defined by $\nu(S) = A$, is a homomorphism.

If $M = 0$, $\beta = 1$.

If $p \neq 2$, ν is onto; if M is nonsingular, $\nu = 0$ and β is onto.

If $p = 2$, $\text{Im } \nu = \{A \in \text{Skew}^r(M) \mid (A\phi A' - A)_{ii} = 0 \text{ for all } i = 1 \cdots r\}$; if M is nonsingular, $\nu = 0$ and $\text{Im } \beta = \{P \in O^r(M) \mid [(I - P)M^{-1}\phi M^{-1}(I - P) - (I - P')M^{-1}]_{ii} = 0 \text{ for } i = 1 \cdots r\}$.

We remark that $\text{Galz}_\phi^0(R, G)$ is a subgroup of $\text{Galz}_\phi(R, G)$ whose product $S \cdot T = (S \otimes_R T)^D$ is the usual product of Galois extensions. This follows

by [CGO, (3.6)] and the fact that since $P = I$ the left and right G -actions on S in $\text{Galz}_\phi^0(R, G)$ coincide.

PROOF. Since $A\phi A' - A$ is skew, $AMA' = A + A'$. Thus $PA' = A' - AMA' = -A$, $P^{-1}A = -A'$, so

$$\begin{aligned} P'MP &= (I - MA')MP \quad (\text{since } M \text{ is symmetric}) \\ &= MP - MA'MP = MP + MP^{-1}AMP = MP + MP^{-1}(I - P)P \\ &= MP + MP^{-1}P - MP^{-1}PP = M. \end{aligned}$$

Thus β is well defined. If $P = I$, $AM = 0$ and $A + A' = AMA' = 0$. Thus ν is well defined.

We show that the map β is an antihomomorphism.

Let S, T be in $\text{Galz}(R, G)$ with left grading matrices P, Q , respectively. Then $ST = \Sigma S_\psi \otimes_\psi T$, where ψ runs through G^* . Since the right grading on ST is induced from that on T , and the left grading is induced from that on S , we can compute the matrix of ST by noticing that $T_{x_i} = {}_{x_i}q_i T$, and $S_{x_i} = {}_{x_i}p_i S$, so that $(ST)_{x_i} = S_{x_i} q_i \otimes T_{x_i}$, and the left grade of

$$S_{x_i} q_i = S_{x_1}^{q_{i1}} S_{x_2}^{q_{i2}} \cdots S_{x_r}^{q_{ir}} \text{ is } \chi^{p_1 q_{i1} + p_2 q_{i2} + \cdots + p_r q_{ir}} = \chi^{q_i P}.$$

So the left grading matrix of ST is QP .

If S, T have left grading matrices $P = Q = I$, then the G -grading on ST is that of $(S \cdot T)_{x_i} = S_{x_i} \otimes_{x_i} T = S_{x_i} \otimes T_{x_i}$, namely, $\sigma^{\alpha_i + \beta_i}$. So ν is a homomorphism.

If $\nu(S) = 0$ then S is trivially graded and $D = 0$, so S is commutative. On the other hand, it is easy to check that any trivially graded commutative Galois extension is in $\text{Galz}^0(R, G)$. Thus the two sequences are exact. Since $P = I - AM$, if $M = 0$, $\beta = 1$. All that is left to prove are the ontteness statements for ν and β .

We begin showing ontteness of β , assuming M is invertible, by picking P in $O(M)$ and defining A by $A = (I - P)M^{-1}$. Then $P'MP = M$, $PM^{-1}P' = M^{-1}$, that is, $M^{-1} = (I - AM)M^{-1}(I - AM') = M^{-1} - A - A' + AMA'$, so $A\phi A' - A = D$ is skew. If $p = 2$ we assume D has zero diagonal. Thus associated to P are matrices A, D . We begin showing ontteness of ν by letting A be such that $AM = 0$ and $A' = -A$. Then associated to A are the skew matrix $D = A\phi A' - A$ and the nonsingular matrix $P = I = I - AM$. If $p = 2$ we assume D has zero diagonal.

Given P, A, D we define an algebra S in $\text{Galz}_\phi(R, G)$ whose structure is described by these matrices as follows.

Let $S = \Sigma_\nu Ru_1^{\nu_1} u_2^{\nu_2} \cdots u_r^{\nu_r}$, where $u_i^m = 1$. Fixing as above a primitive

m th root of unity ζ , define $u_i u_j = \zeta^{d_{ij}} u_j u_i$, extending by linearity. This defines the multiplication on S . Since $d_{ii} = 0$ it is well defined. Define gradings on S by giving u_i the left G^* -grading $\chi^P i$, the right G^* -grading $\chi^{\epsilon i}$, and G -grading $\sigma^{\alpha i}$. It is then clear that S is a G -graded R -algebra and a Galois extension on the right with group G . Since P is invertible, one can find elements v_i in S so that v_i has left grading χ_i and the v_i generate S in the same way as the u_i . So S is a Galois extension on the left with group G . The other relations (1)–(4) of (1.1) all follow from the relationships among the matrices P, A, ϕ, D . This shows the ontteness of ν , and if M is invertible, the ontteness of β , completing the proof of the theorem.

(3.5) REMARK. When R, G satisfy (3.1) the description of $\text{Im}(\pi)$ for G a cyclic p -group given in [CGO, Theorem 4.1] follows easily from the techniques of Theorem (3.4) and the result, to be proved in §4, that, for G a cyclic p -group, $\text{Im}(\pi) = \text{Galz}(R, G)$. For suppose G is a cyclic p -group with generator σ , with the order of $G = p^e = m$. Choose ζ , a primitive p^e th root of unity, so that $\phi(\sigma, \sigma) = \zeta^{p^i}$ for some $i \geq 0$. Note that in applying (3.4) all matrices are 1×1 .

For p odd, (3.4) reads:

$$1 \rightarrow \text{Comm}(R, G) \rightarrow \text{Galz}_\phi(R, G) \xrightarrow{\beta} O^1_{Z/p^e Z}((2p^i)) \rightarrow 0$$

since $\text{Skew}^1_{Z/p^e Z}((2p^i)) = 0$. Now $O^1_{Z/mZ}((2p^i)) = Z/2Z$ unless $\phi = 1$. If $i = 0, \beta$ is onto by (3.4) and we recover [CGO, 4.1]. If $i > 0$ then since $D = A\phi A' - A$ is skew, $D = 0$, so $A(\phi A' - I) = 0$, so $A = 0, P = I$ since $I - \phi A'$ is a unit. So if $i > 0, \beta = 1$ and we recover [CGO, 4.1].

For $p = 2, G$ of order $p^e, e > 1$, set $\phi(\sigma, \sigma) = \zeta^{2^i}$. Now $\{A \in \text{Skew}^1(M) | A\phi A' - A = 0\} = \{0\}$, as is easily checked, so we simply have to compute β . We know $0 = D = A(\phi A - I)$ since cyclic Galois extensions are commutative. If $i \geq 1, \phi A - I$ is a unit and $A = 0, \beta = 1$. If $i = 0, A(A - I) = 0$ so $A = I$ or $A = 0$. If $A = I$, since $M = (2), P = -I$, otherwise $P = I$. So if $i \geq 1, \text{Galz}_\phi(R, G) = \text{Comm}(R, G)$; if $i = 0$ we get that $\text{Im}(\beta) = Z/2Z$ and

$$1 \rightarrow \text{Comm}(R, G) \rightarrow \text{Galz}(R, G) \rightarrow Z/2Z \rightarrow 1$$

is exact, recovering [CGO, 4.1] in this case.

For $p = 2, G = Z/2Z, \phi(\sigma, \sigma) = (-1)^{2^i} (i = 0 \text{ or } 1)$, and $M = 0$, so $\beta = 1$ and the sequence (3.4.2) describes $\text{Galz}_\phi(R, G)$. Then it is easily checked that $\{A \in \text{Skew}^1(M) | A\phi A' - A = 0\}$ is equal to $\{0\}$ if $i = 1$, i.e., ϕ is trivial, but equals $\{0, 1\}$ if $i = 0$, i.e., ϕ is nontrivial, thereby recovering [CGO, 4.1] in this case also.

4. Ontness of π . In this section we prove, under appropriate conditions

on R , that π is onto. This result, together with those of §3, yields a description of $B_\phi(R, G)$ for any $G = \Pi Z_{pe}$.

Throughout this section, the assumptions (3.1) hold, i.e., G is a group of order n and exponent m , and R is a connected commutative ring containing $1/n$ and a primitive m th root of unity.

We begin with some results on the structure of elements of $\text{Gal}_\phi(R, G)$.

(4.1) LEMMA. *With R, G as above, let S be in $\text{Gal}_\phi(R, G)$, and let $K = \{\sigma \in G \mid S_\sigma \neq 0\}$ be the group (2.8) of gradings of S : $S = \sum_{\sigma \in K} S_\sigma$. Let Z be the ungraded center of S and let $H = \{\sigma \in K \mid Z_\sigma \neq 0\}$. Then H is a group and $Z_\sigma = S_\sigma$ for all σ in H .*

PROOF. It suffices to verify both assertions when R is local. In that case, S , being a Galois extension of R with group G , is a twisted group ring $S = \sum_{\chi \in G} Ru_\chi$ with factor set in $U(R)$. The noncommutativity of S is expressed by $u_\chi u_\psi = D(\chi, \psi)u_\psi u_\chi$, where $D: G^* \times G^* \rightarrow U(R)$ is skew and bilinear [6].

Now for any χ, ψ in G^* , $u_\chi u_\psi^\sigma = \phi(u_\chi, u_\psi)D(\psi, \chi)u_\chi u_\psi$; if u_χ has grading σ_χ this becomes $\langle \psi, \sigma_\chi \rangle = \phi(\sigma_\chi, \sigma_\psi)D(\psi, \chi)$, that is, as a function of χ, D only depends on the grading of u . Thus, $H = \{\sigma \in K \mid \langle \psi, \sigma \rangle = \phi(\sigma, \sigma_\psi) \text{ for all } \psi \text{ in } G^*\}$, a group, and if $\sigma \in H$, every u_χ in S_σ , hence all of S_σ (in view of (3.2)) is in Z .

(4.2) LEMMA. *If $S \in \text{Gal}_\phi(R, G)$, R, G as above, if Z is the ungraded center of S and*

$$C = \{x \in S \mid x^\sigma = \phi(\sigma, x)x \text{ for all } \sigma \text{ in } G\},$$

then $Z = C \cdot S_1 \cong C \otimes_R S_1$.

PROOF. We know that S is a Galois extension of R with group G acting on the right, the image of σ in G on $s \in S$ being written s^σ . Define a new action of G on S by $\sigma(s) = \phi(\sigma^{-1}, s)s^\sigma$. Denote G when acting in that way on S by G_r . Notice that $C = \{s \in S \mid \sigma(s) = s \ \forall \sigma \in G_r\}$ by definition. Also, $\{\sigma \mid \sigma(s) = s \text{ for all } s \in S\} = H$. For if $\sigma \in G$ and $\sigma(s) = s$ for all s in S , then in particular $\sigma(s) = s$ for all s in S_1 , but on S_1 , $\sigma(s) = s^\sigma$, so $\sigma \in K$ by (2.8). Now for σ in K, x in $S_\sigma, xs^\sigma = \phi(\sigma, s)sx$, so $x\sigma(s) = sx$. Thus $\sigma(s) = s$ for all s in S iff for all x in S_σ, s in $S, xs = sx$, iff $\sigma \in H$, since H is the group of gradings of the center of S .

Now we observe that, for each σ in $(G/H)_r, \sigma(s) = s^\tau$ for some τ in G . For $\sigma(s) = \phi(\sigma^{-1}, s)s^\sigma$. We know from §3 that $S = \sum_{\chi \in G^*} S_\chi$ such that if $u_\chi \in S_\chi, \sigma \in G$, then $u_\chi^\sigma = \langle \chi, \sigma \rangle u_\chi$. Now $\sigma(u_\chi) = \phi(\sigma^{-1}, \text{gr}(u_\chi)) \langle \chi, \sigma \rangle u_\chi$ (where $\text{gr}(u_\chi)$ is the grade of u_χ). The map $\sigma: G^* \rightarrow U(R)$ given by $\sigma(\chi) = \phi(\sigma^{-1}, \text{gr}(u_\chi)) \langle \chi, \sigma \rangle$ is a homomorphism. Thus $\sigma \in \text{Hom}(G^*, U(R)) =$

G , that is, there exists $\tau_\sigma \in G$ such that $\langle \chi, \tau_\sigma \rangle = \phi(\sigma^{-1}, \text{gr}(u_\chi)) \langle \chi, \sigma \rangle$ for all $\chi \in G^*$. This means that $\sigma(s) = s^{\tau_\sigma}$ for some τ_σ in G . Clearly this map $\sigma \mapsto \tau_\sigma$ is a homomorphism. Let L be the image in G of the map $\sigma \mapsto \tau_\sigma$. Then $L = (G/H)_t$, so that, with respect to the usual right action of G on S , C is the fixed ring of some subgroup L . Thus, by [4, (2.2)], S is a Galois extension of C with group $L = (G/H)_t$, and C is a Galois extension of R with group G/L , and ([4, (4.1)]) $\text{rank}_R C = [G: L] = [H: 1]$.

Now in particular, C is $K/H^1 \cap K = \text{Gal}(Z/S_1)$ -strong [4, (2.1)] since C is G -strong [4, (2.2)] and $H^1 \cap K$ leaves C fixed. (Here K is as in (4.1) or (2.8) and $H^1 = \{\sigma \in G \mid \phi(\sigma, \tau) = 1 \text{ for all } \tau \text{ in } H\}$.) So the restriction of $K/H^1 \cap K$ to C acts as a subgroup of $\text{Gal}(C/R)$ with fixed ring $C \cap S_1$. But $C \cap S_1$ is easily seen to be R . So $\text{Gal}(C/R)$ is the restriction of $K/H^1 \cap K$ to C : $(K/H^1 \cap K)/\bar{J} \cong \text{Gal}(C/R)$, the isomorphism being by restriction, where $\bar{J} \subseteq K/H^1 \cap K$ leaves C fixed. In fact, since $\text{rank}_R C = \text{rank}_{S_1} Z = [H: 1]$, $\bar{J} = \{1\}$. Now $(K/H^1 \cap K)$ acts as a group of automorphisms of CS_1 with fixed ring S_1 , and the Galois elements ([4, (1.3b)], or see §1) for C/R are at the same time Galois elements for $S_1 C/S_1$. So CS_1 is a Galois extension of S_1 with group $(K/H^1 \cap K)$. Also $C \otimes_R S_1$ is a Galois extension of S_1 with group $K/H^1 \cap K$. Thus two applications of [4, (3.4)] to the maps $C \otimes_R S_1 \rightarrow CS_1 \subseteq Z$ show that $C \otimes_R S_1 \cong Z$, proving the lemma.

We are ready for the main result of this section.

(4.3) THEOREM. *If R, G are as above, S is in $\text{Gal}_\phi(R, G)$, H is the group of gradings of the center Z of S , and either H is a direct summand of G or $\text{Pic}(R)$ is $[H: 1]$ -torsion free, then $S \in \text{Im}(\pi)$.*

PROOF. If $C = \{x \in S \mid x^\sigma = \phi(\sigma, x)x \ \forall \sigma \in G\}$, we know by (4.2) that $C \otimes_R S_1 \cong Z$, so for each σ in H , $C_\sigma \otimes_R S_1 \cong S_\sigma$. Thus C_σ is a rank one projective R -module, and from the equality $S_\sigma S_\tau = S_{\sigma\tau}$, one obtains that $(C_\sigma S_1)(C_\tau S_1) = (C_{\sigma\tau} S_1)$, i.e., $C_\sigma C_\tau \otimes_R S_1 = C_{\sigma\tau} \otimes_R S_1$; hence (since R is an R -direct summand of S_1) $C_\sigma C_\tau = C_{\sigma\tau}$.

Thus if $H = \langle \tau_1 \rangle \times \cdots \times \langle \tau_r \rangle$, a product of cyclic groups, and H has order n_H , and if $\text{Pic}(R)$ is n_H -torsion free, each C_τ is free and we may choose $u_i, i = 1 \cdots r$, so that $u_i u_j = u_j u_i, C_{\tau_i} = R u_i$ and $u_i^{n_i} = a_i \in U(R)$ when $n_i =$ the order of τ_i . That is, $C = RH_f$, a commutative twisted group ring with twisting f , a symmetric 2-cocycle representing a class in $H^2(H, U(R))$. (A map $f: G \times G \rightarrow U(R)$ is symmetric if $f(\sigma, \tau) = f(\tau, \sigma)$.)

We suppose, as we shall prove in (4.4), that there exists a symmetric 2-cocycle $g \in H^2(G, U(R))$ such that $\text{res}_{G/H} g = f$, so that $C \cong RH_f \subseteq RG_g$ and RG_g is commutative. Let $A = RG_g \otimes_C S$ with multiplication $sz_\sigma = \phi(\sigma^{-1}, s)z_\sigma s^\sigma$; for

$z_\sigma \in C$, respectively (since RG_g is commutative) for $s \in C$, this agrees with the usual multiplication of C in S , respectively in RG_g . So the multiplication on A is well defined. We must show that A is separable over R , and for this it suffices to observe that C is a Galois extension of R , hence R -separable, and to prove that A is an Azumaya C -algebra. In fact, notice that if $\sigma(s) = \phi(\sigma^{-1}, s)s^\sigma$ is the G_t action on S defined in the proof of (4.2), then $sz_\sigma = z_\sigma\sigma(s)$. This suggests that A may be viewed as a crossed product $\Delta(S, (G/H)_t)_h$ with some factor set $h: G/H \times G/H \rightarrow U(C)$; since S is a Galois extension of C with group $(G/H)_t$ (proof of (4.2)) it would follow that A is an Azumaya C -algebra split by S .

To show $A = \Delta(S, (G/H)_t)_h$ we observe that we can find an $h \in H^2(G/H, U(C))$ so that $RG_g \cong C(G/H)_h$. We do that by letting $v: G/H \rightarrow G$ be a set splitting of the canonical map: $G \rightarrow G/H$, with $v(\bar{1}) = 1$, and we define an R -module isomorphism $\psi: RG_g \rightarrow C(G/H)$ induced by linearity and $\psi(\sigma) = (v(\bar{\sigma})^{-1}\sigma)\bar{\sigma} \in C(G/H)$. This is the identity on H , hence on C , and is an isomorphism of R -modules if it is 1-1. But $\psi_\sigma(\sum_{i \in G} r_i \sigma_i) = 0$ implies $\sum_{\sigma_i \in G} r_i (v(\bar{\sigma}_i)^{-1}\sigma_i)\bar{\sigma}_i = 0$, so for each $\bar{\sigma}$ in G/H , $\sum_{r_i} r_i (v(\bar{\sigma}_i)^{-1}\sigma_i) = 0$ where σ_i runs through the coset of $\sigma \text{ mod } H$. If we set $\sigma_i = \tau_i\sigma$, $\tau_i \in H$, then this sum becomes $\sum_{h \in H} r_i (v(\bar{\sigma})^{-1}\sigma\tau_i) = 0$, i.e., $v(\bar{\sigma})^{-1}\sigma\sum r_i\tau_i = 0$. But $v(\bar{\sigma})^{-1}\sigma\tau_i$ are distinct elements of H for $\tau_i \in H$, so are linearly independent over R ; hence, $r_i = 0$. Thus ψ is an isomorphism of R -modules. Define a product on $C(G/H)$, call it $C(G/H)_h$, to coincide with that on RG_g . Then ψ yields an isomorphism of C -algebras and $C(G/H)_h$ is a commutative twisted group ring with twisting $h: G/H \times G/H \rightarrow C$. Then A is a crossed product as described. (In fact, A may be viewed as a smash product $S \#_C C[G/H]_h$ with respect to the $(G/H)_t$ -action on S and the G/H -grading on $C[G/H]_h$.)

Grade $A = \Delta(S, (G/H)_t)_h = RG_g \otimes_C S$ diagonally. To show that $S = \pi(A)$ we must show $A^{A^1} = S, A^A = A A = R$. First, $S \subseteq A^{A^1}$: for if $s_\sigma z_{\sigma^{-1}} \in A_1$,

$$ss_\sigma z_{\sigma^{-1}} = \phi(\sigma^{-1}, s)s_\sigma s^\sigma z_{\sigma^{-1}} = \phi(\sigma^{-1}, s)s_\sigma \phi(\sigma, s)z_{\sigma^{-1}}(s^\sigma)^{\sigma^{-1}} = s_\sigma z_{\sigma^{-1}}s.$$

If $z_\rho \in A^{A^1}$ then

$$\begin{aligned} z_\rho s_\sigma z_{\sigma^{-1}} &= s_\sigma z_{\sigma^{-1}} z_\rho = s_\sigma z_\rho z_{\sigma^{-1}} \quad \text{since } RG_g \text{ is commutative} \\ &= \phi(\rho^{-1}, s_\sigma)z_\rho s_\sigma^2 z_{\sigma^{-1}} \end{aligned}$$

so $s_\sigma^\rho = \phi(\rho, s_\sigma)s_\sigma$. Since s_σ may be chosen at random from $hS, \rho \in H$, and so $z_\rho \in C \subseteq S$. Thus $A^{A^1} = S$. Now $A^A \subseteq A^{A^1} = S$. So

$$\begin{aligned} A^A &\subseteq \{s \in S \mid z_\sigma s = \phi(\sigma, s)sz_\sigma \text{ for all } \sigma \text{ in } G\} \\ &= \{s \in S \mid z_\sigma s = \phi(\sigma, s)\phi(\sigma^{-1}, s)z_\sigma s^\sigma\} \\ &= \{s \in S \mid s = s^\sigma\} = R; \end{aligned}$$

so $R = A^A$. Also ${}^A A \subseteq A^{A^1} = S$, and a similar argument shows ${}^A A \subseteq R$. Thus ${}^A A = R$.

We have therefore shown that whenever $C = RH_f \subseteq RG_g$, then such an A exists.

In case $G = H \times J$, $J = G/H$ and we can replace $RG_g = C((G/H)_t)_h$ by $C(J)_1$, the usual group ring of J . In that case it is not necessary to assume any conditions on $\text{Pic}(R)$: $A = \Delta(S, J)_1$.

The proof of the theorem is complete as soon as we prove

(4.4) LEMMA. *Let G be a finite abelian group which acts trivially on the abelian group A . If H is a subgroup of G , then $\text{res}: H^2_{\text{sym}}(G, A) \rightarrow H^2_{\text{sym}}(H, A)$ is onto.*

PROOF? We shall write the operation on A as addition. We denote by $H^2_{\text{sym}}(G, A)$ the subgroup of $H^2(G, A)$ consisting of classes represented by symmetric cocycles, those cocycles $f: G \times G \rightarrow A$ satisfying $f(\sigma, \tau) = f(\tau, \sigma)$. Note that coboundaries are always symmetric. From [20, Theorem 2.1 and p. 159] we know that if $H = W_1 \times \dots \times W_r$, then

$$H^2_{\text{sym}}(H, A) \cong \bigoplus_{i=1}^r H^2_{\text{sym}}(W_i, A)$$

the isomorphism being induced by restriction. Similarly for G . Thus it suffices to prove the lemma assuming H is cyclic of order $p^f = c$ (p prime) with generator τ , and we can assume G is a p -group.

Let $G = Z_1 \times \dots \times Z_n$ where $Z_i = \langle \sigma_i \rangle$ is cyclic of order $d_i = p^{e_i}$. Let $\tau = \sigma_1^{r_1} \dots \sigma_n^{r_n}$. If τ has order p^f then $\sigma_i^{r_i}$ has order exactly $p^f = c$ for some i . Fix such an i . Replacing $\sigma_i^{r_i}$ by $\sigma_i^{u r_i}$ where u is some appropriate integer relatively prime to p , we can assume that $r_i c = d_i$. Now $A/cA \cong H^2(H, A)$: an isomorphism is given by sending the class of a to the class of $f_{a,\tau}$, where, setting $[r]$ = the greatest integer $\leq r$,

$$f_{a,\tau}(\tau^\lambda, \tau^\mu) = a[(\lambda + \mu)/c].$$

(That is, $f_{a,\tau}$ is the cup product of $a + cA$ in $\check{H}^0(H, A)$ (Tate cohomology) with the class of $\delta\chi_\tau$ in $H^2(H, \mathbb{Z})$ where $\chi_\tau: H \rightarrow \mathbb{Q}/\mathbb{Z}$ is given by $\chi_\tau(\tau^\lambda) = \lambda/n$ —see [16, p. 141].) So it suffices to show that $f_{a,\tau}: H \times H \rightarrow A$ is

the restriction of a 2-cocycle g on G . We choose such a $g: G \times G \rightarrow A$ as follows:

$$g(\sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}, \sigma_1^{\beta_1} \cdots \sigma_n^{\beta_n}) = g(\sigma_i^{\alpha_i}, \sigma_i^{\beta_i}) = a[(\alpha_i + \beta_i)/d_i].$$

Then

$$\text{res } g(\tau^\lambda, \tau^\mu) = g(\sigma_i^{r_i^\lambda}, \sigma_i^{r_i^\mu}) = a \left[\frac{r_i^\lambda + r_i^\mu}{d_i} \right] = a \left[\frac{\lambda + \mu}{c} \right] = f_{a,\tau}(\tau^\lambda, \tau^\mu).$$

So $\text{res } g = f_{a,\tau}$. Also, g is a 2-cocycle. For g is a 2-cocycle iff $\text{res}_{G \rightarrow Z_i} g$ is a cocycle in $H^2(Z_i, A)$, clearly; and $\text{res}_{G \rightarrow Z_i} g = f_{a,\sigma_i}$ can be shown to be a cocycle either directly or by identifying it as the image of a cup product, as above. That completes the proof of (4.4) and (4.3).

5. Dimodule algebras. In this section we note the relationship between $B_\phi(R, G)$ and a Brauer group $BD(R, G)$ of G -dimodule algebras defined by F. W. Long [12], [13] and apply the results of the previous sections to $BD(R, G)$.

Assume G is a finite abelian group, and $H = RG$. A G -dimodule algebra A is a G -graded algebra and a G -module algebra such that the G -action preserves the grading, that is, the following diagram [13, Definition 3.1(i)] commutes:

$$\begin{array}{ccc} H \otimes A & \xrightarrow{\mu} & A \\ \downarrow 1 \otimes \Delta & & \downarrow \Delta \\ H \otimes A \otimes H & \xrightarrow{\mu \otimes 1} & A \otimes H \end{array}$$

Let H^* be the linear dual of H . We may define an H^* -module structure on A from the H -grading on A via the action $h^*(a) = (1 \otimes h^*)\Delta(a)$.

This H^* -module structure and the H -module structure commute, since they operate on different factors of $\Delta(a)$ for a in A . Thus A is an $H^* \otimes H$ -module. If A is a G -graded algebra, then it is an H -comodule algebra, hence [13, Remark following Definition 2.9] an H^* -module algebra. Thus A is an $H^* \otimes H$ -module algebra. Thus A is an $H \otimes H^*$ -comodule algebra.

So an H -module algebra is an $H \otimes H^*$ -comodule algebra. The converse is equally clear.

Given two G -dimodule algebras A and B their product is the smash product $A \# B$, with multiplication given on homogeneous elements by

$$(a_1 \# b_1)(a_2 \# b_2) = a_1 {}^b_1 a_2 \# b_1 b_2,$$

where if b_1 is homogeneous of grade σ , ${}^b_1 a_2 = \sigma a_2$. In terms of the $(RG)^*$ -comodule structure of A ,

$$\sigma_a = \sum_{(a)} \langle \sigma, a_{(2)} \rangle a_{(1)}.$$

Suppose the order of G is a unit in R , and $\text{Hom}(G, U(R)) = G^* \cong G$, so that $(RG)^* = R(G^*)$. Then $A = \sum_{\chi \in G^*} A_\chi$ such that, for $a \in A_\chi$, $\Delta(a) = a \otimes u_\chi$, and the smash product on homogeneous elements becomes

$$(a_1 \# b_1)(a_2 \# b_2) = a_1 a_2 \langle G\text{-grade}(b_1), G^*\text{-grade}(a_2) \rangle \# b_1 b_2.$$

Let $\phi: (G \times G^*) \times (G \times G^*) \rightarrow U(R)$ be the bilinear map defined by

$$\phi((\sigma_1, \chi_1), (\sigma_2, \chi_2)) = \langle \sigma_1, \chi_2 \rangle.$$

Then $A \# B = A \otimes B$, an instance of the graded tensor product of [CGO].

It is easy to check that G -Azumaya in the sense of Long and graded Azumaya in the sense of [CGO] coincide by observing that the maps F, G of [12, Definition following Theorem 1.3] coincide with the maps η_A, μ_A of [CGO, (2.8)].

Since the Azumaya algebras which are trivial in [CGO, (2.10)] and in [12, Definition preceding Theorem 1.5] coincide, we have, summing up:

(5.1) THEOREM. *If R is a connected commutative ring, G is an abelian group of order n and exponent m , and R contains $1/n$ and a primitive m th root of unity, then Long's Brauer group $BD(R, G)$ is equal to $B_\phi(R, G \times G^*) \cong B_\phi(R, G \times G)$ for ϕ as above.*

We note that when G is a p -group whose r cyclic direct summands all have equal order, then $G \times G$ is also, and the matrix M arising in §3 corresponding to the ϕ of the theorem is nonsingular—it is $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ where I is the $r \times r$ identity matrix.

Summing up our results as applied to Long's group $BD(R, G)$ we have

(5.2) THEOREM. *If $G = \prod_p (\prod_{i=1}^r Z_{p^i} e_p)$ has order n , and R is a connected commutative ring containing $1/n$ and a primitive p^e th root of unity for each $p|n$ and $\text{Pic}(R)$ is p -torsion free for all $p|n$, then $BD(R, G)$ is described by the two exact sequences*

$$0 \rightarrow B(R) \rightarrow BD(R, G) \xrightarrow{\pi} \text{Gal}_\phi(R, G \times G) \rightarrow 0,$$

$$0 \rightarrow \text{Comm}(R, G \times G) \rightarrow \text{Gal}_\phi(R, G \times G) \xrightarrow{\beta} \prod_p O_{Z/p^e Z}^{2r} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where β is onto $O_{Z/p^e Z}^{2r} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ for p odd, and the image of β in $O_{Z/2^e Z}^{2r} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ is

$$\left\{ P \mid \left[(I - P) \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} (I - P') - (I - P) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right]_{ii} = 0 \text{ for all } i = 1, \dots, 2r_2 \right\}.$$

The assumption that $\text{Pic}(R)$ is p -torsion free may be omitted if $r_p = 1$ and p is odd, or if $e_p = 1$.

PROOF. All but the last statement follows from what we have already done.

If $e_p = 1$ then $G_p = \prod_{i=1}^p Z_p$ and every subgroup is a direct summand, so by Theorem (4.3) the assumption on $\text{Pic}(R)$ is not needed.

If $r_p = 1$ and p is odd the p part of the theorem states that for $G = Z_{p^e}$

$$0 \rightarrow \text{Comm}(R, G \times G) \rightarrow \text{Gal}_\phi(R, G \times G) \rightarrow O_{Z_{p^e}}^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow 0$$

is exact.

We show π is onto by showing that for each S in $\text{Gal}_\phi(R, G \times G)$, the gradings H of the center are a direct summand of $G \times G$. We do this by explicitly computing $O_{Z/p^e Z}^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O_{Z/p^e Z}^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so $2ab = 0 = 2cd$, $ad + bc = 1$. From $ab = 0$, we get, in Z , $a = up^r$, $b = up^s$ with $r + s \geq e$, $(uv, p) = 1$. But from $ad + bc = 1$, either $r = 0$ or $s = 0$. Similarly for c and d . So we have only two cases.

Case 1. $a = u$, $d = u^{-1}$, $c = b = 0$ in $Z/p^e Z$.

Case 2. $b = u$, $c = u^{-1}$, $a = d = 0$. We use the notation of §3.

Case 1.

$$P = \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 - u \\ 1 - 1/u & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & u - 1 \\ 1 - u & 0 \end{pmatrix} = A \begin{pmatrix} -u & 0 \\ 0 & -1 \end{pmatrix}.$$

The center is

$$\{rA \mid r \in Z_{p^e}^2, rD = 0\} = \left\{ rA \mid rA \begin{pmatrix} -u & 0 \\ 0 & -1 \end{pmatrix} = 0 \right\} = \{0\}$$

since $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ is invertible.

Case 2.

$$P = \begin{pmatrix} 0 & u \\ 1/u & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -u & 1 \\ 1 & -1/u \end{pmatrix}, \quad D = 0.$$

So the center is $\{rA \mid r \in Z_{p^e}^2\}$ = the row space of A . But the row space of A is isomorphic to Z_{p^e} , so is a direct summand of $G \times G$. That proves the case $r_p = 1, p$ odd.

(5.3) REMARK. Observe that in this last proof we obtained essentially that $O_{Z/p^e Z}^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong D_{p^e - p^{e-1}}$, the dihedral group, where $p^e - p^{e-1}$ is the number of units in $Z/p^e Z$ (cf. [12, Theorem 2.7]). From (5.2) it follows that $BD(C, Z/p^e Z) \cong D_{p^e - p^{e-1}}$. The complexity of $O_{Z/p^e Z}^{2r_p}$, and hence of $BD(C, (Z/p^e Z)^r)$ is much greater if $r_p > 1$. For example, for $p = 3, n = 2, e = 2$, there is a short exact sequence

$$0 \rightarrow (Z/3Z)^6 \rightarrow O_{Z/9Z}^4 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \rightarrow O_{Z/3Z}^4 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \rightarrow 0.$$

$O_{Z/3Z}^4 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ has a subquotient isomorphic to $PSL_2(Z/3Z) \times PSL_2(Z/3Z)$ [1] and the whole group has order $3^8 \times 2^7$. When the target of π is so complicated a direct approach to showing that π is onto (as used in the proof of the last statement of (5.2)) has proved unfruitful.

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