

EXOTIC SINGULAR STRUCTURES ON SPHERES

BY

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ABSTRACT. It is shown how the category of *PL*-manifolds may be obtained from the smooth category by an iterative procedure, viz., first form singular smooth manifolds where smooth seven-spheres are allowed as links. Then, in the new category one has obtained, kill all eight-spheres in similar fashion. Repeating this process ad infinitum (but requiring only finitely many stages in each dimension), one obtains the category of *PL*-manifolds. By taking care that the set of "singular" points is always given enough structure, it is seen that this iterative process corresponds to a skeletal filtration of *BPL* mod *BO*. Also, a geometric interpretation of the Hurewicz map $\pi_*(BPL, BO) \rightarrow H_*(BPL, BO)$ is inferred.

1. **Introduction.** The object of this paper is to re-examine the differences between smooth and *PL*-manifolds via the geometry of *PL*-manifolds whose failure to be smooth may be measured, in some sense, by singularities. The germinal idea is this: From the ideas of Sullivan [a] and Baas [b] we know how to talk about a manifold with singularities; that is, a space which is of the form

$$M^n = M_0^n \cup_{\partial M_0} S$$

where M_0^n is an n -manifold with boundary, S is of the form $P^{n-r} \times cQ^{r-1}$, with P^{n-r} , Q^{r-1} manifolds, $\partial M_0^n = P^{n-r} \times Q^{r-1}$ and c denotes unreduced cone.

Now if M_0^n , P^{n-r} are smooth manifolds and $Q^{r-1} = \Sigma^{r-1}$ is a smoothness structure on the *PL*($r-1$) sphere, we see that M^n is a *PL*-manifold where lack of smoothness resides in the "singular" structure in a neighborhood of P^{n-r} . We can iterate this process, i. e., we can find all *PL*-manifolds constructed in this way, using a certain set of exotic spheres as "allowable" singularities, and then go looking for exotic structures on spheres in this "new" category. We then use some of these spheres as singularities to create yet another category of singular manifolds (and so on), noting the important fact that at every stage the singular manifolds which we introduce retain underlying *PL*-manifold structures.

The natural question, of course, is: does this process terminate? That is, do

Received by the editors March 22, 1974.

AMS (MOS) subject classifications (1970). Primary 57C25; Secondary 57C20, 57D60.

Key words and phrases. Manifold-like category, singular manifold, exotic structures on spheres, thickenings.

⁽¹⁾ The author was partially supported by NSF grant GP-43827.

we ever reach the PL -manifold category in this way, at least in the limit?

The answer is “yes” and, moreover, the process of proving this is closely connected with the construction of a certain relative cell structure from BO to BPL . In addition, we obtain a nice interpretation of the Hurewicz homomorphism

$$h : \pi_n(BPL, BO) \rightarrow H_n(BPL, BO).$$

In particular, $H_n(BPL, BO)$ is viewed as a group of equivalence classes of certain kinds of singular structures on S^{n-1} , where equivalence is given by a certain kind of concordance and the group operation is connected sum. The Hurewicz homomorphism is then seen to be the natural forgetful map from the group of smooth structures on $S^{n-1} = \pi_{n-1}(PL/O) = \pi_n(BPL, BO)$.

A final word about what the category theorists will quite properly regard as an abuse of language. We use terms like “category of smooth manifolds” and “category of PL -manifolds” to denote the entities which topologists have in mind when they make informal use of these terms. We also make a more formal definition (see below) of something which we call a “manifold-like category.” The word “category” in this context should by no means be equated with the object-morphism gadget of formal category-theory.

2. Manifold-like categories. In this section, we shall formalize, to a certain extent, the notion of an “additional structure” on a PL -manifold. In particular, we have in mind a formalism which is a rubric, of sorts, for the naive idea of additional geometric structure, as opposed to the conventional notion that a structure shall be defined as a certain reduction of the PL -tangent bundle. In fact, one of the purposes of this formalism is to set up conditions for geometric structures which enable existence and uniqueness questions to be transformed into bundle reduction or lifting problems.

We shall define the notion of a *manifold-like category* C . C is, first of all, a set whose elements are called C -manifolds. Intuitively, a C -manifold is to be thought of as a PL -manifold together with an additional structure. A good example to keep in mind (the paradigm, in fact, for this construction) is the category of smooth structures on PL -manifolds.

We axiomatize the properties of a manifold-like category as follows:

A(1). For each C -manifold M^n , there is a unique PL -manifold M_{PL}^n . The *dimension* of M^n is n . There is a dimension-lowering boundary operation ∂ so that $(\partial M^n)_{PL} = \partial(M_{PL}^n)$. There is a dimension-preserving involution ι on C -manifolds which commutes with ∂ . Moreover $(\iota M^n)_{PL} = M_{PL}^n$.

A(2). If P^n is a smoothing of a PL -manifold Q^n then P^n determines a unique C -manifold $C(P^n)$ with $(C(P^n))_{PL} = Q^n$. Moreover, $C(\partial P^n) = \partial C(P^n)$ and $\iota C(P^n) = C(P^n)$. Moreover there is a unique C -manifold ϕ^n with $\phi_{PL}^n = \emptyset$.

A-(3). (a) If M^n, N^n are C -manifolds, then there is an operation of disjoint union in C , denoted \amalg with $(M^n \amalg N^n)_{PL} = M^n_{PL} \amalg N^n_{PL}$.

(b) If M^n, N^n are C -manifolds with $\partial M^n = \iota \partial N^n$, there is an operation of union along boundary in C , denoted \cup_{∂} such that $(M^n \cup_{\partial} N^n)_{PL} = M^n_{PL} \cup_{\partial} N^n_{PL}$.

(c) Let M^n, N^n be C -manifolds with

$$\partial M^n = Q^{n-1} \cup_{\partial} P^{n-1}, \quad \partial N^n = \iota Q^{n-1} \cup_{\partial} R^{n-1}.$$

Then we may form $M^n \cup_{\partial} N^n$ with

$$(M^n \cup_{\partial} N^n)_{PL} = M^n_{PL} \cup_{Q^{n-1}_{PL}} N^n_{PL}.$$

All the above union operations commute with ι .

A-(4). There is an operation $I \times$ such that $(I \times M^n)_{PL} = I \times M^n_{PL}$. (Here, the I on the right-hand side is the standard unit interval and \times denotes usual cartesian product.) If P^n is a smoothing of a PL -manifold $I \times C(P^n) = C(I \times P^n)$. $\partial(I \times M^n) = (M^n \amalg \iota M^n) \cup_{\partial} (I \times \partial M^n)$. Finally $\iota(I \times M^n) = I \times \iota M^n$. We define $D^m \times$ as the operation $(I \times)^m$.

We pause here for a definition.

2.1. DEFINITION. Two C -manifolds M^n, N^n are said to be concordant iff there is a C -manifold W^{n+1} with $\partial W^{n+1} = (M^n \amalg \iota N^n) \cup_{\partial} V^n$ such that

$$W^{n+1}_{PL} \cong I \times M^n_{PL} \cong I \times N^n_{PL}$$

and

$$V^n_{PL} \cong I \times \partial M^n_{PL} \cong I \times \partial N^n_{PL}.$$

A-(5) (Regular Neighborhood Property). Let M^n be a C -manifold and let K be a subcomplex of the interior of the PL -manifold M^n_{PL} . Then M^n is concordant to N^n such that $N^n = P^n \cup_{\partial} Q^n$ where P^n_{PL} is a PL -regular neighborhood of K . (The union of P^n and Q^n is along ∂P^n .) Moreover, if $\bar{N}^n = \bar{P}^n \cup_{\partial} \bar{Q}^n$ is another C -manifold concordant to M^n with \bar{P}^n_{PL} a regular neighborhood of K , then there is a concordance W^{n+1} between N^n and \bar{N}^n , which is the union of concordances A^{n+1} between P^n and \bar{P}^n and B^{n+1} between Q^n and \bar{Q}^n (the union being taken along C^n , a concordance between ∂P^n and $\partial \bar{P}^n$), and subject to the condition $I \times K \subseteq A^{n+1}_{PL}$. Furthermore, a relative version of the above holds if K, L is a simplicial pair contained in $M^n_{PL}, \partial M^n_{PL}$.

A-(6) (Cerf Property). If M^n is a C -manifold, M^n is concordant to $M^n_0 \cup C(D^n)$ where D^n is the standard smooth n -disc and the union is along $C(S^{n-1})$. Moreover, this decomposition is unique up to concordance. That is, if D^n_1, D^n_2 are two copies of the smooth disc and $M^n_1 = P^n_1 \cup_{\partial} C(D^n_1)$ is concordant to $M^n_2 = P^n_2 \cup_{\partial} C(D^n_2)$, then there is a concordance W^{n+1} between M^n_1 and M^n_2 such that $W^{n+1} = Q^{n+1} \cup C(E^{n+1})$ where E^{n+1} is a smooth concordance between D^n_1 and D^n_2 .

REMARK. It follows from A-(6) that it will be possible to define connected sum for C -manifolds.

2.2. DEFINITION. A C -structure on a PL -manifold X^n is a C -manifold M^n together with a PL homeomorphism $h : M_{PL}^n \rightarrow X^n$. If $(M^n, h), (N^n, g)$ are two C -structures on A^n they are said to be concordant iff there is a C -structure (W^{n+1}, F) on $I \times X^n$ with $\partial W^{n+1} = (M^n \amalg \iota N^n) \cup V^n$ where V^n is a concordance between ∂M^n and ∂N^n where h, g are given respectively by

$$M_{PL}^n \subseteq W_{PL}^{n+1} \xrightarrow{F} I \times X \xrightarrow{\text{proj}} X^n,$$

$$N_{PL}^n \subseteq W_{PL}^{n+1} \xrightarrow{F} I \times X \xrightarrow{\text{proj}} X^n.$$

Concordance is obviously an equivalence relation on C -structures. Let $S_C(X^n)$ be the set of concordance classes of C -structures on X^n .

REMARK. There is an obvious map $S_C(X^n) \rightarrow S_C(I \times X^n)$. Furthermore, $S_C(S^n)$ becomes an abelian group by virtue of Axiom (6). We call this group θ_C^n . There is a forgetful homomorphism $\theta^n \rightarrow \theta_C^n$ where θ^n is the usual group of smoothings of S^n . Let $[(\Sigma^n, h)]$ be an element of $S_C(S^n)$ where $h : \Sigma_{PL}^n \rightarrow S^n$. We describe $- [(\Sigma^n, h)]$ as follows: By Axiom (4), $\partial(I \times \Sigma^n) = \Sigma^n \amalg \iota \Sigma^n$. So there is an obvious homeomorphism $\iota \Sigma_{PL}^n \subseteq (I \times \Sigma^n)_{PL} \xrightarrow{\text{proj}} \Sigma_{PL}^n \xrightarrow{h} S^n$. Let g be the composition of this homeomorphism with the homeomorphism $S^n \rightarrow S^n$ given by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix}$ in R^{n+1} . The formula is $- [(\Sigma^n, h)] = [(\iota \Sigma^n, g)]$.

There are two important potential properties of a manifold-like category which we wish to examine with some care.

Property (I). Let M^{n+1} be a C -manifold satisfying

- (i) $M_{PL}^{n+1} = I \times X^n$.
- (ii) $\partial M^{n+1} = (I \times P^{n-1}) \cup_{\partial} Q^n$ where P^{n-1} is a C -manifold having P_{PL}^{n-1} a codimension 0 submanifold of $\{0\} \times \partial X^n$.

Then there is a C -manifold W^{n+2} satisfying

- (a) $W_{PL}^{n+2} = I \times M_{PL}^{n+1}$.
- (b) $\partial W^{n+2} = M^{n+1} \cup_{\partial} A^{n+1} \cup_{\partial} (I \times V^n)$ where A^{n+1}, V^n are C -manifolds satisfying:

$$A_{PL}^{n+1} = I \times \partial M_{PL}^{n+1}, \quad V_{PL}^n = \{(0, 1)\} \times X^n,$$

and where $A^{n+1} = B^{n+1} \cup_{\partial} C^{n+1}$ with $B^{n+1} = I \times I \times P^{n-1}$.

What Property (I) asserts is, briefly, this: If the PL -manifold $X^n \times I$ should happen to have a C -structure, then this C -structure is deformable to one coming from a C -structure on X^n via the operation $I \times$; moreover, this deformation can be kept constant along a codimension 0 piece of $\partial(X^n \times I)$ corresponding to a codimension 0 piece of ∂X^n which has already been given a C -structure P^{n-1} ,

the assumption being that $I \times P^{n-1}$ fits into the given C -structure on $I \times X^n$.

Property (I) is modeled on an important property of smoothings of PL -manifolds [c], [d].

Property (II). Let Σ^n be a C -manifold with $\Sigma_{PL}^n \cong S^n$. Then there is a C -manifold Δ^{n+1} with $\partial\Delta^{n+1} = \Sigma^n$ and $\Delta_{PL}^{n+1} \cong D^{n+1}$.

Property (II) says, in effect, that there are no "exotic" spheres in the manifold-like category C .

2.3. LEMMA. *If the manifold-like category C satisfies Properties (I) and (II), then C is the PL -manifold category. That is, given a PL -manifold X^n , $S_C(X^n)$ contains one and only one element.*

PROOF. The proof proceeds via handlebody theory [e]. We shall prove in detail only existence, i. e., that $S_C(X^n)$ is nonvoid. Uniqueness follows from an easy relativization of this proof.

Suppose, then, the existence assertion is true for PL n -manifolds admitting a handlebody decomposition where the highest dimensional handle is of dimension $k - 1 < n$. (Clearly, for $k - 1 = 0$, this holds, thereby starting the induction.) Let X^n be a PL -manifold with a PL -handle decomposition having handles of dimension no higher than k . For the sake of convenience we assume that X^n has but one k -handle (the argument for more such handles involves mere repetition). By the inductive hypothesis, we assume that $X^n = X_0^n \cup (k\text{-handle})$ where $X_0^n = M_{PL}^n$ for some C -manifold M^n . Let $S^{k-1} \subseteq \partial X_0^n$ be the core sphere of the k -handle. By the Regular Neighborhood Axiom (5), we may as well assume that $\partial M^n = P^{n-1} \cup_0 R^{n-1}$ where R_{PL}^{n-1} is a regular neighborhood of S^{k-1} (which has a certain product PL -structure $S^{k-1} \times D^{n-k}$). Thus R^{n-1} determines a C -structure on $S^{k-1} \times D^{n-k}$. By repeated application of Property (I), we may as well assume $R^{n-1} = D^{n-k} \times \Sigma^{k-1}$ for some C -manifold Σ^{k-1} with $\Sigma_{PL}^{k-1} \cong S^{k-1}$ and that the product structure $R_{PL}^{n-1} = S^{k-1} \times D^{n-k}$ thereby obtained is isotopic to the original one. Now, by Property (II), $\Sigma^{k-1} = \partial\Delta^k$ where $\Delta_{PL}^k \cong D^k$. Thus we may form $N^n = M^n \cup_{\partial} (D^{n-k} \times \iota\Delta^k)$ where the union is along $D^{n-k} \times \Sigma^{k-1} = R^{n-1}$. Obviously $N_{PL}^n = X_0^n \cup (k\text{-handle}) = X^n$.

This proves that X^n always admits a C -structure. That such a structure is always unique up to concordance will follow from an easy relative version of this argument, the details of which are left to the reader.

3. Killing exotic spheres. We shall now construct a sequence of manifold-like categories which begins with the smooth category and converges to the category of PL -manifolds. Each of the intermediate categories will satisfy the Axioms (1)–(6) and Lemma 2.3 will be used to demonstrate that their limit is, indeed, the category of PL -manifolds.

We begin, as we have noted, with the smooth manifolds. Consider the first dimension (7, as is well known [f]) in which there is nonstandard smooth structure on the standard PL -sphere. We construct a manifold-like category C_7 as follows. First, pick one smooth structure on S^7 from each concordance class.

3.1. DEFINITION. An n -dimensional C_7 -manifold will consist of a smooth, compact manifold-with-boundary M_0^n , a finite set of compact smooth framed manifolds V_i^{n-8} and a set of diffeomorphisms $h_i : U_i^{n-1} \rightarrow V_i^{n-8} \times \Sigma_i^7$ where the U_i are disjoint, codimension 0 submanifolds of ∂M_0^n and Σ_i^7 is one of the permissible smooth structures on S^7 .

If M^n is given by M_0^n, V_i, h_i then M_{PL}^n arises from the obvious construction, viz.

$$M_{PL}^n = M_0^n \cup_{h_i} \bigcup (V_i \times c\Sigma_i^7)$$

where c denotes cones. The operations $\partial, I \times$ have the obvious meanings. We define the involution ι as follows: If V_i^{n-8} is one of the framed manifolds specified in the definition, we alter V_i^{n-8} to W_i^{n-8} where W_i^{n-8} has the same underlying smooth manifold and is framed by changing the framing of the normal bundle of V_i^{n-8} by the constant matrix $\begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix}$.

We assert that C_7 is a manifold-like category, i. e., it satisfies Axioms (1)–(6) of §2. The verification of most of these points is obvious and will therefore be left to the reader. The only marginally nontrivial axiom to establish for C_7 is the Regular Neighborhood Axiom (5). Let K be a subcomplex of M_{PL}^n for some C_7 -manifold M^n . If we allow K to be moved by an ambient isotopy, it is then easy to see that K may be assumed to be in “general position” with respect to “singularities” V_i . That is,

$$K = L \cup_i \left(\bigcup M_i \times c\Sigma_i^7 \right)$$

where $M_i = K \cap V_i$ and $L = K \cap M_0^n$ with

$$L \cap \left(\bigcup_i V_i \times \Sigma_i^7 \right) = \bigcup_i M_i \times c\Sigma_i^7.$$

Let $R = \bigcup_i R_i$ be a smooth regular neighborhood of $M = \bigcup_i M_i$ in $V = \bigcup_i V_i$. Then $\bigcup_i (R_i \times c\Sigma_i^7)$ is a regular neighborhood of $\bigcup_i (M_i \times c\Sigma_i^7)$ in $\bigcup_i (V_i \times c\Sigma_i^7)$, and $\bigcup_i (R_i \times \Sigma_i^7) = T_1$ is a smooth regular neighborhood of $\bigcup_i (M_i \times \Sigma_i^7)$. Extend T_1 to a smooth regular neighborhood T of L in M_0^n . Then $T \cup \bigcup_i (R_i \times c\Sigma_i^7)$ is a PL -regular neighborhood of K which acquires an obvious C_7 -manifold structure. This proves the existence part of Axiom (5). The uniqueness follows from a relative version of the same argument.

REMARK. It should have been apparent to the reader that we could well have defined a C_7 -manifold differently and, at first glance, more simply, by

merely specifying that the “singularities” V_i^{n-8} of a C_7 -manifold M^n be ordinary smooth manifolds, rather than smooth manifolds with the seemingly superfluous additional structure of a framed normal bundle. In that case, the involution ι would have been trivial, i.e., the identity. Every result that we prove in this section for C_7 , as defined above, could be proved with equal ease for C_7 with this simpler definition. However, the reasons for using the definition with the extra complication of framings on the singularity will emerge in §4.

The most important characteristic of C_7 , beyond satisfying the axioms for a manifold-like category, is given in the following:

3.2. LEMMA. C_7 satisfies Property (I).

PROOF. We shall prove this in the absolute case, i.e., we consider a C_7 -manifold W^{n+1} where $W^{n+1} = I \times X^n$ without trying to keep track of some C_7 -structure on a piece of ∂X^n . The reader should easily ascertain that introducing this complication, while lengthening the details of the proof, will in no way introduce any new essential difficulty.

Our problem, then, is to put a C_7 -structure on $I \times W_{PL}^{n+1}$ which is the given one on $\{0\} \times W_{PL}^{n+1}$ and which is of the form $I \times M^n$ on $\{1\} \times W_{PL}^{n+1}$ where $M_{PL}^n = (1, 0) \times X^n$.

First specify that $W_{PL}^{n+1} = W_0^{n+1} \cup \bigcup_i (V_i^{n-7} \times c\Sigma_i^7)$ where W_0^{n+1} has a certain smoothing and V_i^{n-8} is a smooth framed manifold. We identify X^n with $\{0\} \times X^n \subseteq I \times X^n = W_{PL}^{n+1}$ and assume that X^n is in general position with respect to V_i^{n-7} . (If not, deform the C_7 -structure on W^{n+1} by a general position argument.) This means, in particular, that if $X^n \cap V_i = Q_i^{n-8}$ then $X^n \cap (V_i \times c\Sigma_i^7) = Q_i^{n-8} \times c\Sigma_i^7$. Moreover, Q_i^{n-8} has a collar in V_i^{n-7} as does $Q_i^{n-8} \times c\Sigma_i^7$ in $V_i^{n-7} \times c\Sigma_i^7$ and $Q_i^{n-8} \times \Sigma_i^7$ in $V_i^{n-7} \times \Sigma_i^7 \subseteq \partial W_0^{n+1}$. Calling upon smoothing theory, we may insure that Q_i^{n-8} is a smooth submanifold of V_i^{n-7} (which inherits a framing by preceding the framing inherited from V_i^{n-7} by the inward-pointing normal). This gives rise to a smoothness structure on $Q_i^{n-8} \times \Sigma_i^7$ which extends in turn to the smoothness structure on W_0^{n+1} .

So now, the situation is that Q_i^{n-8} is a smooth submanifold of ∂V_i^{n-7} , framed in its own right, and $Q_i^{n-8} \times \Sigma_i^7$ is a smooth submanifold of ∂W_0^{n+1} (with trivial normal bundle). But $X_0^n = X^n \cap W_0^{n+1}$ has a collar. Therefore, we may adjust the smoothness structure of W_0^{n+1} , leaving it untouched on $\bigcup_i V_i \times \Sigma_i^7$, so that X^n becomes a smooth submanifold of ∂W_0^{n+1} . We are almost done since there is now an obvious C_7 -structure M^n on X^n , coming from X_0^n , Q_i^{n-8} and the obvious gluing of $Q_i^{n-8} \times \Sigma_i^7$ along part of ∂X_0^n . It is now an easy matter to perform a deformation so that the C_7 -structure on $W_{PL}^{n+1} = I \times X^n$ is the product structure $I \times X^n$. This is because of the fact that we may first deform the interior of W_0^{n+1} so that we may

think of the C_7 -structure on W_{PL}^{n+1} as of the form $\bar{W}^{n+1} \cup_{\partial} Z^{n+1}$ (or $\bar{W}^{n+1} \cup_{\partial} Z^{n+1}$ if $\partial X^n = 0$) where \bar{W}_{PL}^{n+1} is a small collar neighborhood of X^n in W_{PL}^{n+1} , Z_{PL}^{n+1} is its complement and \bar{W}^{n+1} is, essentially, $I \times M^n$. We then deform the C_7 -structure on W_{PL}^{n+1} so that \bar{W}^{n+1} "expands" to fill all of W_{PL}^{n+1} . Thus Property (I) is verified.

Having constructed C_7 , we may proceed to an inductive definition of C_8, C_9, \dots, C_n .

First, we pause here to note that if Y^j is a smooth framed manifold and M^n is a C_7 -manifold then we may denote by $Y^j \times M^n$ the obvious C^7 -structure on $Y^j \times M_{PL}^n$.

Now assume C_{k-1} to have been defined so that for any C_{k-1} -manifold N^n and any framed smooth manifold Y^j , $Y^j \times M^n$ is defined as a C_k ($n + j$)-manifold with $(Y^j \times M^n)_{PL} = Y^j \times N_{PL}^n$. Pick one C_{k-1} -structure Σ^k on S^k from each equivalence class.

3.3. DEFINITION. An n -dimensional C_k -manifold is a triple consisting of

(a) A C_{k-1} n -manifold M_0^n where $\partial M_0^n = R^{n-1} \cup_{\partial} (\bigcup_i U_i^{n-1})$ with the $(U_i^{n-1})_{PL}$ mutually disjoint.

(b) A set of compact smooth framed manifolds V_i^{n-k-1} .

(c) A set of C_{k-1} concordances H_i between U_i and $V_i^{n-k-1} \times \Sigma_i^k$ where Σ_i^k is one of the permitted structures on S^k .

M_{PL}^n is, again, the obvious manifold, viz.

$$M_{PL}^n = (M_0^n)_{PL} \cup \bigcup_i H_i \cup \bigcup_i (V_i^{n-k-1} \times c(\Sigma_i^k))_{PL}.$$

We claim that C_k satisfies Axioms (1)–(6) of §2. As before, only Axiom (5) requires any care whatever, and the verification mimics exactly that for the category C_7 . The reader will also easily see that if Y^j is a framed smooth manifold and M^n a C_k -manifold, then there is an obvious way to define $Y^j \times M^n$. Moreover, the proof that C_7 satisfies Property (I) may be carried over bodily with very minor changes to a proof for C_k . Thus we state:

3.4. COROLLARY. C_k satisfies Property (I).

We next define the manifold-like category $C_{\infty} = \varinjlim C_k$ as the union of all the manifold-like categories $C_7 \subseteq C_8 \subseteq \dots \subseteq C_n$. (We may write inclusion signs inasmuch as any C_p -manifold obviously acquires a C_q -structure for $q > p$.) C_{∞} inherits all the important properties we have discussed thus far; it satisfies Axioms (1)–(6) and is thus a manifold-like category. Moreover, it satisfies Property (I).

We are thus led immediately to

3.5. THEOREM. C_{∞} is the PL-manifold category, that is, for each PL-manifold X^n , $S_{C_{\infty}}(X^n)$ has precisely one element.

PROOF. The relevant fact is, of course, Lemma 2.3. We have established that C_∞ satisfies Property (I); this leaves only Property (II) to be verified and the proof will be complete.

Suppose, then, that Σ^j is a C_∞ -manifold with $\Sigma_{PL}^j \cong S^j$. Then, in particular, Σ^j is a C_k -manifold for some $k < j$. But consider the cone on Σ_{PL}^j . This becomes a C_j -manifold in an obvious way. That is, we form the C_j -manifold Δ^{j+1} where $\Delta_0^{j+1} = I \times \Sigma^j$, $U^j = \iota \Sigma^j$ (i. e., $U_{PL}^j = \Sigma_{PL}^j \times \{1\}$), V is a point with the standard framing, and H is also $I \times \Sigma^j$, so that $\Delta_{PL}^{j+1} = I \times \Sigma_{PL}^j \cup I \times \Sigma_{PL}^j \cup c\Sigma_{PL}^j$. (Here the two cylinders $I \times \Sigma_{PL}^j$ are laid end-to-end and the free end of the second becomes the base of the cone $c\Sigma_{PL}^j$.) Obviously $\Delta_{PL}^{j+1} \cong cS^j = D^{j+1}$. Thus Property (II) is verified and the proof of the theorem is complete.

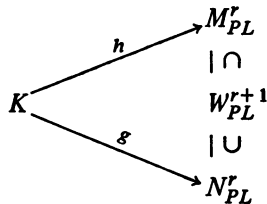
REMARK. As is the case with C_7 , we could define C_8, C_9, \dots , by merely requiring the "singularities" V_i^{n-k-1} of a C_k -manifold M^n to have a smooth structure rather than a smooth, framed structure. Theorem 3.5 would still go through in this case.

4. Classification of C_k -structures. In this section, we construct a sequence of spaces BC_k starting with $BO = BC_1 = BC_2 = \dots = BC_6$ and with limit $BPL = BC_\infty$. BC_k will play a role in characterizing $S_{C_k}(X^n)$ for PL -manifolds X^n analogous to the role of BO in classical smoothing theory.

We construct BC_k by means of Brown's representation theorem for contravariant homotopy functors $[g], [h]$. It should be pointed out that, inasmuch as we are primarily concerned with compact manifolds, we shall ignore distinctions in classifying spaces which cannot be determined by finite complexes. That is, we shall neglect the questions of "phantom" maps and the \lim^1 problem.

Let K be a finite CW -complex.

4.1. DEFINITION. A (C_k, r) -thickening of K consists of a compact C_k -manifold M^r together with a simple homotopy equivalence $h: K \rightarrow M_{PL}^r$. If $(M^r, f), (N^r, g)$ are two (C_k, r) -thickenings of K , we shall call them equivalent iff there is a C_k -concordance W^{r+1} between M^r and N^r so that the diagram



is homotopy commutative.

Denote the set of equivalence classes of (C_k, r) -thickenings of K by $TC_k^r(K)$. We take note that there is an obvious operation $I \times: TC_k^r(K) \rightarrow TC_k^{r+1}(K)$ defined

by $I \times: (M^r, f) \rightarrow (I \times M^r, (\frac{1}{2}) \times f)$. In addition, there is a forgetful map $TC_k^r(K) \rightarrow TC_{k+1}^r(K)$. We let $TC_k(K)$ be the direct limit of the sequence

$$\dots \xrightarrow{I \times} TC_k^r(K) \xrightarrow{I \times} TC_{k+1}^r(K) \xrightarrow{I \times} \dots$$

That is, $TC_k(K)$ is the set of stable equivalence classes of C_k -thickenings of K .

4.1. LEMMA. TC_k is the restriction to finite CW-complexes of a representable homotopy functor to the category of sets.

PROOF. We first note that TC_k is, in fact, a contravariant homotopy functor from finite CW-complexes to sets. For let an element in $TC_k(K)$ be represented by $g: K \xrightarrow{\cong} M_{PL}^r$ where M^r is a C_k -manifold, and let $f: L \rightarrow K$ be some map where L is a finite CW-complex. We may assume without loss of generality that r is very large compared with $\dim L$. Therefore the composition $g \circ f$ may be factored as

$$L \xrightarrow{h} X^r \subseteq M_{PL}^r$$

where X^r is a codimension 0 submanifold of M_{PL}^r , h is a simple homotopy equivalent. The Regular Neighborhood Axiom A-(5) allows us to put on X^r a unique (up to concordance) C_r -structure, designated N^r .

We define $f^*[(M^r, g)] = [(N^r, h)]$ where $[\]$ denotes equivalence class in TC_k . One may easily verify that f^* is well defined and depends only on the homotopy class of f .

Without any difficulty, we may think of the domain of TC_k to be pointed finite CW-complexes, and we proceed to verify the Wedge Axiom and the Mayer-Vietoris Axiom (cf. [g], [h], [i]) for TC_k .

The Wedge Axiom states that if $K = \bigvee_{i=1}^j K_i$ then the natural map

$$TC_k(K) \rightarrow \prod_{i=1}^j TC_k(K_i)$$

is an isomorphism. So let $u \in \prod_i TC_k(K_i)$, with the i th coordinate given by $[(M_i^r, g_i)]$. (We assume that r is the same for each i and very much larger than $\dim K = \max \dim K_i$.) By the Cerf Axiom A-(6), we may assume that $\partial M_i^r = C_k(D_i^{r-1}) \cup_{\partial} P_i$ where D_i^{r-1} is a copy of the standard smooth $(r-1)$ -disc. Pick j disjoint $(r-1)$ -discs E_i^{r-1} in the boundary of the standard r -disc D_i^r . We may identify E_i^{r-1} with D_i^{r-1} and thus $C(D_i^{r-1}) = C(E_i^{r-1}) = \iota C(E_i^{r-1})$, since ι is trivial on C_k -manifolds arising from smooth manifolds. We may then form $(\prod_i M_i^r) \cup_{\partial} C(D^r) = M^r$. Obviously $M_{PL}^r = \#(M_i^r)_{PL}$ where $\#$ denotes connected sum along $\bar{\partial}$ boundary. Thus $M_{PL}^r \simeq \bigvee (M_i^r)_{PL} \simeq \bigvee K_i = K$. If we denote this homotopy equivalence by $g: K \rightarrow M_{PL}^r$, and the inclusion $K_i \subseteq \bigvee K_i$ by α_n then clearly $\alpha_n^*[(M^r, g)] = [(M_i^r, g_n)]$. This proves that $TC_k(K) \rightarrow \prod_i TC_k(K_i)$ is onto.

To show that the correspondence is one-to-one, observe first, that if M^r is concordant in C_k to \bar{M}^r and N^r is concordant to \bar{N}^r then $M^r \# N^r$ is concordant to $\bar{M}^r \# \bar{N}^r$. Here, we are recalling the Cerf Axiom (6) which both allows us to define $\#$ in C_k and suffices to verify the assertion above. Now suppose that (M^r, g) is a (C_k, r) -thickening of K where r is very large compared to $\dim K$. The reader may easily check that no generality is lost by assuming that $M^r = \# M_i^r$ where g is the wedge of $g_i: K_i \xrightarrow{\cong} (M_i^r)_{PL}$. Suppose also that (N^s, h) is a (C_k, s) -thickening of K with $(N^s, h) = \# (N_i^s, h_i)$ and $[(N_i^s, h_i)] = [(M_i^r, g_i)]$ for all i . We may assume that $s = r$ and that (N_i^r, h_i) is actually equivalent to (M_i^r, g_i) as a (C_k, r) -thickening. The first remark of this paragraph concerning connected sums shows that (M^r, g) is equivalent to (N^r, h) as a (C_k, r) -thickening, hence as a stable C_k -thickening. Thus, any element in $TC_k(\bigvee_i K_i)$ is determined by its image in $\Pi_i TC_k(K_i)$. Thus the Wedge Axiom holds.

As for the Mayer-Vietoris Axiom, this states that if, with relation to the diagram

$$\begin{array}{ccc} TC_k(K \cup L) & \xrightarrow{a} & TC_k(K) \\ \downarrow b & & \downarrow c \\ TC_k(L) & \xrightarrow{d} & TC_k(K \cap L) \end{array}$$

there are elements $u \in TC_k(K)$, $v \in TC_k(L)$ with $cu = dv$, then there is an element $w \in TC_k(K \cup L)$ such that $aw = u$, $bw = v$.

To prove this, let (M^r, g) represent u and (N^r, h) represent v . We may assume that g, h embed $K \cap L$ in $\partial M_{PL}^r, \partial N_{PL}^r$ respectively. By Axiom A-(5), the regular neighborhood X^{r-1} of $K \cap L$ in ∂M_{PL}^r acquires a C_k -structure Q^{r-1} . Similarly, the regular neighborhood Y^{r-1} of $K \cap L$ in ∂N_{PL}^r acquires a C_k -structure R^{r-1} . Without loss of generality, we may assume that the $(C_k, r-1)$ thickenings $(Q^{r-1}, g|(K \cap L)), (R^{r-1}, h|(K \cap L))$ are $(C_k, r-1)$ equivalent via a concordance W^r . Form the C_k -manifold $P^r = M^r \cup_{\partial} W^r \cup_{\partial} N^r$, the union attaching Q^{r-1} and R^{r-1} to either end of W^r . There is an obvious homotopy equivalence $f: K \cup L \rightarrow P_{PL}^r$ and clearly, if $w = [(P^r, f)]$ then

$$aw = [(M^r, g)] = u, \quad bw = [(N^r, h)] = v.$$

We also need the following countability condition:

4.2. PROPOSITION. $TC_k(S^n)$ is countable.

This will follow from

4.3. LEMMA. Let X^n be a PL-manifold and let Q^{n-1} be a C_k -manifold with Q_{PL}^{n-1} a codimension 0 submanifold of ∂X^n . Then, up to concordance, there are only countably many C_k -manifolds M^n with $M_{PL}^n = X^n$ and $\partial M^n = P^{n-1} \cup_{\partial} Q^{n-1}$.

PROOF OF LEMMA. The proof proceeds by induction on k , exploiting the fact that classical smoothing theory tells us the result is true for $C_k = \text{smooth manifolds}$, i. e., $k \leq 6$. Let us consider, therefore, a PL -manifold X^n such that the C_k -manifold Q^{n-1} has Q_{PL}^{n-1} a codimension 0 submanifold of ∂X^n . We must show that there are at most countably many ways of extending this C_k -structure to all of X^n . First of all, note that the specification of a C_k -structure on a PL -manifold Y^n involves the selection of a codimension $(k + 1)$ framed submanifold. That is, the smooth manifold which is the "singularity", away from which the C_k -manifold is C_{k-1} , is, in an obvious way, a framed submanifold of the underlying PL -manifold. Thus, in extending the C_k -structure on Q_{PL}^{n-1} we first have to extend some framed submanifold W^{n-k-2} of Q_{PL}^{n-1} to a framed submanifold V^{n-k-1} on X^n . We may think of framed submanifolds of X^n as simplicial maps of some compatible triangulation of X^n to the sphere S^{k+1} (the manifold is the inverse image of the barycenter of a $(k + 1)$ -simplex of S^{k+1}), and thus we may think of the extension of the framed submanifold W to a framed submanifold V as a simplicial extension to X^n of a simplicial map $Q_{PL}^{n-1} \rightarrow S^{k+1}$. Obviously, there are at most countably many such extensions. Next, the C_k -structure on Q^{n-1} involves a smooth, framed structure on W^{n-k-2} . (Framing in this case means a smooth framing of the stable smooth normal bundle determined by the choice of smoothing of W .) This structure must be extended to V . Up to concordance, there are only countably many ways to do this. Next, we note that for each component W_i of W^{n-k-2} , we have picked a C_{k-1} -structure Σ_i^k on the "linking" sphere of W_i in Q_{PL}^{n-1} . We must extend this choice to the components of V^{n-k-1} (where it is not already determined by what happens to the W_i). The inductive hypothesis tells us that there are only countably many concordance classes of C_{k-1} -structures on S^k , and thus, since we may use only one Σ^k from each concordance class, there are only countably many choices that have to be made.

The effect of all the choices we have made thus far is to give us a C_{k-1} -structure on $(V \times S^k) \cup (Q_{PL}^{n-1} - (W \times D^{k+1}))$. We now must extend this to a C_{k-1} -structure on the rest of $X^n - (V \times D^{k+1})$. But, by the induction hypothesis, there are, up to concordance, only countably many ways of doing this.

To summarize then, any concordance class of C_k -structures on X^n extending Q^{n-1} will be determined by one of a countable number of possible choices from a certain set of invariants. This proves Lemma 4.3. Proposition 4.2 follows immediately.

We now extend the homotopy functor TC_k from finite CW -complexes to all CW -complexes. We do this in the crudest possible way, i. e.,

$$TC_k(K) = \varprojlim TC_k(K_i)$$

where K_i ranges over all finite subcomplexes. Brown's theorem enables us to assert that the functor TC_k has a classifying space BC_k . This completes the proof of the theorem.

REMARK. We may talk about a sequence of "maps"

$$BO \rightarrow BC_7 \rightarrow BC_8 \rightarrow \dots \rightarrow BPL.$$

The reason for the quotation marks is that the maps $BO \rightarrow BC_k$ and $BC_k \rightarrow BPL$ may not be quite honest inasmuch as they are only truly realized on finite subcomplexes of their respective domains. However, since our primary concern is with finite complexes, i. e., compact C_k -manifolds, ignoring this nicety causes no problems.

REMARK. As we have been observing all along, we could have obtained the result above if we had defined C_k -manifolds by allowing singularities of C_{k-1} -manifolds along smooth manifolds, rather than insisting upon smooth framed manifolds. However, the next result gives a clear motivation for the more exacting definition.

4.4. THEOREM. BC_k has the homotopy type of BC_{k-1} with $(k + 1)$ -cells adjoined. In particular, there will be one $(k + 1)$ -cell for each element in $S_{C_{k-1}}(S^k)$.

PROOF. We are really only concerned with finite approximations to BC_{k-1} , BC_k , etc., and so we shall prove the result in that context.

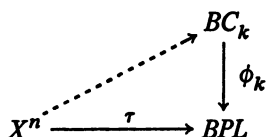
Therefore, let K, L be a finite pair approximating BC_k, BC_{k-1} . If we wish, we may think of K as a high-dimensional PL -manifold with a certain C_k -structure; L is a codimension 0 submanifold of K which misses the highest order singularities, so that its C_k -structure is effectively a C_{k-1} -structure. By the definition of C_k -manifold we see that K has the form $(\bigcup_i V_i \times c\Sigma_i^k) \cup K_0$ where K_0 has a certain C_{k-1} -structure. Since L , with its C_{k-1} -structure, is a codimension 0 submanifold of K_0 , it follows that K_0 is an even better finite approximation to BC_{k-1} than L is. We therefore may as well assume that $L = K_0$. Assume now that the V_i are indexed so that $\Sigma_i^k = \Sigma_j^k$ if and only if $i = j$. That is, we consider together as one manifold all the pieces of the singular part which have the same C_{k-1} -sphere as link. Let n be the dimension of K viewed as a manifold. We claim that $I \times K^n$ may be embedded as a codimension 0 submanifold of a C_k -manifold \bar{K}^{n+1} such that the singular parts \bar{V}_i^{n-k} of \bar{K}^{n+1} are at least $([(n - k)/2] - 1)$ -connected. To see this do surgery starting from one end $1 \times V_i$ of $I \times V_i$ so that the result is $([(n - k)/2] - 1)$ -connected and is still framed. Call the result \bar{V}_i^{n-k} . Let H_i denote a collar $(I \times \Sigma_i^k)$. Then it is easily seen that we

may assume that $(I \times V_i) \times (c\Sigma_i^k \cup H_i)$ intersects $I \times K_0$ in a "collar neighborhood" $(I \times V_i) \times H_i$. If we set $W_i = \text{closure } (\bar{V}_i - (I \times V_i))$, we can form the C_k -manifold $P^{n+1} = W_i \times (c\Sigma_i^k \cup H_i)$ whose C_k -structure is obvious. We then set $\bar{K}^{n+1} = (I \times K^n) \cup_{\bar{a}} P^{n+1}$.

It is obvious that (\bar{K}, K_0) is an even better approximation of (BC_k, BC_{k-1}) than (K, K_0) . So we state the following paradigm. BC_k may be approximated arbitrarily well by a C_k -manifold K^n whose nonsingular (i. e., C_{k-1}) part K_0^n is an arbitrarily good approximation of BC_{k-1} , and such that each component V_i^{n-k-1} of the singular part is arbitrarily highly connected (since n is arbitrarily high). Since V_i^{n-k-1} is arbitrarily close to being contractible, $V_i^{n-k-1} \times \Sigma_i^k = K_0^n \cap (V_i^{n-k-1} \times c\Sigma_i^k)$ is arbitrarily close to the homotopy type of S^k . We leave it to the reader to show that we may force the approximating C_k -manifold K^n to have V_i 's corresponding to any finite subset of $S_{C_{k-1}}(S^k)$. It follows from the foregoing that to approximate BC_k arbitrarily well, we need only consider $K_0 \cup (p_i \times c\Sigma_i^k)$ where $p_i \in V_i$. That is, to approximate BC_k , we approximate BC_{k-1} and then add $(k + 1)$ -cells. This proves the theorem.

REMARK. If e_i^{k+1} is a cell of $BC_k \text{ mod } BC_{k-1}$ corresponding to $[\Sigma_i^k] \in S_{C_{k-1}}(S^k)$, then the attaching map is the map $S^k \rightarrow BC_{k-1}$ classifying the C_{k-1} thickening of S^k determined by $(\Sigma_i^k)_{PL} \xrightarrow{\cong} S^k$.

4.5. PROPOSITION. *Let X^n be a PL-manifold; $S_{C_k}(X^n)$ is in 1-1 correspondence with homotopy classes of liftings*



where τ classifies the stable tangent bundle of X^n and ϕ_k is the natural map.

This follows from results of Rourke [j], or the reader may prove it directly.

4.6. PROPOSITION. *The map ϕ_k is $(k + 1)$ -connected.*

PROOF. By the results of §3, any PL n -thickening of a finite complex K^j has a C_r -structure for large enough r . We claim that if $j < r + 1$ the PL n -thickening may be given a C_{r-1} -structure. For if $h : K^j \rightarrow X^n$ is the PL-thickening, X^n has a j -dimensional spine (which we may assume to contain $\text{im } h$). The C_r -structure on X^n gives a C_{r-1} -structure on the complement of a tubular neighborhood of a submanifold of codimension $r + 1$. We may assume, therefore, that the spine of X^n misses this tubular neighborhood, by general position considerations. So let \bar{X}^n be a small regular neighborhood of the spine lying wholly within the C_{r-1} part of X^n . By the Regular Neighborhood Axiom (5), \bar{X}^n acquires a

C_{r-1} -structure. But the thickening of K^j into \bar{X}^n is PL -equivalent to the original thickening into X^n . By induction, we then see that any PL -thickening of a j -dimensional complex pulls back to a C_k -thickening of $j \leq k + 1$. Since BPL classifies stable PL -thickenings TPL , it is obvious that $\phi_k : BC_k \rightarrow BPL$ induces an epimorphism of homotopy groups in dimensions up to $k + 1$. A relative version of the same argument shows that ϕ_k induces a monomorphism in dimensions up to k . Thus the proposition is proved.

5. The homology groups of (BPL, BO) . The purpose of this section is to make a few points about the integral homology groups $H_*(BPL, BO)$ in the light of the foregoing results. As a first step, consider the fact that Theorem 4.4 immediately implies that we may think of $H_*(BPL, BO)$ as the homology of a chain complex Γ_* whose generators in dimension $k + 1$ are in 1-1 correspondence with $SC_{k-1}(S^k)$. We wish to characterize the boundary operator δ of Γ_* .

A typical generator of Γ_{k+1} is a k -sphere Σ^k with a C_{k-1} -structure. This structure involves a multiply-punctured sphere Σ_0^k with a C_{k-2} -structure, together with a identification of each boundary component of Σ_0^k with a C_{k-2} -structure on the $(k - 1)$ -sphere of the form $p \times \Sigma^{k-1}$, where p is a framed point and Σ^{k-1} is the C_{k-2} -structure chosen to represent its concordance class. Thus, each boundary component of Σ_0^k is associated to the appropriate Σ^{k-1} . (N.B.: This association does not depend on the framing of the point.) Moreover, since $p \times \Sigma^{k-1}$ is thought of as a structure on the standard sphere, $(p \times \Sigma^{k-1})_{PL}$ inherits an orientation. On the other hand, $(\Sigma_0^k)_{PL}$ is a codimension zero submanifold of the standard k -sphere, therefore it acquires an orientation, as do all the boundary components. Thus, depending on how a given boundary component is identified with $p \times \Sigma^{k-1}$, the two orientations may differ or agree. In the former case, call the boundary component positive; in the latter, negative. Now, given Σ^k , if $\{\partial_i \Sigma_0^k\}$ are the distinct boundary components, define $\delta \Sigma^k = \sum \pm \Sigma_i^{k-1}$, where $\partial_i \Sigma_0^k$ is identified with $p \times \Sigma_i^{k-1}$ and where the sign is positive or negative according as the boundary component is positive or negative. We leave it to the reader to check that $\delta^2 = 0$.

Now note that a smooth sphere Σ^k is naturally a C_{k-1} -sphere. Moreover, it is obvious that $\delta \Sigma^k = 0$.

5.1. PROPOSITION. *Let the smooth sphere Σ^k correspond to the element $\alpha \in \pi_k(PL/O) \cong \pi_{k+1}(BPL, BO)$; then the cycle Σ^k of Γ_{k+1} represents the homology class in $H_{k+1}(BPL, BO)$ which is the image of α under the Hurewicz homomorphism.*

The proof is left to the reader.

6. **Some spectral sequences.** At this point, we append some remarks concerning the relation between the categories we have constructed and bordism. Let $D_{m,n}$ be defined as follows: Consider C_m -manifolds M^{m+n+1} with smooth boundary. Call M^{m+n+1} equivalent to N^{m+n+1} iff there is a C_m -manifold W^{m+n+2} with $\partial W^{m+n+2} = (M^{m+n+1} \amalg N^{m+n+1}) \cup_0 V^{m+n+1}$ where V^{m+n+1} is smooth. Equivalence classes of such manifolds form a group $D_{m,n}$ under disjoint union.

Set $E_{m,n} = \bigoplus_{\alpha} (\Pi_n^s)_{\alpha}$ where Π_n^s denotes the group of framed bordism classes of framed manifolds and where α ranges over the elements of $S_{C_{m-1}}(S^m)$.

There is an obvious map $i : D_{m-1,n+1} \rightarrow D_{m,n}$. We can define a map $j : D_{m,n} \rightarrow E_{m,n}$ as follows: A C_m $(m+n+1)$ -manifold M^{m+n+1} has a "singular set" consisting of disjoint smooth manifolds V_i^n , each conceived of as a framed manifold, and each having a C_{m-1} m -sphere Σ_i^m as link. If we consider all the n -manifolds with the same link as one manifold (i.e., the disjoint union), we can define j on the equivalence class of M^{m+n+1} as having $[V_i^n]$ as the Σ_i^m th coordinate, the α coordinate being zero if α is not among the Σ_i^m . We leave it to the reader to verify that j is well defined.

We also define $k : E_{m,n} \rightarrow D_{m-1,n}$ by taking $k[V_{\alpha}^n] = [\amalg_{\alpha} V_{\alpha}^n \times \Sigma_{\alpha}^m]$ where $[]$ on the right-hand side denotes equivalence class in $D_{m-1,n}$. (Note that each product $V_{\alpha}^n \times \Sigma_{\alpha}^m$ has a C_{m-1} -structure.) Again, verification that k is well defined is routine.

If we let $D = \sum_{m,n} D_{m,n}$, $E = \sum_{m,n} E_{m,n}$ it is also routine to see that

$$(1) \quad \begin{array}{ccc} D & \xrightarrow{i} & D \\ & \searrow k & \swarrow j \\ & & E \end{array}$$

is an exact couple.

6.1. **PROPOSITION.** *The spectral sequence of the exact couple (1) converges to $E_{m,n}^{\infty}$ where $\sum_{m+n=p} E_{m,n}^{\infty}$ is a graded object associated to a filtration of the group $N_{p+1}^{rel} =$ group of bordism classes of PL-manifolds with smooth boundary.*

This is easily seen since (1) is a regular ∂ -couple $[k]$, and $D_{p+1,-1}$ is easily seen to be N_{p+1}^{rel} .

In the last section, we introduced a chain complex Γ_* , where $\Gamma_{m+1} =$ free abelian group on $S_{C_{m-1}}(S^m)$. We now introduce Δ_* , a new chain complex where $\Delta_{m+1} = \Gamma_{m+1}$ but where the boundary operator ϵ has a different meaning. Let Σ^m be a generator of Δ_{m+1} . Then the boundary components of the nonsingular part of Σ^m each are associated with a $C_{m-2}(m-1)$ -sphere Σ_i^{m-1} . The singular point p_i with link Σ_i^{m-1} is thought of as a framed, i.e., oriented, point. Let

$e\Sigma^m = \sum \pm \Sigma_i^{m-1}$, with sign determined according to whether p_i is positively or negatively oriented.

We may now easily prove

6.2. COROLLARY. *The E^2 term of the spectral sequence of (1) is given by $E_{m,n}^2 = H_{m+1}(\Delta_* \otimes \Pi_n^s)$.*

We leave to the reader details of the proof, noting only that $E_{m,n} = \Gamma_{m+1} \otimes \Pi_n^s$.

We may remark that, in the construction of the manifold-like categories C_k , we could have dropped the condition that singular sets be framed smooth manifolds, requiring merely that they be smooth. Call the manifold-like categories constructed in this fashion \bar{C}_k ; again $\bar{B}\bar{C}_k$ exist and $\varinjlim \bar{B}\bar{C}_k = BPL$. (However, this is not a skeletal filtration.) Let $\bar{\Gamma}^*$ be constructed as a chain complex in analogy to Γ^* .

Set $\bar{D}_{m,n}$ to be the group defined like $D_{m,n}$ but with C_m replaced with \bar{C}_m throughout. Likewise define $\bar{E}_{m,n}$ as $\bigoplus_{\alpha} (N_n)_{\alpha}$ where N_n denotes smooth bordism and α ranges over $S_{\bar{C}_{m-1}}(S^m)$.

In addition set $D'_{m,n}$ to be defined like $\bar{D}_{m,n}$ with "manifold" replaced by "oriented manifold" and bordisms required to be oriented. Set $E'_{m,n} = \bigoplus_{\alpha} (\Omega_n)_{\alpha}$ where Ω_n denotes oriented smooth bordism and α again ranges over $S_{\bar{C}_{m-1}}(S^m)$. Then we have exact couples

(2)
$$\begin{array}{ccc} \bar{D} & \xrightarrow{i} & \bar{D} \\ & \swarrow k & \searrow j \\ & \bar{E} & \end{array},$$

(3)
$$\begin{array}{ccc} D' & \xrightarrow{i'} & D' \\ & \swarrow k' & \searrow j' \\ & E' & \end{array}$$

6.3. COROLLARY. *In the spectral sequence of (2) (resp. (3)), $\Sigma_{m+n=p} E_{m,n}^{\infty}$ is a graded object associated to N_{p+1}^{rel} (resp. Ω_{p+1}^{rel}). In addition, $E_{m,n}^2 = H_{m+1}(\bar{\Gamma}^* \otimes N_n)$ (resp. $H_{m+1}(\bar{\Gamma}^* \otimes \Omega_n)$).*

In point of fact, the spectral sequences of (1), (2), (3) are thinly-disguised versions of spectral sequences arising from filtrations of $MPL \text{ mod } MO$ (in the case of (1) and (2)) and $MSPL \text{ mod } MSO$ (in the case of (3)). In (1) the filtration of MPL comes from that of the base space BPL via $BO \subseteq BC_1 \subseteq \dots \subseteq BPL$. In (2), the relevant filtration is

$$BO \subseteq \bar{B}\bar{C}_1 \cdots \bar{B}\bar{C}_k \subseteq BPL$$

and in (3),

$$BSO \subseteq \widetilde{\bar{B}\bar{C}}_1 \cdots \widetilde{\bar{B}\bar{C}}_k \cdots BSPL$$

where \sim denotes 2-fold covering by the orientation sheaf.

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