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GROUPS OF FREE INVOLUTIONS OF HOMOTOPY $S^{[n/2]} \times S^{[(n+1)/2]}$'s

BY

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ABSTRACT. Let M be an oriented n-dimensional manifold which is homotopy equivalent to $S^l \times S^{n-l}$, where l is the greatest integer in $n/2$. Let Q be the quotient manifold of M by a fixed point free involution. Associated to each such Q are a unique integer k mod $2^{\varphi(l)}$, called the type of Q, and a cohomology class ω in $H^1(Q; Z_2)$ which is the image of the generator of the first cohomology group of the classifying space for the double cover of Q by M. Let $I_n(k)$ be the set of equivalence classes of such manifolds Q of type k for which $\omega^{l+1} = 0$, where two such manifolds are equivalent if there is a diffeomorphism, orientation preserving if k is even, between them. It is shown in this paper that if $n \geqslant 6$, then $I_n(k)$ can be given the structure of an abelian group. The groups $I_8(k)$ are partially calculated for k even.

1. **Introduction.** Suppose M is an oriented n-dimensional manifold which is homotopy equivalent to $S^{[n/2]} \times S^{[(n+1)/2]}$, and ρ is a fixed point free involution of M. Define Q to be the quotient of M by the equivalence relation $x \sim x$ and $x \sim \rho(x)$. Notice that if ρ is orientation preserving, Q inherits an orientation from that of M. Otherwise, Q is not orientable.

We say that two such involutions $\rho: M \longrightarrow M$ and $\rho': M' \longrightarrow M'$ are equivalent if there is a diffeomorphism between Q and Q' which also preserves orientation in the case ρ and ρ' are orientation preserving. Define J_n to be the set of equivalence classes of these involutions. We want to study these sets J_n.

In more generality, suppose M is a simply connected stably parallelizable, oriented n-dimensional manifold with the same integral homology as $S^{[n/2]} \times S^{[(n+1)/2]}$, and ρ is a fixed point free involution of M. Set Q to be the quotient of M, as above. Define $l = [n/2]$.

We can embed l-dimensional projective space P_l in Q so that the inclusion induces an isomorphism of the fundamental groups. The restriction of the

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normal bundle of Q to the embedded P_l does not depend on the embedding chosen, providing the embedding induces an isomorphism of fundamental groups (see [18]). Thus, each Q determines a unique integer k mod $2^{\phi(l)}$ such that $\nu(Q)|P_l$ is stably $k\xi_l$, where ξ_l is the canonical line bundle over P_l, and $2^{\phi(l)}$ is the order of ξ_l in $\widetilde{KO}(P_l)$. Call k the type of Q.

Since $M \longrightarrow Q$ is a principal Z_2-bundle, it determines (up to homotopy) a classifying map $g\colon Q \longrightarrow BZ_2 = P_\infty$. Define $\omega \in H^1(Q; Z_2)$ to be the image under g^* of the generator of $H^1(P_\infty; Z_2) = Z_2$.

Define I_n to be the set of elements of J_n which have $\omega^{l+1} = 0$. For technical reasons, we shall study the sets I_n instead of J_n. In many cases, $I_n = J_n$. For example, if $n = 2l$ and l is even, then I_n and J_n are equal.

Let I_n^+ be the set of orientation preserving involutions in I_n, and I_n^- be the set of involutions in I_n which do not preserve orientation. Define $I_n(k)$ to be the set of elements in I_n which have type k. Observe that I_n is the disjoint union of the sets $I_n(k)$ for $k = 1, 2, \cdots, 2^{\phi(l)}$.

We may also define $I_n^+(k)$ and $I_n^-(k)$ in the obvious way. Clearly, the involutions in $I_n(k)$ will be orientation preserving if and only if k is even.

In §2, we shall show that $I_n(k)$ can be given the structure of an abelian group. In §3, we shall calculate $J_8^+ = I_8^+$, and show that it is the disjoint union of groups $A \cup (Z_2 + Z_2) \cup Z_2 \cup (Z_2 + Z_2 + Z_2)$, where A is $0, Z_2$, or Z_4. In a future paper, we shall develop an exact sequence relating the groups $I_n(k)$ to certain Lashof cobordism groups and the Wall surgery groups $L_n(Z_2, (-1)^k)$.

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2. The involution groups $I_n(k)$.

2.1 *The involution group theorem.* Instead of the definition of $I_n(k)$ given in the Introduction, we shall use the following definition of $I_n(k)$, which we shall see later is an equivalent definition.

DEFINITION. $I_n(k)$ is the set of (oriented) diffeomorphism classes of n-dimensional manifolds Q such that

(i) Q has a double cover M which is a simply connected, stably parallelizable, n-dimensional manifold with the same integral homology as $S^l \times S^{n-l}$.

(ii) Q has type k.

(iii) $\omega^{l+1} = 0$.

In this §2, we shall prove:

THEOREM 1. *If* $n \geqslant 6$, *then* $I_n(k)$ *can be given the structure of an abelian group.*

2.2. *The pasting construction.* Let γ be an $(n-l)$-dimensional bundle over P_l which is stably equivalent to $(2^{\phi(l)} - l - 1 - k)\xi_l$. We shall make the choice of γ more definite later. The disk and sphere bundles of γ will be denoted by $E(\gamma)$ and $S(\gamma)$, respectively.

If ψ is any diffeomorphism of $S(\gamma)$ onto itself, then we may construct a manifold $P(\psi)$ by using the pasting construction, as follows: Let $E_1(\gamma)$ and $E_2(\gamma)$ be two disjoint copies of $E(\gamma)$. We may take collars $S_i(\gamma) \times [0, 1]$ of the boundary of $E_i(\gamma)$, identifying $S_i(\gamma)$ with $S_i(\gamma) \times \{0\}$. Then $P(\psi)$ is the manifold $E_1(\gamma) \cup_\psi E_2(\gamma)$ formed by taking the union of $E_1(\gamma)$ and $E_2(\gamma)$ and identifying $(x, t) \in S_1(\gamma) \times [0, 1]$ with $(\psi(x), 1-t) \in S_2(\gamma) \times [0, 1]$.

Notice that we may consider $P(\psi)$ to be the union of $E_1(\gamma)$ and $E_2(\gamma)$ with only the identification of $x \in S_1(\gamma)$ with $\psi(x) \in S_2(\gamma)$, providing we give $P(\psi)$ the differentiable structure indicated in the first construction.

PROPOSITION 1. *Suppose* $Q \in I_n(k)$. *Then* $Q = P(\psi)$ *for some diffeomorphism* ψ *of* $S(\gamma)$ *with itself.*

PROOF. Let M be the double cover of Q, and $g: Q \longrightarrow P_\infty$ be the classifying map for the double cover. Since this is a nontrivial cover, g_* takes the generator of $\pi_1(Q)$ to the generator of $\pi_1(P_\infty)$. Thus, we have

$$P_l \xrightarrow{f} Q \xrightarrow{g} P_\infty,$$

where the induced maps on fundamental groups are all isomorphisms. Then f_* and g_* also induce isomorphisms in the first homology groups. This will show that the maps

$$g^*: H^1(P_\infty; Z_2) \longrightarrow H^1(Q; Z_2) \quad \text{and} \quad f^*: H^1(Q; Z_2) \longrightarrow H^1(P_l; Z_2)$$

are isomorphisms. Denote the generators of $H^1(P_\infty; Z_2)$, $H^1(Q; Z_2)$, and $H^1(P_l; Z_2)$ by x, ω, and y, respectively. Let X represent P_l, Q, or P_∞.

Now consider the coefficient sequence

$$0 \longrightarrow Z_2 \longrightarrow \overline{Z_2 + Z_2} \longrightarrow Z_2 \longrightarrow 0,$$

where $\pi_1(X) = Z_2$ acts on $\overline{Z_2 + Z_2}$ by changing factors. If the double cover of X is denoted by $\overline{X}$, this sequence induces long exact sequences

$$H^m(X; Z_2) \longrightarrow H^m(\bar{X}; Z_2) \longrightarrow H^m(X; Z_2) \xrightarrow{\beta^*} H^{m+1}(X; Z_2) \longrightarrow \cdots.$$

In the case $X = P_\infty$, then $\bar{X} = S^\infty$ and $H^m(\bar{X}; Z_2) = 0$ for all $m \geqslant 1$. Thus, β^* is an isomorphism for $m \geqslant 1$. Since the generators of $H^m(P_\infty; Z_2)$ and $H^{m+1}(P_\infty; Z_2)$ are x^m and x^{m+1}, β^* must be cup product with x. Since $g^*(x) = \omega$ and $(g \circ f)^*(x) = y$, it follows that β^* is cup product with ω if $X = Q$ and cup product with y if $X = P_l$, by naturality of β^*.

From the same exact sequence as above, with $X = P_l$ and $\bar{X} = S^l$, since y^m generates $H^m(P_l; Z_2)$, we have $\beta^*\colon H^{l-1}(P_l; Z_2) \longrightarrow H^l(P_l; Z_2)$ is an isomorphism and $H^{l+1}(P_l; Z_2) = 0$, so $H^l(S^l; Z_2) \longrightarrow H^l(P_l; Z_2)$ is an isomorphism. Thus, we have a diagram

$$\begin{tikzcd}
H^l(S^l; Z_2) \arrow[d, "\cong"'] & H^l(M; Z_2) \arrow[l, "\bar{f}^*"'] \arrow[d] \\
H^l(P_l; Z_2) \arrow[d] & H^l(Q; Z_2) \arrow[l, "f^*"'] \arrow[d, "\omega\cup"] \\
0 & H^{l+1}(Q; Z_2)
\end{tikzcd}$$

Here $\bar{f}$ is a map covering f. Since $\omega^{l+1} = 0$ by the definition of $I_n(k)$, $\omega^l \in H^l(Q; Z_2)$ is mapped to zero in the downward exact sequence and, hence, comes from some $y \in H^l(M; Z_2)$. But $f^*(\omega^l) = y^l \neq 0$, so $\bar{f}^*(y)$ must be the nonzero element of $H^l(S^l; Z_2)$. Thus, $\bar{f}^*$ is onto.

Dually, $\bar{f}_*\colon H_l(S^l; Z_2) \longrightarrow H_l(M; Z_2)$ must be one-to-one.

We have $H_l(M) = Z$ or $Z + Z$. If $[S^l]$ is the generator of $H_l(S^l)$, then $\bar{f}_*[S^l] = rz$, where $z \in H_l(M)$ is indivisible. Let u be the generator of $H_l(S^l; Z_2)$. If r were even, then $\bar{f}_*(u)$ would be 0, contradicting that $\bar{f}_*\colon H_l(S^l; Z_2) \longrightarrow H_l(M; Z_2)$ is one-to-one. Thus, r must be odd.

We want to choose f so that r will be one.

We have a commutative diagram

$$\begin{tikzcd}
S^l \arrow[r, "\bar{f}"] \arrow[d, "\bar{\pi}"] & M \arrow[d, "\pi"] \\
P_l \arrow[r, "f"] & Q
\end{tikzcd}$$

where π is the map which identifies any element $x \in M$ with $\rho(x)$. Since $\bar{\pi}$ is the usual double cover, it identifies antipodal points on S^l. Denote the antipodal map by (-1).

The antipodal map has degree $(-1)^{l+1}$. Since $\bar{f}$ is equivariant, we have that

$$\rho_*(\bar{f}_*[S^l]) = \bar{f}_*(-1)_*[S^l] = \bar{f}_*((-1)^{l+1}[S^l]) = (-1)^{l+1}\bar{f}_*[S^l].$$

Therefore $r\rho_*(z) = \rho_*(rz) = r(-1)^{l+1}z$, so

$$\rho_*(z) = (-1)^{l+1}z.$$

Let $y = sz \in H_l(M) \cong \pi_l(M)$, where s is the integer $(1-r)/2$. Let h: $S^l \longrightarrow M$ represent y. Choose a cell in S^l which is disjoint from its image under (-1). Then by pinching the boundary of this cell to a point, and pinching the image of the boundary under (-1) to a point, we obtain a map $\theta\colon S^l \longrightarrow S^l \vee S^l \vee S^l$ which is equivariant with respect to the obvious involutions. Define $F\colon S^l \longrightarrow M$ to be $(h \vee \bar{f} \vee \rho \circ h \circ (-1)) \circ \theta$.

Since $\bar{f}$ is equivariant, F is equivariant. The map F covers a map of P_l into Q which induces an isomorphism on fundamental groups, since F does. We may homotope F to an embedding by homotoping the map it covers to an embedding, as we did for f.

Then

$$\begin{aligned} F_*[S^l] &= \bar{f}_*[S^l] + h_*[S^l] + \rho_* h_*(-1)_*[S^l] \\ &= (r+s)z + \rho_*((-1)^{l+1}sz) \\ &= (r+2s)z = z, \quad \text{since } 2s = 1-r. \end{aligned}$$

Thus, we could have assumed $r = 1$.

Thus, we have an embedding $f\colon P_l \longrightarrow Q$ whose double cover $\bar{f}\colon S_l \longrightarrow M$ has the property that $\bar{f}_*[S_l]$ is an indivisible element of $H_l(M)$.

Now consider $M - \bar{f}(S^l)$. By Lefschetz duality, $H_q(M - \bar{f}(S^l)) \cong H^{n-q}(M, \bar{f}(S^l))$. The cohomology of M and S^l is the same as the homology, and from what we have found for homology, $H^l(M) \longrightarrow H^l(S^l)$ is an isomorphism for $n = 2l+1$ and onto for $n = 2l$. Using this in the cohomology sequence of the pair $(M, \bar{f}(S^l))$, we find that $H_q(M - \bar{f}(S^l)) = Z$ if $q = 0$ or l, and $H_q(M - \bar{f}(S^l)) = 0$ otherwise.

We now have essentially the same situation as before. Since the double cover of $Q - f(P_l)$ is nontrivial, $\pi_1(Q - f(P_l)) = Z_2$, and since it is in a zero group, $\omega^{l+1} = 0$ for $\omega \in H^1(Q - f(P_l))$. Thus, we can find an embedding $f'\colon P_l \longrightarrow Q - f(P_l)$ such that its covering $\bar{f}'\colon S^l \longrightarrow M - \bar{f}(S^l)$ is an embedding and an isomorphism in homology.

Then $\bar{f}'$ is a homotopy equivalence, and since ρ acts as the antipodal map on $\bar{f}'(S^l)$, its projection $f'\colon P_1 \longrightarrow Q - f(P_l)$ is also a homotopy equivalence.

Since the normal bundle of Q restricted to P_l embedded in Q is stably $k\xi_l$, the normal bundle of P_l is stably equivalent to $(2^{\phi(l)} - l - 1 - k)\xi_l$. If $n = 2l + 1$, then γ has dimension $l + 1$, so γ is unique up to bundle equivalence. Thus, $\gamma = \nu(f(P_l)\colon Q) = \nu(f'(P_l)\colon Q)$, so we may embed two disjoint copies of $E(\gamma)$ in Q.

Suppose $n = 2l$. Choose γ to be the bundle stably equivalent to $(2^{\phi(l)} - l - 1 - k)\xi_1$ such that the twisted Euler class is zero if the binomial coefficient

$$\binom{2^{\phi(l)} - l - 1 - k}{l} \equiv 0 \pmod{2},$$

and the twisted Euler class is one if

$$\binom{2^{\phi(l)} - l - 1 - k}{l} \equiv 1 \pmod{2}.$$

Recall that $f_*[S^l] = (1, 0)$ or $(1, -1)$ in a basis chosen to be symplectic with respect to the intersection pairing. In the first case the twisted Euler class of $\nu(f(P_l)\colon Q)$ is zero, and in the second case it is one. Thus, $\gamma = \nu(f(P_l)\colon Q)$. Define $\gamma' = \nu(f'(P_l)\colon Q)$. Then γ and γ' are stably equivalent. We want to show that their cell bundles are diffeomorphic.

Since $n = 2l$, γ and γ' have dimension l. If l is even, then they are equivalent if they are both stably equivalent, and $\tilde{\chi}(\gamma) = \tilde{\chi}(\gamma')$, where $\tilde{\chi}$ is the twisted Euler class (see [14]).

Recall the basic facts about the twisted Euler class. If η is an l-dimensional bundle with base space B and $\bar{B}$ is the orientation double cover of B determined by η, then Z_2 acts on $\bar{B}$ in the obvious manner, and Z_2 acts on Z with the nontrivial element changing signs. Let Z^ω denote the orientation Z_2-module of η. That is, if η is nonorientable, $Z^\omega = \bar{Z}$ is the nontrivial Z_2-module on Z, and if η is orientable, $Z^\omega = Z$ is the trivial Z_2-module on Z. Then $\tilde{\chi}(\eta) \in H^l(B; Z^\omega)$.

The twisted Euler class is natural for bundle maps, and equals the usual Euler class $\chi(n)$ in the orientable case.

We must examine two possibilities, l even or odd. First, l even.

If l and k are both even, then the fact that γ is equivalent to γ' is shown in [18].

If l is even and k is odd, then γ and γ' are orientable. From the coefficient sequence

$$0 \longrightarrow Z \xrightarrow{2} Z \longrightarrow Z_2 \longrightarrow 0,$$

we see that the induced map $H^l(P_l) \longrightarrow H^l(P_l; Z_2)$ is an isomorphism. Thus, in this case the twisted Euler class is just the mod 2 Euler class, which is the lth Stiefel-Whitney class. Since this is a stable invariant, it is the same for γ and γ'. Thus, $\widetilde{\chi}(\gamma) = \widetilde{\chi}(\gamma')$.

Now suppose l is odd. If k is even , then by the Lefschetz fixed point theorem and Curtis and Reiner's theorem [5], $H_l(M) = Z + Z$ as a Z_2-module. We may assume that $\bar{f}$ represents either $(1, 0)$ or $(1, 1)$. Since $\bar{f}$ and $\bar{f}'$ are disjoint, their intersection is zero. Since $(1, 0) \cdot (a, b) = b$ and $(1, 1) \cdot (a, b) = -a + b$, we see from the indivisibility of $\bar{f}'$ that $\bar{f}$ and $\bar{f}'$ represent the same element, so they are homotopic.

The double cover $\pi\colon S^l \longrightarrow P_l$ induces a monomorphism $\pi^*\colon H^l(P_l; Z + Z) \longrightarrow H^l(S^l; Z + Z)$. Now look at the commutative diagram

$$\begin{tikzcd}
S^l \arrow[r, "\bar{f}"] \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & M \arrow[d] \\
P_l \arrow[r, "f"] & Q
\end{tikzcd}$$

and the corresponding diagram for f'. Since $\bar{f}$ and $\bar{f}'$ are homotopic, the class of their difference cochain, $d(\bar{f}, \bar{f}') \in H^l(S^l; \pi_l(M))$, is 0.

By naturality, we have

$$\begin{tikzcd}
0 = d(\bar{f}, \bar{f}') \arrow[d] & \in H^l(S^l; \pi_l(M)) \arrow[d, "\cong"] \\
0 = d(\tilde{f}, \tilde{f}') & \in H^l(S^l; \pi_l(Q)) \\
d(f, f') \arrow[u] & \in H^l(P_l; \pi_l(Q)) \arrow[u, "\pi^*"']
\end{tikzcd}$$

We have already seen that the map π^* above is a monomorphism, so $d(f, f') = 0$. Thus, f and f' are homotopic, so by Haefliger's theorem, they are isotopic.

Next, suppose l is odd and k is odd. By the Lefschetz fixed point theorem, $\rho_*|H_l(M)$ has trace zero, so by Curtis and Reiner's theorem, $H_l(M) = \overline{Z + Z}$ or $\bar{Z} + Z$. First, consider the case $H_l(M) = \overline{Z + Z}$. Then $H_l(M; Z_2) = H^l(M; Z_2) = \overline{Z_2 + Z_2}$. There is a spectral sequence associated to the covering $M \longrightarrow Q$ with $E_2^{p,q} = H^p(P_l; H^q(M; Z_2))$. In this case, the sequence shows that $\omega^{l+2} \neq 0$, contrary to the definition of $I_n(k)$.

Thus, we must have $H_l(M) = Z + \bar{Z}$. Clearly, $\bar{f}$ and $\bar{f}'$ can both be taken to represent $(1, 0) \in H_l(M)$. The coefficient sequence

$$0 \longrightarrow \bar{Z} \xrightarrow{2} \bar{Z} \longrightarrow Z_2 \longrightarrow 0$$

gives a natural isomorphism $H^l(P_l; \bar{Z}) \cong H^l(P_l; Z_2)$, so the twisted Euler class

$$\begin{aligned}\tilde{\chi}(\gamma) = \omega_l(\gamma) &= \omega_l[(2^{\phi(l)} - l - 1 - k)\xi_l] \\ &= \binom{2^{\phi(l)} - l - 1 - k}{l} x^l\end{aligned}$$

where x is the first Stiefel-Whitney class of ξ_l.

Case I.

$$\binom{2^{\phi(l)} - l - 1 - k}{l} \equiv 0 \pmod 2.$$

In this case, γ has an everywhere nonzero cross section. Thus, there is a cross section of $S(\gamma)$. We may use this to push the embedding $f\colon P_l \longrightarrow E(\gamma)$ to an isotopic embedding $f_0\colon P_l \longrightarrow Q - \mathring{E}(\gamma)$. This map f_0 will be homotopic to f' if it is a homotopy equivalence, since $f'\colon P_l \longrightarrow Q - \mathring{E}(\gamma)$ is a homotopy equivalence.

To see that f_0 is a homotopy equivalence, first notice that, by the choice of f, the double cover $\bar{f}_0$ is a homotopy equivalence. Thus, f_0 induces isomorphisms of all homotopy groups of dimension greater than one. It also induces an isomorphism of fundamental groups, since otherwise the classifying map of P_l, given by $P_l \longrightarrow Q - \mathring{E}(\gamma) \longrightarrow P_\infty$, would be homotopically trivial. If that were the case, then the double cover of P_l would be two disjoint copies of P_l, rather than the S^l which it actually is. Thus, f_0 is a homotopy equivalence, so f and f' are homotopic and hence isotopic.

Case II.

$$\binom{2^{\phi(l)} - l - 1 - k}{l} \equiv 1 \pmod 2.$$

Since $\bar{f}_*[S^l]$ is indivisible and invariant under ρ_*, it represents a generator of Z in $H_l(M) = Z + \bar{Z}$. Let $\bar{v} = \bar{f}_*[S^l]$, and let $\bar{u}$ be a generator of $\bar{Z}$ in $H_l(M)$. Then $\{\bar{u}, \bar{v}\}$ is a basis for $H_l(M)$. Since the intersection matrix must be antisymmetric and unimodular, it follows that $\bar{u} \cdot \bar{u} = \bar{v} \cdot \bar{v} = 0$ and $\bar{u} \cdot \bar{v} = -\bar{v} \cdot \bar{u} = s$, where $s = \pm 1$. Let $\bar{g}\colon S^l \longrightarrow M$ be a smooth map representing $\bar{u}$. By the Whitney procedure, we may assume that $\bar{g}$ is an embedding that meets $\bar{f}$ transversally in exactly one point. We may also assume that $\bar{g}$ and $\rho\bar{g}$ are transverse. Since l is odd and ρ is orientation reversing, the intersection signs of a and ρa are the same for any $a \in \bar{g}(S^l) \cap \rho\bar{g}(S^l)$. Since $\bar{u} \cdot \bar{u} = 0$, the sum of the intersection signs is zero. Let $a, b \in \bar{g}(S^l) \cap \rho\bar{g}(S^l)$ with opposite signs, so $\{a, \rho(a)\}$ and $\{b, \rho(b)\}$ are disjoint. Run a path α_1 from a to b in $\bar{g}(S^l)$. Then we have a closed loop $\pi \circ \alpha_1 - \pi \circ \rho \circ \alpha_1$ in $\pi \circ \bar{g}(S^l)$ running through the two self-intersections $\pi(a)$ and $\pi(b)$. If this loop does not bound a disk, then the loop constructed using α_1' from a to $\rho(b)$ will, and the Whitney procedure may be used to eliminate $\pi(a)$ and $\pi(b)$. After a finite number of steps, we may assume that $g = \pi \circ \bar{g}$ is an embedding, so $\bar{g}$ and $\rho\bar{g}$ are disjoint.

Notice that if $v \cdot v$ were zero, the Euler class of the normal bundle of f would be zero. But this Euler class is not zero by our assumption that

$$\binom{2^{\phi(l)} - l - 1 - k}{l} \equiv 1 \qquad (\text{mod } 2).$$

Thus, $v \cdot v = 1$. Since the matrix of intersection numbers of u and v must be nonsingular, it follows that $u \cdot u = 0$.

Since the embedding $\pi \circ \bar{u}$ is covered by two disjoint embeddings of S^l in M, and M is stably parallelizable, the normal bundle $\nu(\pi \circ \bar{u})$ is trivial. Let $8\colon S^l \longrightarrow Q$ be the "figure-eight" immersion. That is, 8 is an immersion, with exactly one self-intersection, of S^l into a cell. The normal bundle of this immersion is the same as $\tau(S^l)$, the tangent bundle of S^l. Using the symbol # to denote connected sum, we have

$$\nu(f \# \pi \circ \bar{u}) = \nu(f) \quad \text{and} \quad \nu(f \# \pi \circ \bar{u} \# 8) = \nu(f) + \nu(S^l).$$

This second normal bundle is classified by

$$P_l \longrightarrow P_l \vee S^l \xrightarrow{\nu(f) \vee \tau(S^l)} BO(l).$$

where the first map is obtained by pinching the boundary of an l-cell in P_l to a point.

The self-intersections of $f_1 \# \pi \circ \bar{u} \# 8$ are 0 modulo 2, and this map from P_l to Q induces an isomorphism in fundamental groups. Thus, by the Whitney theorem, this map is regularly homotopic to an embedding h.

We have obtained an embedding $h\colon P_l \longrightarrow Q$ such that $\nu(h) = \nu(f) + \tau(S^l)$; and h represents the homology class $v + u$. Since this implies that $h \cdot f = 0$, we may isotope h to be disjoint from f. Thus, we may assume f and h are disjoint.

Since $H^l(P_l; \pi_l(SO/SO(l)) = Z_2$, if follows from obstruction theory that there are just two reductions to l-plane bundles of $(2^{\phi(l)-l-1-k})\xi_l$. If one of these is γ, then the other must be $\gamma + \tau(S^l)$, given by

$$P_l \longrightarrow P_l \vee S^l \xrightarrow{\gamma \vee \tau(S^l)} BO(l).$$

But $\gamma = \nu(f)$, so the other is $\gamma' = \nu(h) = \nu(f) + \tau(S^l) = \gamma + \tau(S^l)$.

Notice that f and f', the two embeddings of P_l in Q we have constructed, are disjoint, so $f \cdot f' = 0$. But $v \cdot v \neq 0$, so if f represents v, then f' must represent $v + u$. Thus, we may identify f' with h, which justifies using γ' for both $\nu(f')$ and $\nu(h)$.

Let δ denote the unique $l + 1$-plane bundle over P_l stably equivalent to $(2^{\phi(l)} - l - 1 - k)\xi_l$. We have two disjoint embeddings of P_l in $S(\delta)$, which we may still denote by f and f'. We may thicken our embeddings to disjoint embeddings, also denoted by f and f', of $E(\gamma)$ and $E(\gamma')$ in Q. Since $S(\delta) - f(P_l)$ is homotopy equivalent to P_l, we have that $S(\delta) - f(E(\gamma))$ is homotopy equivalent to $E(\gamma')$.

If we write $S(\delta)'$ for $S(\delta) - f(\mathring{E}(\gamma)) - f'(\mathring{E}(\gamma'))$, then $S(\delta)' - f(S(\gamma))$ is homotopy equivalent to $S(\gamma')$. Since the Whitehead group of Z_2 is 0, $S(\delta)'$ is an s-cobordism between $f(S(\gamma))$ and $f'(S(\gamma'))$. Thus, $S(\delta)'$ is diffeomorphic to $S(\gamma) \times I$, and $S(\gamma)$ is diffeomorphic to $S(\gamma')$.

Thus, we have that $S(\delta)$ is diffeomorphic to $E(\gamma) \cup E(\gamma')$, where the union is taken by identifying points using a diffeomorphism of their boundaries. But δ equals γ plus a trivial line bundle, so $S(\delta)$ also is diffeomorphic to $E(\gamma) \cup E(\gamma)$, where the boundaries are simply identified. Thus, we have $E(\gamma) \cup E(\gamma')$ diffeomorphic to $E(\gamma) \cup E(\gamma)$, and we may arrange the first $E(\gamma)$ in $E(\gamma) \cup E(\gamma)$ so that the diffeomorphism is the identity on first components. Removing $\mathring{E}(\gamma)$ from each, we see that $E(\gamma)$ is diffeomorphic to $E(\gamma')$ by a diffeomorphism which restricts to a diffeomorphism of the boundaries. This completes Case II.

Thus, in all cases, $E(\gamma)$ and $E(\gamma')$ are diffeomorphic. By repeating the above argument, we may embed two disjoint copies of $E(\gamma)$ in Q, and show that Q with the interiors of these embeddings removed is an s-cobordism and, hence, is diffeomorphic to $S(\gamma) \times I$. Hence, Q is diffeomorphic to $E(\gamma) \cup_\psi E(\gamma)$, where $\psi: S(\gamma) \longrightarrow S(\gamma)$ is some diffeomorphism.

This completes the proof of Proposition 1.

Next, we shall examine the double cover $\overline{P(\psi)}$ of $P(\psi)$ for arbitrary diffeomorphisms ψ from $S(\gamma)$ to itself.

Note that $E(\gamma)$ has as its double cover $S^l \times D^{n-l}$. Thus, the double cover of $E(\gamma) \cup_\psi E(\gamma)$ is $S^l \times D^{n-l} \cup_{\overline{\psi}} S^l \times D^{n-l}$, where $\overline{\psi}$ is a diffeomorphism of $S^l \times S^{n-l-1}$ which covers ψ.

Suppose $n = 2l$. In this case, the Mayer-Vietoris sequence for $S^l \times D^l \cup_{\overline{\psi}} S^l \times D^l = M$ shows that $H_l(M) = Z + Z$, and the only other nonzero homology groups are $H_0(M) = H_{2l}(M) = Z$.

Thus, if $n = 2l$, $\overline{P(\psi)}$ has the homology of $S^l \times S^l$.

Now suppose $n = 2l + 1$. As above, the Mayer-Vietoris sequence for $S^l \times D^{l+1} \cup_{\overline{\psi}} S^l \times D^{l+1} = M$ shows easily that $H_0(M) = H_{2l+1}(M) = Z$ and $H_q(M) = 0$ for $q \neq 0, l, l + 1$, or $2l + 1$. The center part of the sequence is

$$0 \longrightarrow H_{l+1}(M) \longrightarrow H_l(S^l \times S^l) \xrightarrow{i_*} H_l(S^l \times D^{l+1}) + H_l(S^l \times D^{l+1})$$
$$\longrightarrow H_l(M) \longrightarrow 0.$$

Let x_1 and x_2 denote the obvious generators of $H_l(S^l \times S^l) = Z + Z$, and y_1 and y_2 denote the generators of the two $H_l(S^l \times D^{l+1})$. The map i_* is induced by the inclusion $i_1 : S^l \times S^l \longrightarrow S^l \times D^{l+1}$ into the first term and by $i_1 \circ \overline{\psi}$ into the second term. The map $i_{1*}: H_l(S^l \times S^l) \longrightarrow H_l(S^l \times D^{l+1})$, is clearly given by $i_{1*}(x_1) = y_1, i_{1*}(x_2) = 0$.

Since $\overline{\psi}$ is a diffeomorphism, $\overline{\psi}_*: H_l(S^l \times S^l) \longrightarrow H_l(S^l \times S^l)$ is an isomorphism. Suppose $\overline{\psi}_* x_1 = ax_1 + bx_2, \overline{\psi}_* x_2 = cx_1 + dx_2$. Then the map i_* is given by

$$i_*(x_1) = y_1 + ay_2, \qquad i_*(x_2) = cy_2.$$

There are three possibilities for the homology of M, depending on the integer c.

If $|c| > 1$, then ry_2 is not in the image of i_* for $0 < r < |c|$, but $|c|y_2$ is in the image. Thus, $H_{l+1}(M) = 0$, and $H_l(M) = Z_{|c|}$.

If $|c| = 1$, then i_* is onto, so $H_l(M)$ must be 0. Also, i_* is one-to-one, so $H_{l+1}(M) = 0$. Thus, in this case, M has the homology of S^{2l+1}.

Finally, suppose $c = 0$. Since $\overline{\psi}_*$ is an isomorphism, its matrix is invertible over the integers. For $c = 0$, the diagonal entries are $1/a$ and $1/d$. Thus, a and d are 1 or -1. Thus, $i_*(x_1) = y_1 \pm y_2$, $i_*(x_2) = 0$. In the sequence, this makes $H_1(M) = H_{l+1}(M) = Z$, so M has the homology of $S^l \times S^{n-l}$.

Define Diff $S(\gamma)$ to be the group of diffeomorphisms of $S(\gamma)$ onto itself, with operation composition of maps. Define $\text{Diff}^+ S(\gamma)$ to be the subset of Diff $S(\gamma)$ consisting of those diffeomorphisms ψ such that the double cover $\overline{P(\psi)}$ of $P(\psi)$ has the homology of $S^l \times S^{n-l}$. From the above discussion, we see that $\psi \in \text{Diff}\, S(\gamma)$ is in $\text{Diff}^+ S(\gamma)$ if and only if either

n is even, or n is odd and $\bar{\psi}_*\colon H_l(S^l \times S^l) \longrightarrow H_l(S^l \times S^l)$ has matrix of the form $\left[\begin{smallmatrix} a & 0 \\ b & d \end{smallmatrix}\right]$. But the set of such matrices is a group under multiplication, so $\mathrm{Diff}^+ S(\gamma)$ is a subgroup of $\mathrm{Diff}\, S(\gamma)$.

Suppose Q is in the image under P of $\mathrm{Diff}^+ S(\gamma)$. Then Q satisfies all of the hypotheses for manifolds in $I_n(k)$, except that it is not obvious whether $\omega^{l+1} = 0$.

Denote the double cover of Q by M. There is a cohomology spectral sequence for the covering map $M \longrightarrow Q$. The sequence has $E_2^{p,q} = H^p(P_\infty; H^q(M; Z_2))$, and E_∞ gives $H^*(Q; Z_2)$ (see [8]).

If $n = 2l + 1$, the spectral sequence shows that either $H^l(Q; Z_2) = H^{l+1}(Q; Z_2) = Z_2$, in which case $\omega^{l+1} = 0$, or $H^l(Q; Z_2) = H^{l+1}(Q; Z_2) = Z_2 + Z_2$, in which case $\omega^{l+1} \neq 0$. Suppose the latter holds for Q. Since Q is in the image of P, $Q = E(\gamma) \cup_\psi E(\gamma)$. Denoting the Thom space of γ by $T(\gamma)$, the cohomology sequence for the pair $(Q, E(\gamma))$ yields $H^{l+1}(T(\gamma); Z_2) \longrightarrow Z_2 + Z_2 \longrightarrow 0$. But by the Thom isomorphism theorem, $H^{l+1}(T(\gamma); Z_2) = Z_2$, so this is impossible. Thus, the only possible case is $\omega^{l+1} = 0$.

If $n = 2l$, the spectral sequence shows that if $H^l(M) \neq \overline{Z + Z}$, then $H^*(Q; Z_2)$ is generated by $\omega \in H^1(Q; Z_2)$ and $x \in H^l(Q; Z_2)$, with the relations $\omega^{l+1} = 0$, $x^2 = 0$. Thus, in this case, $\omega^{l+1} = 0$. Notice that this same argument shows that $J_{2l}(k) = I_{2l}(k)$ if l is even or k is even. If $H^l(M) = \overline{Z + Z}$, then the spectral sequence shows that $H^l(Q; Z_2) = Z_2 + Z_2$, and the same argument as used in the $n = 2l + 1$ case shows that this is impossible.

It follows that $P(\mathrm{Diff}^+ S(\gamma)) = I_n(k)$, and since every manifold in $I_n(k)$ can be obtained by the pasting construction, $P^{-1}(I_n(k)) = \mathrm{Diff}^+ S(\gamma)$.

We can now show that the two definitions of $I_n(k)$ are equivalent. It is obvious that if Q is the quotient of a homotopy $S^l \times S^{n-l}$ by a fixed point free involution, Q has type k, and $\omega^{l+1} = 0$ for Q, then Q is a member of $I_n(k)$ as defined at the beginning of this chapter. We must show that if $Q \in I_n(k)$ as defined in this chapter, then the double cover M of Q is homotopy equivalent to $S^l \times S^{n-l}$.

If $n = 2l + 1$, then $M = S^l \times D^{l+1} \cup_{\bar{\phi}} S^l \times D^{l+1}$, where $\bar{\phi}_*\colon H_l(S^l \times S^l) \longrightarrow H_l(S^l \times S^l)$ has matrix

$$\begin{pmatrix} \pm 1 & 0 \\ b & \pm 1 \end{pmatrix}.$$

It follows that there is an embedded S^{l+1} in M representing the generator of $H_{l+1}(M) = Z$. Of course, there is also an embedded S^l in M representing the generator of $H_l(M)$. By the Whitney procedure, we may assume these spheres intersect only once, and transversally. Thus, these spheres give an embedding $S^l \vee S^{l+1} \subset M$. Then

$$H_i(M - S^l \vee S^{l+1}) = H^{2l+1-i}(M, S^l \vee S^{l+1}) = 0 \quad \text{for } i \neq 0,$$

and since $M - S^l \vee S^{l+1}$ is simply connected, it follows that $M - S^l \vee S^{l+1}$ is contractible. Thus, M is homotopy equivalent to $E(S^l \vee S^{l+1}) \cup D^{2l+1}$, where the disk is attached along $\partial E(S^l \vee S^{l+1}) = S^{2l}$, so M is homotopy equivalent to $S^l \times S^{l+1}$.

Similarly, if $n = 2l$, M is homotopy equivalent to $S^l \times S^l$. It follows that both definitions of $I_n(k)$ are equivalent.

2.3. *$\lambda(n, k)$-cobordism.* We want to define an abelian semigroup $I'_n(k)$. Since surgery will be used in the definition, we shall begin by stating the surgeries which are permissible.

Let $k\xi \colon P_\infty \longrightarrow BO$ be the classifying map of k times the canonical line bundle over P_∞. The lth stage of the Moore-Postnikov decomposition is defined by a commutative diagram

$$\begin{tikzcd}
& F' \arrow[dl] \arrow[dr] & \\
F \arrow[rr] \arrow[d] & & P_\infty \arrow[d] \arrow[dd, bend left=90, "k\xi"] \\
F'' \arrow[rr] & & P[l] \arrow[d, "\lambda(n, k)"] \\
& & BO
\end{tikzcd}$$

where F is the fibre of $k\xi$ made into a fibration, F'' is the fibre of $\lambda(n, k)$, and F' is the fibre of the maps $P_\infty \longrightarrow P[l]$ and $F \longrightarrow F''$. Also, there are isomorphisms $\pi_i(F) \longrightarrow \pi_i(F'')$ for $i \leqslant l$, and $\pi_i(F') \longrightarrow \pi_i(F)$ for $i > l$.

Thomas [16] shows that such a decomposition exists whenever the fundamental group of the classifying space acts trivially on the fibre F. If k is even, this is certainly true, since $k\xi$ is orientable, so we may classify it by mapping to BSO, which has trivial fundamental group. It is not so obvious, however, for k odd.

First, observe that the fibre F can be considered to be SO. We want to show that the action of $\pi_1(BO) = Z_2$ on SO is trivial.

Define $J \in O$ to be the infinite diagonal matrix whose first k diagonal elements are -1, and whose other diagonal elements are all 1. Let E be a contractible space on which 0 acts freely on the right. The action of J on E generates the following commutative diagram:

$$\begin{tikzcd}
SO \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "\cdot J"] & E/SO = BSO \arrow[d, "\cdot J"] \\
SO \cdot J \arrow[r] & E \arrow[r] & E/SO
\end{tikzcd}$$

The map $E \longrightarrow BSO$ is a fibration with fibre SO. Consider Z_2 to be $\{1, J\}$. Dividing the contractible space E by the Z_2-action must give P_∞ as quotient. Dividing BSO by the action of Z_2 gives $E/O = BO$. The resulting map $P_\infty \longrightarrow BO$ is the classifying map $k\xi$.

Thus, the quotient of the above diagram by the action of $\{1, J\}$ gives $SO \longrightarrow P \xrightarrow{k\xi} BO$, where SO is the fibre of the fibration $k\xi$. Hence, we have

$$\begin{tikzcd}
SO \arrow[d] & SO \arrow[l] \arrow[d] \\
P_\infty \arrow[d] & E \arrow[l, "{[\,]}"'] \arrow[d] \\
BO & BSO \arrow[l, "\pi"']
\end{tikzcd}$$

Let $g\colon I \longrightarrow BO$ represent the nontrivial element of $\pi_1(BO)$, where I is the closed unit interval. Let $\tilde{g}\colon I \longrightarrow P_\infty$ be the lift of g. We obtain a commutative diagram

$$\begin{tikzcd}
I \times SO \arrow[r, "G"] \arrow[d] \arrow[rr, bend left, "\bar{G}"] & P_\infty \arrow[d] & E \arrow[l, "{[\,]}"'] \arrow[d, "\pi"] \\
I \arrow[r, "g"] \arrow[ur, "\tilde{g}"] \arrow[urr, "\bar{g}"'] \arrow[rr, bend right, "g'"'] & BO & BSO \arrow[l]
\end{tikzcd}$$

where the maps are defined as follows:

Map $\bar{g}$ is the lift of $\tilde{g}$, and has the properties $\bar{g}(0) = *$ (the base point of E), $\bar{g}(1) = * \cdot J$.

Map $g' = \pi \circ \bar{g}$.

Map G is defined by $\bar{G}(t, \sigma) = \bar{g}(t) \cdot \sigma$, and $G(t, \sigma) = [\bar{G}(t, \sigma)]$.

Let $i\colon SO \longrightarrow P_\infty$ be the inclusion of the fibre over the base point. That is, $i(\sigma) = G(0, \sigma) = [* \cdot \sigma]$. The action of the nontrivial element of $\pi_1(BO)$ on SO takes σ to

$$\begin{aligned} i^{-1}(G(1, \sigma)) &= i^{-1}([\bar{g}(1) \cdot \sigma]) = i^{-1}([* \cdot J \cdot \sigma]) \\ &= i^{-1}([* \cdot J \cdot \sigma \cdot J]) = J \cdot \sigma \cdot J. \end{aligned}$$

Thus, the action of $\pi_1(BO)$ on SO is conjugation by J.

Since the matrix $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is homotopic to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and k is odd, conjugation by J is homotopic to conjugation by j, where j is the infinite diagonal matrix with first element -1 and all other diagonal elements 1. If A is a matrix whose only nonzero entry in the first row and first column is a 1 in their intersection, then $jAj = A$. Let $i\colon SO \longrightarrow SO$ take the matrix $[B]$ to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right]$, and $i_n\colon SO(n) \longrightarrow SO(n+1)$ be the corresponding map. Then for $m + 1 < n$, we have

$$\begin{tikzcd} 0 = \pi_{m+1}(SO(n+1)/SO(n)) \arrow[d] & \\ \pi_m(SO(n)) \arrow[r, "\cong"] \arrow[d, "i_{n*}", "\cong"'] & \pi_m(SO) \arrow[d, "i_*"] \\ \pi_m(SO(n+1)) \arrow[r, "\cong"] \arrow[d] & \pi_m(SO) \\ 0 = \pi_m(SO(n+1)/SO(n)) & \end{tikzcd}$$

By taking n large, we see that i is a homotopy equivalence. It follows that conjugation by j is homotopic to the identity.

Thus, $\pi_1(BO)$ acts trivially on $F = SO$, so the Moore-Postnikov decomposition for $k\xi\colon P_\infty \longrightarrow BO$ exists.

Suppose Q is manifold whose stable normal bundle is classified by $\nu(Q)$: $Q \longrightarrow BO$. In order to do surgery, we require that there be a unique homotopy class of liftings $\nu'(Q)$,

$$\begin{tikzcd} & P[l] \arrow[d, "\lambda(n, k)"] \\ Q \arrow[ur, "\nu'(Q)"] \arrow[r, "\nu(Q)"'] & BO \end{tikzcd}$$

We further require that the cobordism manifold realizing each surgery be a $\lambda(n, k)$-cobordism, that is, the cobordism manifold has such a lifting of its normal bundle which restricts to $\nu'(Q)$ on Q.

Let $S^r \longrightarrow Q$ be an embedded sphere on which we want to do surgery, $r \leqslant l$. Since $\pi_i(F') = 0$ for $i \leqslant l-1$, there is no obstruction to lifting the map $S^r \longrightarrow P[l]$ to P_∞. But any map $S^r \longrightarrow P_\infty$ with $r > 1$ is homotopically trivial, so the restriction of the normal bundle of Q to S^r is stably trivial. Thus, $\nu(S^r\colon Q)$, the normal bundle of S^r *in* Q, is stably trivial, and trivial if $r < n/2$. By Levine [11] and Lashof surgery [15], if $S^r \longrightarrow Q$ has $\nu(S^r\colon Q)$ trivial, there is a unique S^r-surgery that is a $\lambda(n, k)$-cobordism. In the case $r = 1$, we do require $S^1 \longrightarrow P[l]$ to be trivial, since it is not automatically so.

2.4. *The semigroup* $I'_n(k)$. Now we begin to define the abelian semigroup $I'_n(k)$.

We shall start by defining an operation on two elements of $I_n(k)$.

Suppose $Q_1, Q_2 \in I_n(k)$. First, we perform a 0-surgery to obtain their connected sum, which we denote $Q_1 \#_0 Q_2$. By van Kampen's theorem, $\pi_1(Q_1 \#_0 Q_2)$ is the free group with two generators a and b, corresponding to the generators of $\pi_1(Q_1)$ and $\pi_1(Q_2)$, with the relations $a^2 = b^2 = 1$.

The double cover of $Q_1 \#_0 Q_2$ is a double connected sum, $M_1 \widetilde{\#}_0 M_2$. That is, M_1 and M_2 are joined along two n-cells rather than one. We can think of this as joining M_1 and M_2 by a handle $S^{n-1} \times D^1$, forming a simply connected manifold, and then adding another handle. By van Kampen's theorem again, $\pi_1(M_1 \widetilde{\#}_0 M_2) = Z$.

Let $h\colon S^1 \longrightarrow Q_1 \#_0 Q_2$ be an embedding of a sphere which represents ab. Since a and b both map to the nontrivial element in $BZ_2 = P_\infty$, ab maps to the trivial element. Thus, the double cover of this $h(S^1)$ is trivial and consists of two disjoint copies of S^1. Therefore, h lifts to $\bar{h}$, an embedding of S^1 in $M_1 \widetilde{\#}_0 M_2$, and since ab is indivisible, $\bar{h}$ must represent a generator of $\pi_1(M_1 \widetilde{\#}_0 M_2)$.

Performing surgery on $h(S^1) \subset Q_1 \#_0 Q_2$ kills $ab \in \pi_1(Q_1 \#_0 Q_2)$, making the fundamental group Z_2. This corresponds to performing two surgeries in the covering manifold $M_1 \widetilde{\#}_0 M_2$. The first kills the entire fundamental group, and the second surgery must add a generator to the second homotopy group, making this group Z.

Denote the surgered manifold by $Q_1 \#_1 Q_2$ and its double cover by $M_1 \widetilde{\#}_1 M_2$.

From the exact sequences for the pairs

$$(M_1 \widetilde{\#}_r M_2, M_1 \widetilde{\#}_{r-1} M_2 - (S^r \times D^{n-r}) - \rho(S^r \times D^{n-r})) \qquad \text{for } r = 0, 1$$

(where $\widetilde{\#}_{-1}$ is disjoint union), we see that the Z_2-action on $H_1(M_1 \widetilde{\#}_0 M_2)$ is multiplication by -1, and the action on $H_2(M_1 \mathbin{\tilde{\#}_1} M_2)$ is trivial. That is,

$$H_1(M_1 \mathbin{\tilde{\#}_0} M_2) = \bar{Z}, \qquad H_2(M_1 \mathbin{\tilde{\#}_1} M_2) = Z.$$

Then we may choose a map $S^2 \longrightarrow M_1 \mathbin{\tilde{\#}_1} M_2$ generating $H_2(M_1 \mathbin{\tilde{\#}_1} M_2)$, project it to a map $S^2 \longrightarrow Q_1 \#_1 Q_2$, homotope this to an embedding and surger this S^2. Continuing this process, we obtain manifolds $M_1 \mathbin{\tilde{\#}_r} M_2, r < l-1$, with homology up to the middle dimension given by

$$\begin{aligned} H_0(M_1 \mathbin{\tilde{\#}_r} M_2) &= Z, \\ H_{r+1}(M_1 \mathbin{\tilde{\#}_r} M_2) &= Z \quad \text{for } r \text{ even}, \bar{Z} \text{ for } r \text{ odd}, \\ H_l(M_1 \mathbin{\tilde{\#}_r} M_2) &= H_l(M_1) + H_l(M_2), \end{aligned}$$

and

$$H_q(M_1 \mathbin{\tilde{\#}_r} M_2) = 0 \quad \text{if } q < l \text{ and } q \neq 0 \text{ or } r+1.$$

We may also form $M_1 \mathbin{\tilde{\#}_{l-1}} M_2$, but the homology groups are not as obvious.

Define

$$\begin{aligned} M- &= M_1 \mathbin{\tilde{\#}_{l-2}} M_2 - (S^{l-1} \times D^{n-l+1}) - \rho(S^{l-1} \times D^{n-l+1}) \\ &= M_1 \mathbin{\tilde{\#}_{l-1}} M_2 - (D^l \times S^{n-l}) - \rho(D^l \times S^{n-l}). \end{aligned}$$

Suppose $n = 2l$. By Lefschetz duality,

$$H_{l+1}(M-) \cong H_{l-1}(M_1 \mathbin{\tilde{\#}_{l-1}} M_2, S^l + \rho S^l),$$

and it follows from the fact that $H_{l-1}(M_1 \mathbin{\tilde{\#}_{l-1}} M_2) = 0$ that $H_{l+1}(M-) = 0$. The exact homology sequence for the pair $(M_1 \mathbin{\tilde{\#}_{l-2}} M_2, M-)$ shows $H_l(M-) = Z + Z + Z + Z + Z$, which we denote by $5Z$. Finally, the sequence for $(M_1 \mathbin{\tilde{\#}_{l-1}} M_2)$ shows that $H_l(M_1 \mathbin{\tilde{\#}_{l-1}} M_2) = 6Z$.

Similarly, if $n = 2l + 1$, we obtain $H_l(M_1 \mathbin{\tilde{\#}_{l-1}} M_2) = 3Z$.

Thus, we obtain $Q_1 \#_{l-1} Q_2$ with double cover $M_1 \mathbin{\tilde{\#}_{l-1}} M_2$ having homology

$$H_q(M_1 \mathbin{\tilde{\#}_{l-1}} M_2) = \begin{cases} Z & \text{for } q = 0, n, \\ 6Z & \text{for } q = l \text{ if } n = 2l, \\ 3Z & \text{for } q = l \text{ or } l+1 \text{ if } n = 2l+1, \\ 0 & \text{otherwise.} \end{cases}$$

The action of Z_2 on $H_l(M_1 \mathbin{\widetilde{\#}_{l-1}} M_2)$ is not obvious.

Note that in each case we are performing surgery on a specific element in a homotopy group of dimension less than l. Thus, these surgeries are uniquely defined, and the results do not depend on the embeddings of spheres chosen. In other words, Q_1 and Q_2 uniquely determine $Q_1 \#_{l-1} Q_2$. If we were to continue the surgeries and surger an embedded S^l, this would not necessarily be the case.

Denote $Q_1 \#_{l-1} Q_2$ by $Q_1 + Q_2$. This is a well-defined operation on any two elements of $I_n(k)$, but the sum is not in $I_n(k)$. Using the same method, we can define finite sums of elements of $I_n(k)$. Let Q_0 denote P(identity). We shall say two finite sums Q and Q' are equivalent if there are nonnegative integers r and s such that $Q + rQ_0 = Q' + sQ_0$. Define $I'_n(k)$ to be the set of equivalence classes.

The operation of addition on elements of $I_n(k)$ is clearly commutative. To see that it is associative, suppose $P(\theta)$, $P(\phi)$, and $P(\psi)$ are elements of $I_n(k)$. Notice that to obtain $E(\gamma) \cup_\theta E(\gamma) + E(\gamma) \cup_\phi E(\gamma)$, for example, we only surger spheres of dimension less than l, so we can move our embeddings of spheres so that they miss a tubular neighborhood of a P_l embedded in each of the two manifolds. Thus, we can perform our surgeries in just one of the $E(\gamma)$'s in each. Thus, we may write

$$E(\gamma) \underset{\theta}{\cup} E(\gamma) + E(\gamma) \underset{\phi}{\cup} E(\gamma) = E(\gamma) \underset{\theta}{\cup} (E(\gamma) + E(\gamma)) \underset{\phi}{\cup} E(\gamma).$$

Then

$$\begin{aligned} E(\gamma) \underset{\theta}{\cup} E(\gamma) &+ E(\gamma) \underset{\phi}{\cup} E(\gamma) + E(\gamma) \underset{\psi}{\cup} E(\gamma) \\ &= E(\gamma) \underset{\theta}{\cup} (E(\gamma) + E(\gamma)) \underset{\phi}{\cup} (E(\gamma) + E(\gamma)) \underset{\psi}{\cup} E(\gamma), \end{aligned}$$

regardless of the order in which the two additions are performed.

It follows that addition of equivalence classes in $I'_n(k)$, defined in the obvious way, is well defined, associative, and commutative. The class of Q_0 is an identity.

Thus, $I'_n(k)$ is an abelian semigroup with identity.

2.5. *The homomorphism* Φ. We have defined a map P of $\mathrm{Diff}^+S(\gamma)$ onto $I_n(k)$. There is the obvious inclusion of $I_n(k)$ into $I'_n(k)$. Denote the composition of these maps by Φ.

The purpose of this section is to prove the following theorem.

THEOREM 2. *If* $n \geq 6$, *the map* $\Phi\colon \mathrm{Diff}^+S(\gamma) \longrightarrow I'_n(k)$ *is a homomorphism.*

We begin by examining the operation of addition on elements of $I'_n(k)$ in more detail. Recall that addition of representatives can be carried out by doing surgeries on only an $E(\gamma)$ in each.

There are canonical embeddings $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_l \subset E(\gamma)$. Recall $\nu(P_l: E(\gamma)) = \gamma$. Define $\beta_r = \nu(P_r: E(\gamma)) = \gamma | P_r + \nu(P_r: P_l)$, for $r \leqslant l$.

Since $P_l \subset E(\gamma)$ is a homotopy equivalence, the canonical embedding $P_l \longrightarrow P_\infty$ gives a unique homotopy class $\nu'(E(\gamma))$ of liftings of the normal bundle of $E(\gamma)$:

$$\begin{tikzcd}
P_l \arrow[r] \arrow[dd] & P_\infty \arrow[d] \arrow[dd, bend left=60, "k\xi"] \\
 & P[l] \arrow[d, "\lambda(n,k)"] \\
E(\gamma) \arrow[ur, "\nu'(E(\gamma))"] \arrow[r, "\nu(E(\gamma))"] & BO
\end{tikzcd}$$

Since γ is stably $(2^{\phi(l)} - l - 1 - k)\xi_l$ and $\nu(P_r: P_l) = (l - r)\xi_r$, these bundles map stably to bundles over P_∞, so the classifying maps of their stable normal bundles lift uniquely to $P[l]$. Thus, there is a lifting of $\nu(P_r) = \nu(E(\gamma))|_{P_r} + \beta_r$. We may also use this normal bundle to define a connected sum of $E(\gamma) \times 0$ and $E(\gamma) \times I$ along P_r by removing the interior of $E(\beta_r)$ from each, joining along $S(\beta_r) \times I$, and rounding corners. Denote the connected sum by $E(\gamma) \#_{P_r} E(\gamma)$. The liftings above give us a uniquely defined lifting of $\nu(E(\gamma) \#_{P_r} E(\gamma))$ to $P[l]$.

PROPOSITION 2. *For* $1 \leqslant r \leqslant l$, *there is an embedded* $S^r \subset E(\gamma) \#_{P_{r-1}} E(\gamma)$ *such that surgery of* S^r *gives* $E(\gamma) \#_{P_r} E(\gamma)$.

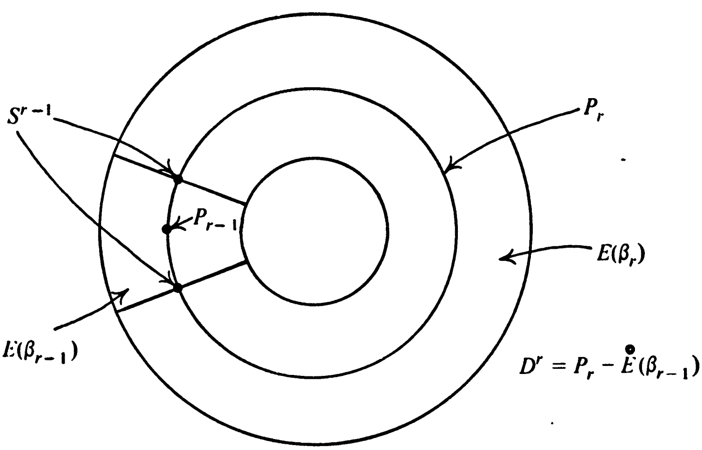

FIGURE 1. $E(\beta_r)$

PROOF. Consider $P_{r-1} \subset P_r$. Then $P_r - \mathring{E}(\beta_{r-1}) = D^r$, and D^r meets $E(\beta_{r-1})$ in $S^{r-1} \subset S(\beta_{r-1})$. This gives an embedded $S^{r-1} \times I \subset S(\beta_{r-1}) \times I$. Thus, we have

$$
\begin{aligned}
S^r &\subset E(\gamma) \#_{P_{r-1}} E(\gamma) \\
&= (E(\gamma) - \mathring{E}(\beta_{r-1})) \times 0 \cup S(\beta_{r-1}) \times I \cup (E(\gamma) - \mathring{E}(\beta_{r-1})) \times 1
\end{aligned}
$$

defined to be $D^r \times 0 \cup S^{r-1} \times I \cup D^r \times 1$. This is the sphere on which we shall perform surgery.

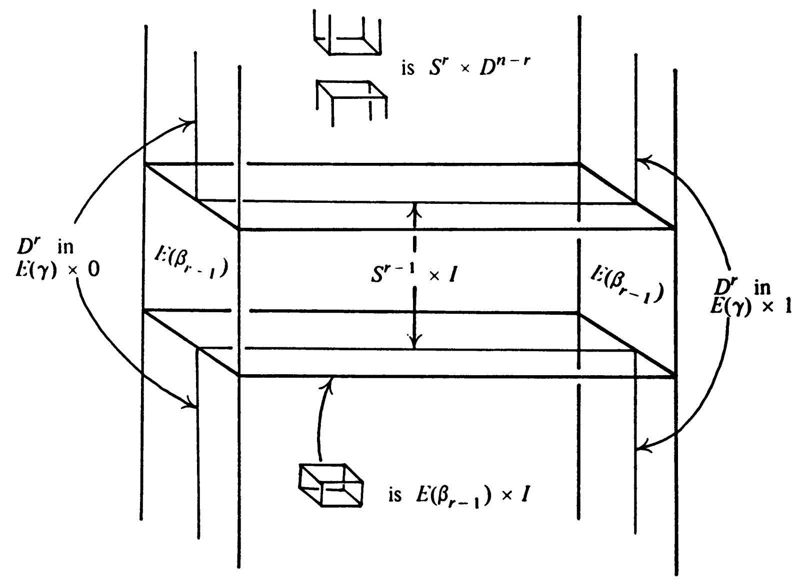

FIGURE 2. Part of $E(\gamma) \#_{P_{r-1}} E(\gamma)$

If $r < n/2$, we have already seen that we can perform surgery on S^r. Even if $r = l$, it is obvious from the construction that the normal bundle of S^l is trivial, so it can be surgered.

Next, we need to show that the result of surgering the above $S^r \subset E(\gamma) \#_{P_{r-1}} E(\gamma)$ is $E(\gamma) \#_{P_r} E(\gamma)$.

The connected sum $E(\gamma) \#_{P_{r-1}} E(\gamma)$ is essentially the boundary of the manifold formed by joining the $E(\beta_{r-1})$ in each $E(\gamma)$ by a tube $E(\beta_{r-1}) \times I$. We must add a handle $S^r \times D^{n-r}$ to this, with S^r as described above. We may identify $D^r \times D^{n-r}$ with $E(\beta_r) - \mathring{E}(\beta_{r-1})$. The result will have the $E(\beta_r)$ in each $E(\gamma)$ joined by a tube $E(\beta_r) \times I$ (see Figure 2). Thus, the result of doing the surgery is the connected sum along P_r, $E(\gamma) \#_{P_r} E(\gamma)$.

This completes the proof of Proposition 2.

Now consider $E(\gamma)\ \#_{P_l}\ E(\gamma)$. This is formed by removing $\overset{\circ}{E}(\beta_l) = \overset{\circ}{E}(\gamma)$ from each $E(\gamma)$ and connecting what is left by a tube $S(\gamma) \times I$. The result is clearly $S(\gamma) \times I$. Thus, $E(\gamma)\ \#_{P_l}\ E(\gamma) = S(\gamma) \times I$.

The operation of connected sum along P_r on $E(\gamma)$'s, as described above, defines an operation on elements of $I_n(k)$ by

$$\left[E(\gamma) \underset{\theta}{\cup} E(\gamma)\right]\ \#_{P_r}\ \left[E(\gamma) \underset{\phi}{\cup} E(\gamma)\right] = E(\gamma) \underset{\theta}{\cup} [E(\gamma)\ \#_{P_r}\ E(\gamma)] \underset{\phi}{\cup} E(\gamma).$$

We may think of $Q_1\ \#_{P_r}\ Q_2$ as consisting of $Q_1 - E(\beta_r)$ and $Q_2 - E(\beta_r)$, joined by a tube $S(\beta_r) \times I$. Thus, the double cover of $Q_1\ \#_{P_r}\ Q_2$, which we denote by $M_1\ \widetilde{\#}_{P_r}\ M_2$, consists of $M_1 - (S^r \times D^{n-r})$ and $M_2 - (S^r \times D^{n-r})$, joined by a tube $S^r \times S^{n-r-1} \times I$. By the Mayer-Vietoris sequence, the homology groups of $M_1\ \widetilde{\#}_{P_r}\ M_2$ are the same as the homology groups of $M_1\ \#_r\ M_2$ for $r \leqslant l-1$.

Clearly, $Q_1\ \#_{P_0}\ Q_2 = Q_1\ \#_0\ Q_2$, since both are connected sums along a point. The surgery which takes $Q_1\ \#_{P_0}\ Q_2$ to $Q_1\ \#_{P_1}\ Q_2$ must be the unique surgery which takes $Q_1\ \#_0\ Q_2$ to $Q_1\ \#_1\ Q_2$, since the surgery kills the correct homology group in the double cover. Similarly, we find $Q_1\ \#_{P_r}\ Q_2 = Q_1\ \#_r\ Q_2$ for all $r \leqslant l-1$.

Notice that $E(\gamma) \cup_\theta E(\gamma)\ \#_{P_l}\ E(\gamma) \cup_\phi E(\gamma)$ is $E(\gamma) \cup_{\theta \times 0} S(\gamma) \times I \cup_{\phi \times 1} E(\gamma)$, which is diffeomorphic to $E(\gamma) \cup_{\phi \circ \theta} E(\gamma)$. Thus, $P(\theta) + P(\phi)$ is a connected sum along P_{l-1}, and there is an S^l in $P(\theta) + P(\phi)$ which we may surger to obtain $P(\phi \circ \theta)$.

In order to show that Φ is a homomorphism, we shall show that $P(\theta) + P(\phi)$ is diffeomorphic to $P(\phi \circ \theta) + P(\text{identity})$. Since $P(\phi \circ \theta) + P(\text{identity})$ represents the same element of $I'_n(k)$ as $P(\phi \circ \theta)$, this will be sufficient.

By the above discussion, there are embedded S^l's in $P(\theta) + P(\phi)$ and in $P(\phi \circ \theta) + Q_0$ such that surgering them takes both manifolds to $P(\phi \circ \theta)$. Thus, there are two surgeries of embedded S^{n-l-1}'s in $P(\phi \circ \theta)$ which reverse the two original surgeries. We shall show that these two reverse surgeries are actually the same.

If n is even, this is easy. Any embedding of $S^{n-l-1} = S^{l-1}$ in $P(\phi \circ \theta)$ is homotopically trivial, since $\pi_{l-1}(P(\phi \circ \theta)) = 0$, and any two embeddings of S^{l-1} are isotopic. Since $l - 1 < l$, there is a unique framing of the embedded S^{l-1} leading to an allowable cobordism. Thus, there is essentially only one surgery of an embedded S^{l-1}, so the two results of surgering S^{l-1} are diffeomorphic.

For the rest of this section we shall assume n is odd. Thus, we are surgering two embedded S^l's in $P(\phi \circ \theta)$.

Denote either $P(\theta) + P(\phi)$ or $P(\phi \circ \theta) + Q_0$ by Q, and its double cover by $\bar{Q}$. Let X be the manifold realizing the cobordism between Q and $P(\phi \circ \theta)$, and let $\bar{X}$ be its double cover, which is a cobordism between $\bar{Q}$ and $\overline{P(\phi \circ \theta)}$, the double cover of $P(\phi \circ \theta)$. Then the S^l we are surgering in $P(\phi \circ \theta)$ bounds a D^{l+1} in X. We have the following commutative diagram:

$$\begin{tikzcd}
\overline{P(\phi \circ \theta)} \arrow[d] & \subset & \bar{X} \arrow[d] \\
P(\phi \circ \theta) & \subset & X \\
\cup & & \cup \\
S^l \arrow[uuu, bend left] & \subset & D^{l+1} \arrow[uuu, bend right]
\end{tikzcd}$$

The homology sequences for the pairs $(\bar{X}, \bar{Q})$ and $(\bar{X}, \overline{P(\phi \circ \theta)})$ show $H_{l+1}(\bar{X}) = 3Z$, $H_l(\bar{X}) = Z$, and $\partial\colon H_{l+1}(\bar{X}, \overline{P(\phi \circ \theta)}) \longrightarrow H_l(\overline{P(\phi \circ \theta)})$ is a zero map. Since the class of S^l in $H_l(\overline{P(\phi \circ \theta)})$ maps to the zero class in $H_l(\bar{X})$, the class of S^l is in the image of ∂, which is 0. Thus, S^l represents the zero homology class in $H_l(\overline{P(\phi \circ \theta)})$, so it represents the zero homotopy class in $\pi_l(\overline{P(\phi \circ \theta)})$ and, hence, in $\pi_l(P(\phi \circ \theta))$. Thus, the spheres we are surgering are both homotopically trivial.

In some cases, we shall see that the spheres are isotopically trivial, so that we are surgering isotopic spheres. In the remaining cases, we shall see that there is a diffeomorphism, orientation preserving when necessary, of $P(\phi \circ \theta)$ onto itself which takes one of the embedded spheres that we wish to surger onto the other one. Thus, in these cases, we are surgering, up to diffeomorphism, the same spheres.

There are four cases to consider, which we may treat in pairs. Either ρ is orientation preserving or reversing, and each of these can occur with l even or odd.

First, suppose either ρ preserves orientation and l is odd, or ρ reverses orientation and l is even.

To show each sphere is isotopically trivial, it is sufficient to show that each embedding $g\colon S^l \longrightarrow P(\phi \circ \theta) \cong P(\phi \circ \theta) \times \{0\}$ on which we wish to perform surgery extends to an embedding $G\colon D^{l+1} \longrightarrow P(\phi \circ \theta) \times I$ (see [9]). Each map g is covered by two disjoint embeddings $\bar{g}, \rho\bar{g}\colon S^l \longrightarrow \overline{P(\phi \circ \theta)} \times \{0\}$. By Haefliger's theorem, if $l \geqslant 2$, these extend to embeddings $\bar{G}, \rho\bar{G}\colon D^{l+1} \longrightarrow \overline{P(\phi \circ \theta)} \times I$. The difficulty is that $\bar{G}$ and $\rho\bar{G}$ may not be disjoint and, hence, the immersion G may not be an embedding.

Thus, we have two embeddings $\bar{G}, \rho\bar{G}$, and we are interested in their intersection. We may assume that they intersect transversely in a finite number of points. Restricted to ∂D^{l+1}, $\bar{G}$ and $\rho\bar{G}$ are disjoint embeddings of S^l in $P(\phi \circ \theta) \times \{0\}$. Thus, we have classes $[\bar{G}(D^{l+1})]$, $[\rho\bar{G}(D^{l+1})]$ in $H_{l+1}(\overline{P(\phi \circ \theta)} \times I, \overline{P(\phi \circ \theta)} \times \{0\}) = 0$. It follows that the intersection number $[\bar{G}(D^{l+1})] \cdot [\rho\bar{G}(D^{l+1})] = 0$. We want to use the Whitney procedure to remove the intersections equivariantly.

Suppose x and y are distinct intersection points of $\bar{G}(D^{l+1})$ and $\rho\bar{G}(D^{l+1})$, and that the intersection numbers at x and y have opposite signs. If $y \neq \rho(x)$, we may use the Whitney procedure to remove the double points and at the same time equivariantly remove the double points $\rho(x)$ and $\rho(y)$. (See [20].) Note, however, that in the two cases we are considering, the intersection numbers at x and $\rho(x)$ are the same. Thus, $y \neq \rho(x)$, so we may use this method to remove all double points. Finally, we obtain an embedding $G: D^{l+1} \longrightarrow P(\phi \circ \theta) \times I$, as required.

In the other two cases, the intersection numbers at x and $\rho(x)$ have opposite signs, so this method fails.

Suppose, then, that either ρ is orientation preserving and l is even, or ρ reverses orientation and l is odd.

A generic immersion is one with only a finite number of double points, each of which is self-transverse, and no triple points. Suppose $e_0, e_1: S^l \longrightarrow P(\phi \circ \theta)$ are the two homotopic embeddings on which we want to perform surgery. The homotopy may be realized as a generic immersion $f: S^l \times I \longrightarrow P(\phi \circ \theta) \times I$ such that $f|S^l \times \{i\}: S^l \times \{i\} \longrightarrow P(\phi \circ \theta) \times \{i\}$ is $e_i \times \{i\}$ for $i = 0, 1$. We want to remove the double points of f.

We begin by defining a self-intersection invariant for generic immersions $f: S^l \times I \longrightarrow P(\phi \circ \theta) \times I$. Let $x_1, \cdots, x_r, y_1, \cdots, y_r$ be the double points in $S^l \times I$, that is, $f(x_j) = f(y_j)$ for $j = 1, \cdots, r$. Fix a base point $*$ in $S^l \times \{0\}$. Choose paths α_j, disjoint except at $*$, from x_j to y_j by way of $*$. By taking a path from $f(*)$ to the base point of $P(\phi \circ \theta)$, each $F \circ \alpha_j$ defines an element of $\pi_1(P(\phi \circ \theta) \times I) = Z_2$. Thus, the immersion f determines an element $\Sigma_{j=1}^{r} f \circ \alpha_j$ of $Z(\pi_1(P(\phi \circ \theta)))$. This, in turn, determines a well-defined element $\iota(f)$ of $Z_2(\pi_1(P(\phi \circ \theta)))$. If we denote the elements of $\pi_1(P(\phi \circ \theta))$ by 1 and σ, then $\iota(f)$ can be $0, 1, \sigma$, or $1 + \sigma$.

If $\iota(f) = 0$, then we may use the Whitney procedure to remove pairs of double points which determine the same element of $\pi_1(P(\phi \circ \theta))$. Thus, in this case, we may make f an embedding.

If $\iota(f) = 1$, we may take a standard "figure eight" immersion of S^{l+1} in a cell with one self-intersection, and make this immersion in $P(\phi \circ \theta) \times (0, 1)$ disjoint from $f(S^l \times I)$. Taking a connected sum of the two immersions gives a new immersion f' which agrees with f on $S^l \times \{0\}$ and $S^l \times \{1\}$, and with $\iota(f') = 0$. Thus, we can again make f an embedding. Similarly, if $\iota(f) = 1 + \sigma$, we may change f so that $\iota(f) = \sigma$.

Thus, we must only examine the case where $\iota(f) = \sigma$. To remove the double point in this case, we must construct a "generalized figure eight" immersion.

First, we must define a self-intersection invariant for a different case than the one for which it is defined above. Suppose $g\colon S^{l+1} \longrightarrow N$, where N is a manifold of dimension $2l + 2$, is a generic immersion with exactly one double point. Let $g(x) = g(y)$ be the double point. Then, as before, choose a path α from x to y, and $g \circ \alpha$ determines a conjugacy class in $\pi_1(N)$. Interchanging x and y and reversing α gives the inverse conjugacy class. Define $\iota(g)$ to be the union of these conjugacy classes.

Suppose l is even, $l \geqslant 4$. Then we shall construct a generic immersion $g\colon S^{l+1} \longrightarrow S^1 \times D^{2l+1}$ with exactly one double point such that $\iota(g) = \{1, -1\} \subset Z = \pi_1(S^1 \times D^{2l+1})$.

Consider D^{2l} to be a subset of $R^l \times R^l$, and define an involution $r\colon S^1 \times D^{2l} \longrightarrow S^1 \times D^{2l}$ by $r(x, y, z) = (-x, z, y)$. Define E to be the quotient of $S^1 \times D^{2l}$ by the action of r. Then E is the disk bundle of $l\xi_1 + l$. Since l is even, E is equivalent to $S^1 \times D^{2l}$.

Define g_1 to be the composition of the inclusion $S^1 \times D^l \times \{0\} \subset S^1 \times D^{2l}$ with the quotient by the action of r. Thus, we have an immersion $g_1\colon S^1 \times D^l \longrightarrow S^1 \times D^{2l}$, the restriction of g_1 to $S^1 \times S^{l-1}$ is an embedding of $S^1 \times S^{l-1}$ in $S^1 \times S^{2l-1}$, and g_1 has self-transverse self-intersection $S^1 \times \{0\}$. We may take

$$g_1|\{1\} \times S^{l-1}\colon \{1\} \times S^{l-1} \longrightarrow \{1\} \times S^{2l-1}$$

to be the standard embedding, where $1 \in S^1$.

Next, make g_1 into a generic immersion

$$g_2\colon S^1 \times D^l \longrightarrow S^1 \times D^{2l+1} = S^1 \times D^{2l} \times I$$

with one double point by using a smooth function $S^1 \times D^l \longrightarrow [0, 1]$ which distinguishes all but one pair of antipodal points of $S^1 \times \{0\}$. We may assume that $g_2|\{1\} \times D^l\colon \{1\} \times D^l \longrightarrow \{1\} \times D^{2l+1}$ is the standard embedding.

Let $S^1 \times D^l \times *$ and $S^1 \times D^{2l+1} \times *$ be copies of $S^1 \times D^l$ and $S^1 \times D^{2l+1}$. Since $l > 2$, by obstruction theory and Haefliger's theorem, the embedding $(g_2|S^1 \times S^{l-1}) \times *$ can be extended to an embedding

$$h\colon S^1 \times D^l \times * \longrightarrow S^1 \times D^{2l+1} \times *.$$

Again, we may assume that $h|\{1\} \times D^l \times *$ is the standard embedding. Then

$$g_2 \cup h\colon (S^1 \times D^l) \cup (S^1 \times D^l \times *) \longrightarrow (S^1 \times D^{2l+1}) \cup (S^1 \times D^{2l+1} \times *)$$

gives an immersion $g_3\colon S^1 \times S^l \longrightarrow S^1 \times S^{2l+1}$ with exactly one double point such that

(i) if x, y is the double point and α is a path from x to y, then the conjugacy class of $g_3 \circ \alpha$ is either $\{1\}$ or $\{-1\}$.

(ii) $g_3|\{1\} \times S^l\colon \{1\} \times S^l \longrightarrow \{1\} \times S^{2l+1}$ is the standard embedding.

By (ii), we may surger $\{1\} \times S^l$ to obtain an immersion $g_4\colon S^{l+1} \longrightarrow S^1 \times S^{2l+1}$ with a single double point $g_4(x) = g_4(y)$. The path α may be chosen not to meet $\{1\} \times S^l$. It follows that $\iota(g_4) = \{1, -1\}$. Finally, we may move $g_4(S^{l+1})$ to be disjoint from $S^1 \times$ point $\subset S^1 \times S^{2l+1}$, so $g_4(S^{l+1}) \subset S^1 \times D^{2l+1} \subset S^1 \times S^{2l+1}$. Thus, g_4 determines the required immersion $g\colon S^{l+1} \longrightarrow S^1 \times D^{2l+1}$ with $\iota(g) = \{1, -1\}$.

If l is odd, $l \geqslant 3$, then the same argument gives a generic immersion $g\colon S^{l+1} \longrightarrow E(\xi_1 + 2l)$ such that $\iota(g) = \{1, -1\}$.

Now, let $f\colon S^l \times I \longrightarrow P(\phi \circ \theta) \times I$ be the generic immersion considered above which we are trying to make an embedding. The only case which remains to be considered is that in which f has just one double point and $\iota(f) = \sigma$. Then we just add the proper generalized figure eight immersion constructed above, take a connected sum, and apply the Whitney procedure to remove the double point. Thus, if $l \geqslant 3$, we may realize the homotopy between e_1 and e_2 by an embedding $f\colon S^l \times I \longrightarrow P(\phi \circ \theta) \times I$.

By Hudson [9], for $l \geqslant 3$, there is a diffeomorphism $\mathcal{D}\colon P(\phi \circ \theta) \times I \longrightarrow P(\phi \circ \theta) \times I$ such that $\mathcal{D}(x, 0) = (x, 0)$ and $\mathcal{D}(e_1(x), 1) = (e_2(x), 1)$. Define $D\colon P(\phi \circ \theta) \longrightarrow P(\phi \circ \theta)$ by $\mathcal{D}(x, 1) = (D(x), 1)$. Then $D(e_1(x)) = e_2(x)$, so D is the required diffeomorphism.

Thus, the two spheres we must surger in $P(\phi \circ \theta)$ to obtain $P(\theta) + P(\phi)$ and $P(\phi \circ \theta) + Q_0$ are essentially the same. It only remains to be shown that both of these spheres have the same framing.

Let P denote $\overline{P(\phi \circ \theta)}$ and N denote either $\overline{P(\theta) + P(\phi)}$ or $\overline{P(\phi \circ \theta) + Q_0}$. Then P can be surgered to N by surgering two homotopically trivial spheres. We know

$$H_q(P) = \begin{cases} Z, & q = 0, l, l+1, 2l+1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$H_q(N) = \begin{cases} Z, & q = 0, 2l+1, \\ 3Z, & q = l, l+1, \\ 0, & \text{otherwise}. \end{cases}$$

First, suppose that $n = 2l + 1$, where l is odd. We want to show that if we surger the S^l's in P with the wrong framing, we will obtain a manifold with homology different from that of N. Let P' be the result after surgering two spheres in P, say the ones given by embeddings $\psi: S^l \times D^{l+1} \longrightarrow P$ and $\rho\psi$, where ψ *and* $\rho\psi$ are disjoint and homotopic to 0.

Let $P_0 = P - \psi(S^l \times D^{l+1})^\circ - \rho\psi(S^l \times D^{l+1})^\circ$. If λ denotes the element of $H_l(P)$ corresponding to $\psi|S^l \times 0$, and λ' denotes the element of $H_l(P')$ corresponding to $\psi|0 \times S^l$, we obtain the Kervaire-Milnor [10] exact sequences

$$\begin{tikzcd}
 & & H_{l+1}P' \arrow[d, "\cdot\lambda' + \cdot\rho\lambda'"] & & \\
 & & Z + Z \arrow[d, "\epsilon"] & & \\
H_{l+1}P \arrow[r, "\cdot\lambda + \cdot\rho\lambda"] & Z + Z \arrow[r, "\epsilon'"] & H_lP_0 \arrow[r, "i"] \arrow[d, "i'"] & H_lP \arrow[r] & 0 \\
 & & H_lP' \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where ϵ' is the boundary map $\epsilon': H_{l+1}(P, P_0) = Z + Z \longrightarrow H_l(P_0)$, etc.

Since λ is actually 0 in our case, the horizontal sequence is

$$0 \longrightarrow Z + Z \longrightarrow H_l(P_0) \longrightarrow Z \longrightarrow 0,$$

so $H_l(P_0) = Z + Z + Z = 3Z$. Also, $\epsilon' = \epsilon'(1, 1)$ corresponds to the meridians $\psi(* \times S^l) + \rho\psi(* \times S^l)$ of tori $\psi(S^l \times S^l)$ and $\rho\psi(S^l \times S^l)$, where $*$ is a base point.

Suppose that ψ has the framing that makes $P' = N$. In this case, the vertical sequence becomes

$$3Z \longrightarrow 2Z \xrightarrow{\epsilon} 3Z \longrightarrow 3Z \longrightarrow 0.$$

Thus, ϵ in H_lP_0 is 0.

Now suppose we modify ψ to ψ_α, where $\alpha: S^l \longrightarrow SO(l + 1)$, defined by $\psi_\alpha(u, v) = \psi(u, \alpha(u)v)$, for $u \in S^l, v \in D^{l+1}$. The effect of this is to replace ϵ by $\epsilon_\alpha = \epsilon + j_*(\alpha)\epsilon'$, where $j_*: \pi_l(SO(l + 1)) \longrightarrow \pi_l(S^l)$. The integer $j_*(\alpha)$ can be any multiple of 2.

In this case, the vertical sequence becomes

$$H_{l+1}P' \longrightarrow 2Z \xrightarrow{\epsilon_\alpha} 3Z \longrightarrow H_lP' \longrightarrow 0,$$

where $\epsilon_\alpha = \epsilon + j_*(\alpha)\epsilon' = j_*(\alpha)\epsilon'$. Thus, if $j_*(\alpha) = 2m$, ϵ_α: $2Z \longrightarrow 3Z$ takes (1, 1) to $(2m, 2m, 0)$. It follows that $H_lP' = Z + Z_{2m} + Z_{2m}$. But H_lN has no torsion, so P' can have the homology of N only if $m = 0$. Thus, there is only one framing which gives P'. It follows that N is unique so $P(\phi \circ \theta) + Q_0$ and $P(\theta) + P(\phi)$ are diffeomorphic.

Next, suppose that l is even. Since P is diffeomorphic to $S^{2l+1} \# P \# S^{2l+1}$, we may consider the two trivial l-spheres we must surger to obtain N each to be embedded in an S^{2l+1}. The surgery of an S^l embedded in S^{2l+1} yields the total space of a bundle over S^{l+1} with fibre S^l. By obstruction theory, we see that there are at most two nonequivalent such bundles. One is the trivial bundle with total space $S^l \times S^{l+1}$, which we shall denote by M, and the other is the sphere bundle of the tangent bundle of S^{l+1}, whose total space we shall denote by Λ. Thus, $N = M \# P \# M$ or $N = \Lambda \# P \# \Lambda$. We want to show that only the first of these occurs, regardless of whether N is $\overline{P(\theta) + P(\phi)}$ or $\overline{P(\phi \circ \theta) + Q_0}$.

We begin by defining an invariant which will distinguish Λ from M. Let α be an embedding of S^{l+1} in some manifold X, which may be Λ, M, or N. The normal bundle of S^{l+1} in X is classified by a map of S^{l+1} into $BSO(l)$ and, hence, by an element of $\pi_{l+1}(BSO(l)) = \pi_l(SO(l))$. Since this normal bundle is stably trivial, this element of $\pi_l(SO(l))$ comes from an element of $\pi_{l+1}(SO/SO(l))$. Finally, this element is mapped to an element, say $\nu(\alpha) \in \pi_{l+1}(SO/SO(l+1)) = Z_2$ by the projection $SO/SO(l) \longrightarrow SO/SO(l+1)$. Of course, it is not obvious that $\nu(\alpha)$ is well defined, since an element in $\pi_l(SO(l))$ may have more than one pre-image in $\pi_{l+1}(SO/SO(l))$.

If $l > 13$, then the Barratt-Mahowald theorem [4] shows that

$$\pi_l(SO(l)) \cong \pi_l(SO) + \pi_{l+1}(SO/SO(l)),$$

and, hence, an element in $\pi_l(SO(l))$ which maps to 0 in $\pi_l(SO)$ has a unique pre-image. If $l \equiv 4 \pmod 8$, then $\pi_{l+1}(SO) = 0$, so the map $\pi_{l+1}(SO/SO(l)) \longrightarrow \pi_l(SO(l))$ is one-to-one. If $l = 8$, then $\pi_8(SO(8)) = Z_2 + Z_2 + Z_2$, and again $\pi_{l+1}(SO/SO(l)) \longrightarrow \pi_l(SO(l))$ is one-to-one. If $l = 10$, then $\pi_{11}(SO/SO(10)) = Z_4$, and since $\pi_{10}(SO(10)) \neq 0$, we see that even if an element in $\pi_{10}(SO(10))$ had two pre-images, both would map to the same element in $\pi_{11}(SO/SO(11)) = Z_2$. Thus $\nu(\alpha)$ is well-defined for all even $l \geqslant 4$ except $l = 6$. If $l = 6$, only M can occur, so we do not need to define $\nu(\alpha)$.

Thus, we have ν: $\pi_{l+1}(X) \longrightarrow Z_2$. If α, β are disjoint embeddings of S^{l+1} in X, then $\nu(\alpha + \beta) = \nu(\alpha) + \nu(\beta)$. If α amd β are not disjoint, then there is an immersion δ of S^{l+1} in a cell such that $\alpha + \beta + \delta$ is isotopic to the disjoint connected sum of α and β, and $\nu(\delta) = 0$. (See [20].) Thus, $\nu(\alpha + \beta) = \nu(\alpha + \beta) + \nu(\delta) = \nu(\alpha + \beta + \delta) = \nu(\alpha) + \nu(\beta)$. It follows that the mapping ν is a homomorphism.

We shall see that $\nu\colon \pi_{l+1}(\Lambda) \longrightarrow Z_2$ is onto, but $\nu\colon \pi_{l+1}(M) \longrightarrow Z_2$ is the zero map.

Since $l+1$ is odd, both bundles have a cross section, which we shall denote by c in both cases. Let $\nu'(c)$ denote the classifying map of the normal bundle of this cross section of Λ. Then $\nu'(c) + 1$ classifies $\tau(S^{l+1})$, and $\nu'(c) + 2$ is trivial. Thus, the inclusion-induced map $\pi_l(SO(l)) \longrightarrow \pi_l(SO(l+1))$ takes $\nu'(c)$ to the class of $\tau(S^{l+1})$, which is nontrivial. It follows from the commutativity of the diagram

$$\begin{tikzcd}
\pi_{l+1}(SO/SO(l)) \arrow[r] \arrow[d] & \pi_l(SO(l)) \arrow[d] \\
\pi_{l+1}(SO/SO(l+1)) \arrow[r] & \pi_l(SO(l+1))
\end{tikzcd}$$

that any pre-image of $\nu'(c)$ must map to the nonzero element of $\pi_{l+1}(SO/SO(l+1)) = Z_2$. Thus, $\nu\colon \pi_{l+1}(\Lambda) \longrightarrow \pi_{l+1}(SO/SO(l+1))$ is onto.

On the other hand, $\pi_{l+1}(M) = Z + Z_2$ is generated by the trivial section $c\colon S^{l+1} \longrightarrow S^{l+1} \times S^l$, and by the Hopf map $\alpha\colon S^{l+1} \longrightarrow S^l \subset S^{l+1} \times S^l$. Since c is a trivial cross section, $\nu(c)$ is zero. We have

$$\begin{tikzcd}
S^{l+1} \arrow[r, "\alpha"] \arrow[rr, bend right, "\nu'(\alpha)"'] & S^l \arrow[r] \arrow[rr, bend left, "0"] & BSO(l) \arrow[r] & BSO(l+1)
\end{tikzcd}$$

Thus, $\nu'(\alpha)$ maps to zero in $\pi_{l+1}(BSO(l+1)) = \pi_l(SO(l+1))$. It follows from the exactness of the horizontal sequences and the commutativity of the diagram

$$\begin{tikzcd}
\pi_{l+1}(SO) \arrow[r] \arrow[d, equal] & \pi_{l+1}(SO/SO(l)) \arrow[r] \arrow[d] & \pi_l(SO(l)) \arrow[d] \\
\pi_{l+1}(SO) \arrow[r] & \pi_{l+1}(SO/SO(l+1)) \arrow[r] & \pi_l(SO(l+1))
\end{tikzcd}$$

that $\nu(\alpha)$ is 0. Thus, $\nu\colon \pi_{l+1}(M) \longrightarrow Z_2$ is the zero map.

Since P is homotopy equivalent to $S^l \times S^{l+1}$, it follows by taking generators $S^{l+1} \longrightarrow P$ for $\pi_{l+1}(P)$ which factor through the $(l+1)$-skeleton $S^l \vee S^{l+1}$ that ν is also zero on $\pi_{l+1}(P)$. Thus, it will be sufficient to show that ν is zero on $\pi_{l+1}(N)$.

Recall N is the double cover of a connected sum of two manifolds in $I_n(k)$, say A and B, along P_{l-1}. Equivalently, N is a connected sum of the double covers, $\bar{A}$ and $\bar{B}$, of A and B along S^{l-1}. Thus, N is a union of $\bar{A}-S^{l-1}$ and $\bar{B}-S^{l-1}$ along $S^{l-1}\times S^{l+1}\times(0,1)$. Since $\bar{A}$ and $\bar{A}-S^{l-1}$ are simply connected, and the first nonzero group $\pi_i(\bar{A},\bar{A}-S^{l-1})$ occurs when $i=l+2$, we have an exact sequence

$$\pi_{l+1}(\bar{A}-S^{l-1})\longrightarrow\pi_{l+1}(\bar{A})\longrightarrow 0.$$

We may define ν on $\pi_{l+1}(\bar{A}-S^{l-1})$ in the obvious manner, and obtain a commutative diagram

$$\begin{tikzcd}
\pi_{l+1}(\bar{A}-S^{l-1}) \arrow[r] \arrow[dr, "\nu"'] & \pi_{l+1}(\bar{A}) \arrow[r] \arrow[d, "\nu"] & 0 \\
 & Z_2 &
\end{tikzcd}$$

Since $\bar{A}$ is homotopy equivalent to $S^l\times S^{l+1}$, as before ν is zero on $\pi_{l+1}(\bar{A})$. I claim it must also be zero on $\pi_{l+1}(\bar{A}-S^{l-1})$.

Indeed, suppose S^{l+1} is embedded in $\bar{A}-S^{l-1}$. Then we may take a tubular neighborhood of S^{l+1} in $\bar{A}$ which is entirely contained in $\bar{A}-S^{l-1}$. Thus, the normal bundle of S^{l+1} in $\bar{A}-S^{l-1}$ is the same as that of S^{l+1} in $\bar{A}$, so ν is the same in both cases. It follows that ν is zero on $\pi_{l+1}(\bar{A}-S^{l-1})$, and similarly is zero on $\pi_{l+1}(\bar{B}-S^{l-1})$.

Since S^{l-1} in $\bar{A}-S^{l-1}$ and $\bar{B}-S^{l-1}$ is homotopically trivial, the union of $\bar{A}-S^{l-1}$ and $\bar{B}-S^{l-1}$ along S^{l-1} is homotopic to $(\bar{A}-S^{l-1})\vee S^l\vee(\bar{B}-S^{l-1})$. By examining the homology of N and $(A-S^{l-1})\cup_{S^{l-1}}(\bar{B}-S^{l-1})$, we see that

$$H_i\ (N,(\bar{A}-S^{l-1})\cup_{S^{l-1}}(\bar{B}-S^{l-1}))=0\quad\text{for } i\leqslant l+1,$$

so the map $\pi_{l+1}(\bar{A}-S^{l-1}\cup_{S^{l-1}}\bar{B}-S^{l-1})\longrightarrow\pi_{l+1}(N)$ is onto.

Also, $\pi_{l+1}(\bar{A}-S^{l-1}\cup_{S^{l-1}}\bar{B}-S^{l-1})$ equals $\pi_{l+1}(\bar{A}-S^{l-1})+\pi_{l+1}(S^l)+\pi_{l+1}(\bar{B}-S^{l-1})$. We have already seen that ν is zero on the first and last of these groups, and that $\nu(\alpha)=0$, where α is the Hopf map, the generator of $\pi_{l+1}(S^l)$. Thus, ν is zero on $\pi_{l+1}(\bar{A}-S^{l-1}\cup_{S^{l-1}}\bar{B}-S^{l-1})$, and we have a commutative diagram

$$\begin{tikzcd}
\pi_{l+1}(\bar{A}-S^{l-1}\cup_{S^{l-1}}\bar{B}-S^{l-1}) \arrow[r] \arrow[dr, "0"'] & \pi_{l+1}(N) \arrow[r] \arrow[d, "\nu"] & 0 \\
 & Z_2 &
\end{tikzcd}$$

It follows that ν is 0 on N, so N must be the connected sum $M \# P \# M$ in both cases. Thus, the framing is the same for the S^l's surgered to get either $P(\theta) + P(\phi)$ or $P(\phi \circ \theta) + Q_0$ from $P(\phi \circ \theta)$ and, hence, both surgeries are the same. Finally, we conclude that $P(\theta) + P(\phi) = P(\phi \circ \theta) + Q_0$, so Φ: $\mathrm{Diff}^+ S(\gamma) \longrightarrow I'_n(k)$ is a homomorphism for $n \geqslant 6$, establishing Theorem 2.

Notice that Φ is also onto. In fact, it follows from the fact that the class of $P(\theta) + P(\phi)$ equals the class of $P(\phi \circ \theta) + Q_0$ in $I'_n(k)$, that each class in $I'_n(k)$ has a representative which is a member of $I_n(k)$.

Thus, $I'_n(k)$ is a group, not just a semigroup.

Furthermore, not only does each class in $I'_n(k)$ have a representative in $I_n(k)$, but this representative is unique. It is sufficient to show that if $Q_1 \#_{P_{l-1}} Q_0$ is diffeomorphic to $Q_2 \#_{P_{l-1}} Q_0$, where Q_0 is P(identity), then Q_1 and Q_2 are diffeomorphic.

Suppose, then, that $Q_1 \#_{P_{l-1}} Q_0$ is diffeomorphic to $Q_2 \#_{P_{l-1}} Q_0$. We may assume the diffeomorphism takes $E(\gamma) \#_{P_{l-1}} E(\gamma) \subset Q_1 \#_{P_{l-1}} Q_0$ to $E(\gamma) \#_{P_{l-1}} E(\gamma) \subset Q_2 \#_{P_{l-1}} Q_0$. Extending the connected sum along P_l takes $Q_1 \#_{P_{l-1}} Q_0$ to Q_1 and $Q_2 \#_{P_{l-1}} Q_0$ to Q_2. If there were only one way to extend the connected sum, we would have Q_1 and Q_2 diffeomorphic. There are, however, two ways to extend it . We shall see that the two extensions give diffeomorphic manifolds.

Let $E(\gamma) \subset Q_i$ for $i = 0, 1$ and 2. We may form the connected sum along P_l of Q_j and $Q_0, j = 1$ or 2, either by using the identity $E(\gamma) \longrightarrow E(\gamma)$ or another linear diffeomorphism σ: $E(\gamma) \longrightarrow E(\gamma)$. Observe that $Q_0 = S(\gamma + 1)$. There is an obvious diffeomorphism $S(\sigma + 1)$: $S(\gamma + 1) \longrightarrow S(\gamma + 1)$ which restricts to σ: $E(\gamma) \longrightarrow E(\gamma)$. Thus, if we form the connected sum $Q_j \#_{P_l} Q_0$, $j = 1$ or 2, by using the identity or σ, there is a diffeomorphism $\mathrm{id} \# S(\sigma + 1)$ between them. Hence, there is, up to diffeomorphism, only one connected sum along P_l of Q_j and $Q_0, j = 1$ or 2, as required.

Thus, the inclusion of $I_n(k)$ in $I'_n(k)$ is a bijection. We may give $I_n(k)$ the abelian group structure which makes this map an isomorphism.

This completes the proof of Theorem 1.

2.6. *Consequences of the involution group theorem.* Now we shall examine some of the consequences of the fact that Φ is a homomorphism and $I_n(k)$ is an abelian group.

If α is a diffeomorphism of $E(\gamma)$ onto itself, let $\partial\alpha$ be the restriction of α to $S(\gamma)$. Define the group $G(\gamma)$ to be the image of ∂: $E(\gamma) \longrightarrow S(\gamma)$.

We may observe (e.g., see Kai Wang's thesis [17]) that $P(\phi)$ is diffeomorphic to $P(\psi)$ if and only if ϕ and ψ are in the same double coset in $G(\gamma)\backslash \mathrm{Diff}\, S(\gamma)/G(\gamma)$. To see this, notice that if $P(\phi)$ is diffeomorphic to $P(\psi)$, then we may choose a diffeomorphism $A: E(\gamma) \cup_\phi E(\gamma) \longrightarrow E(\gamma) \cup_\psi E(\gamma)$ in such a way that it preserves first and second components. Denote the restriction of A to the first $E(\gamma)$ by α, and its restriction to the second $E(\gamma)$ by β. It follows that $\partial\alpha \circ \phi = \psi \circ \partial\beta$, so ϕ is in the same double coset as ψ. Since each of these steps is reversible, the converse is also true.

We can now prove some corollaries of Theorem 1. Each is true for the dimensions in which Φ is a homomorphism.

COROLLARY 1. *$I_n(k)$ is isomorphic to* $\mathrm{Diff}^+S(\gamma)/G(\gamma)$.

PROOF. Since $\Phi: \mathrm{Diff}^+S(\gamma) \longrightarrow I_n(k)$ is a homomorphism onto $I_n(k)$, it is sufficient to show that $G(\gamma)$ is the kernel of Φ.

Suppose $\Phi(\theta) = Q_0 = P(i)$, where $i: S(\gamma) \longrightarrow S(\gamma)$ is the identity map. It follows from Kai Wang's lemma that there are $\sigma, \tau \in G(\gamma)$ such that $\theta = \sigma \circ i \circ \tau = \sigma \circ \tau$, so $\theta \in G(\gamma)$. On the other hand, if $\theta \in G(\gamma)$, then it is in the same double coset as i, so $P(\theta) = P(i) = Q_0$.

COROLLARY 2. *Any commutator in* $\mathrm{Diff}^+S(\gamma)$ *extends to* $E(\gamma)$.

PROOF. Since $I_n(k)$ is abelian, $G(\gamma)$ contains the commutator subgroup of $\mathrm{Diff}^+S(\gamma)$.

3. Calculation of I_8^+.

3.1. *The method.* In §3, we shall make a partial calculation of I_8^+.

Each element $Q \in I_n(k)$ has a double cover $S^l \times S^{n-l}$ which is classified by a map $Q \longrightarrow P_\infty$. If the normal bundle of Q pulls back from $k\xi$, then there is an alternate method available for computing $I_n(k)$. Of course, the normal bundle does not always pull back from $k\xi$ (see [19]). Wells has shown in [18] that $\nu(Q)$ is stably k times the canonical line bundle when $n \equiv 8, 12 \pmod{16}$, and k is even. We shall examine the case $n = 8$, using the methods of [18].

Suppose $n = 2l$, k is even, and $Q \in I_n(k)$ has normal bundle which is stably a multiple of the canonical line bundle. Then the classifying map of the double cover of Q induces a commutative diagram

$$\begin{array}{ccc} \nu(Q) & \longrightarrow & k\xi + m \\ \downarrow & & \downarrow \\ Q & \xrightarrow{f} & P_\infty \end{array}$$

By the Pontrjagin-Thom construction, this gives a (not necessarily well-defined) element $\widetilde{\alpha}(Q) \in \pi_{m+k+2l}(T(k\xi + m)) = \pi^s_{k+2l}(T(k\xi))$ for m sufficiently large.

We want to examine the indeterminancy in the choice of $\tilde{\alpha}(Q)$. The problem is that there may be more than one bundle map covering $f\colon Q \longrightarrow P_\infty$.

As before, let Q_0 be the identity of $I_n(k)$. Since Q_0 bounds, the Thom construction gives zero for a suitable choice of bundle map $F\colon \nu(Q_0) \longrightarrow k\xi + m$ covering $f\colon Q_0 \longrightarrow P_\infty$. Each map $\beta\colon Q_0 \longrightarrow SO(q)$ defines a twisting of added trivial bundles by defining

$$F_\beta\colon \nu(Q_0) + q \longrightarrow k\xi + m + q,$$

$$F_\beta(x, y) = (F(x), \beta(\pi(x)) \cdot y),$$

where $\pi\colon \nu(Q_0) + q \longrightarrow Q_0$ is the projection. Thus, we have defined a map

$$j: \widetilde{KO}^{-1}(Q_0) = \lim_{q\to\infty} \,[Q_0, SO(q)] \longrightarrow \pi^s_{k+2l}(T(k\xi)).$$

Let $\Lambda_{n,k}$ denote the cokernel of j. Then the projection $\alpha(Q)$ of $\tilde{\alpha}(Q)$ in $\Lambda_{n,k}$ is well defined.

The map j factors through $\pi^s_{k+2l}(T(k\xi_l))$, giving

$$\begin{tikzcd} \widetilde{KO}^{-1}(Q_0) \arrow[r, "j_l"] \arrow[rr, bend right, "j"'] & \pi^s_{k+2l}(T(k\xi_l)) \arrow[r] & \pi^s_{k+2l}(T(k\xi)) \end{tikzcd}$$

For any manifold M, let SM denote the suspension of M. We may now prove:

THEOREM 3. *$\widetilde{KO}^{-1}(Q_0)$ is isomorphic to $\widetilde{KO}^{-1}(P_l) + \widetilde{KO}^{-1}(T(\gamma))$. The kernel of $j_l\colon \widetilde{KO}^{-1}(Q_0) \longrightarrow \pi^s_{k+2l}(T(K\xi_l))$ is $\widetilde{KO}^{-1}(P_l) + \ker J$, where J is the map $\widetilde{KO}^{-1}(T(\gamma)) \cong KO(ST(\gamma)) \longrightarrow \tilde{J}(ST(\gamma))$.*

PROOF. If $\beta \in \widetilde{KO}^{-1}(Q_0)$, then $\beta\colon Q_0 \longrightarrow SO(q)$ for some q. Let $E = E(\gamma + 1)$, so the boundary of E is $S(\gamma + 1) = Q_0$. Then we have

$$\begin{tikzcd} \nu(E) \supset \nu(Q_0) \arrow[r, "\bar{\pi}"] \arrow[d] & k\xi_l + m \arrow[d] \\ E \supset Q_0 \arrow[r, "\pi"] & P_l \end{tikzcd}$$

Since the Thom construction depends only on the cobordism class of a manifold, $j_l(\beta) = 0$ if and only if there exists a manifold X such that $\partial X = Q_0$, and $\bar{\pi} \times \beta$ extends to a mapping $\Gamma\colon \nu(X) + q \longrightarrow k\xi_l + q$. Since E is homotopy equivalent to P_l, Γ induces a map $\bar{\Gamma}\colon \nu(X) \longrightarrow \nu(E)$.

Define $B(\beta) = E \times R^q \cup_\beta E \times R^q$, where $(x, t) \in M \times R^q$ in the first factor is identified with $(x, \beta(x) \cdot t)$ in the second. Then $\bar{\Gamma} \cup$ identity defines a map $\nu(X \cup E) \longrightarrow \nu(E \cup E) + B(\beta)$, where $X \cup E$ and $E \cup E$ are joined by the identity on Q_0. Since k is even, Q_0 is orientable, so the map $X \longrightarrow E$ has degree one. Thus, $X \cup E \longrightarrow E \cup E$ has degree one.

By [3], the Thom space of any compact connected differentiable manifold is S-reducible. Thus, $T(\nu(X \cup E))$ is S-reducible, and the above degree one map makes $T(\nu(E \cup E) + B(\beta))$ S-reducible. Let δ be a bundle such that $B(\beta) + \delta$ is trivial. Then the S-dual of $T(\nu(E \cup E) + B(\beta))$ is $T(\delta)$, so $T(\delta)$ is S-coreducible. It follows from [3] that δ, and hence $B(\beta)$, is fibre homotopically trivial.

Since the above steps are all reversible, we conclude that $j_l(\beta) = 0$ if and only if $B(\beta)$ is fibre homotopically trivial. Thus, we must calculate the map $B\colon \widetilde{KO}^{-1}(M_1) \longrightarrow \widetilde{KO}(E \cup E)$.

Let the inclusion $c\colon P_l \longrightarrow Q_0$ be a cross section of the bundle $\pi\colon Q_0 \longrightarrow P_l$. Then the K-theory exact sequence for the pair (Q_0, P_l) yields a split short exact sequence

$$0 \longrightarrow \widetilde{KO}^{-1}(Q_0/P_l) \longrightarrow \widetilde{KO}^{-1}(Q_0) \longrightarrow \widetilde{KO}^{-1}(P_l) \longrightarrow 0.$$

Since $Q_0 = E(\gamma) \cup E(\gamma)$ and $E(\gamma)$ is homotopy equivalent to P_l, $\widetilde{KO}^{-1}(Q_0/P_l) = \widetilde{KO}^{-1}(T(\gamma))$. Thus, $\widetilde{KO}^{-1}(Q_0) \cong \widetilde{KO}^{-1}(P_l) + \widetilde{KO}^{-1}(T(\gamma))$.

The suspension isomorphism gives a factorization of B by:

$$\begin{tikzcd}
\widetilde{KO}^{-1}(Q_0) \arrow[r, "\cong"] \arrow[dr, "B"'] & \widetilde{KO}(SQ_0) \arrow[d, "\overline{B}"] \\
 & \widetilde{KO}(E \cup E)
\end{tikzcd}$$

The map $\overline{B}$ is induced by first taking the map $E \longrightarrow CQ_0$ given by pinching the centers of all disks to a point, then using this to define $E \cup E \longrightarrow CQ_0 \cup CQ_0$. This same map induces a map between short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{KO}(SQ_0/SP_l) \arrow[r] \arrow[d] & \widetilde{KO}(SQ_0) \arrow[r] \arrow[d, "\overline{B}"] & \widetilde{KO}(SP_l) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{KO}(E \cup E/P_l) \arrow[r] & \widetilde{KO}(E \cup E) \arrow[r] & \widetilde{KO}(P_l) \arrow[r] & 0
\end{tikzcd}$$

The map $E \cup E/P_l \longrightarrow SQ_0/SP_l$ is homotopic to the identity on $T(\gamma + 1)$ and, hence, induces an isomorphism in the $\widetilde{KO}$-groups. The map $P_l \longrightarrow SP_l$ is homotopically trivial.

Thus, $\widetilde{KO}(E \cup E) = \widetilde{KO}^{-1}(T(\gamma)) + \widetilde{KO}(P_l)$, and the map

$$B\colon \widetilde{KO}^{-1}(Q_0) = \widetilde{KO}^{-1}(T(\gamma)) + \widetilde{KO}^{-1}(P_l)$$

$$\longrightarrow \widetilde{KO}(E \cup E) = \widetilde{KO}^{-1}(T(\gamma)) + \widetilde{KO}(P_l)$$

is the identity on the $\widetilde{KO}^{-1}(T(\gamma))$ term and zero on the other term. This completes the proof of Theorem 3.

3.2. *The calculation.* We are now ready to consider the case $n = 8, k$ even. As an aid to calculation, notice that $T(r\xi_s) = P_{r+s}/P_{r-1}$.

Denote γ for the case $n = 8, k = k$, by γ_k. Then $\gamma_k = \nu(P_4; Q_0)$, and

$$\begin{aligned}
\widetilde{KO}^{-1}(T(\gamma_2)) &= \widetilde{KO}^{-1}(T(\xi_4 + 3)) = \widetilde{KO}^{-4}(P_5) = Z_4,\\
\widetilde{KO}^{-1}(T(\gamma_4)) &= \widetilde{KO}^{-1}(T(7\xi_4 - 3)) = \widetilde{KO}^{-6}(P_{11}/P_6) = Z_2 + Z_2,\\
\widetilde{KO}^{-1}(T(\gamma_6)) &= \widetilde{KO}^{-1}(T(5\xi_4 - 1)) = \widetilde{KO}(P_9/P_4) = Z_4,\\
\widetilde{KO}^{-1}(T(\gamma_8)) &= \widetilde{KO}^{-1}(T(3\xi_4 + 1)) = \widetilde{KO}^{-2}(P_7/P_2) = Z_2 + Z_2.
\end{aligned}$$

The $\widetilde{KO}$-groups for the stunted projective spaces were computed by using the K-theory exact sequence for pairs (P_r, P_s) and determining maps from the K-theory spectral sequence.

Next, we must calculate $\widetilde{J}(ST(\gamma_k))$ for each k. By Quillen's theorem [13], $\widetilde{J}(ST(\gamma_k)) = \widetilde{J}''(ST(\gamma_k))$. To find this, we must know the action of the Adam's ψ-functions on $\widetilde{KO}^{-1}(T(\gamma_k))$.

Claim.

$$\psi^m(x) = \begin{cases} 0, & m \text{ even} \\ x, & m \text{ odd} \end{cases} \quad \text{for } x \in \widetilde{KO}^{-1}(T(\gamma_k)), \quad k = 2, 4, 6, \text{ or } 8.$$

PROOF. $k = 2$. In this case, $\widetilde{KO}^{-1}(T(\gamma_2)) = \widetilde{KO}^{-4}(P_5) = Z_4$. There is a commutative diagram

$$\begin{array}{ccc}
\widetilde{KO}(S^4P_5) & \xrightarrow{\psi^m} & \widetilde{KO}(S^4P_5) \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\
\widetilde{K}(S^4P_5) & \xrightarrow{\psi^m_c} & \widetilde{K}(S^4P_5),
\end{array}$$

and c is an isomorphism (see [6]). On $\widetilde{K}(S^4P_5)$,

$$\psi^m_c x = \begin{cases} 0, & m \text{ even}, \\ m^2x, & m \text{ odd}. \end{cases}$$

But if m is odd, $m^2x \equiv x \pmod 4$, establishing the claim in this case.

$k = 6$. $\widetilde{KO}^{-1}(T(\gamma_6)) = \widetilde{KO}(P_9/P_4) = Z_4$. The action in this case is given by Adams [2].

$k = 4$ or 8. $\widetilde{KO}^{-1}(T(\gamma_4)) = \widetilde{KO}^2(P_{11}/P_6) = Z_2 + Z_2$. There are natural isomorphisms

$$\widetilde{KO}^2(P_{11}/P_6) \cong \widetilde{KO}^2(P_{11}) \cong \widetilde{KO}^2(P_{11}/P_8) = \widetilde{KO}^2(T9\xi_2)$$
$$= \widetilde{KO}^2(T(\xi_2 + 8)) = \widetilde{KO}^{-6}(T\xi_2) = \widetilde{KO}^{-6}(P_3).$$

On the other hand,

$$\widetilde{KO}^{-1}(T(\gamma_8)) = \widetilde{KO}^{-2}(P_7/P_2) \cong \widetilde{KO}^{-2}(P_7/P_4) = \widetilde{KO}^{-2}(T(5\xi_2))$$
$$= \widetilde{KO}^{-2}(T(\xi_2 + 4)) = \widetilde{KO}^{-6}(T\xi_2) = \widetilde{KO}^{-6}(P_3).$$

Thus, in both these cases we need to determine the action of ψ^k on $\widetilde{KO}^{-6}(P_3)$.

Consider the fibration

$$\begin{tikzcd} S^1 \arrow[r] & P_3 \arrow[d, "\pi"] \\ & S^2 \end{tikzcd}$$

induced from

$$\begin{tikzcd} SO(2) \arrow[r] & SO(3) \arrow[d] \\ & S^2 \end{tikzcd}$$

This fibration induces a spectral sequence in the category of Ψ-groups, with Ψ-action from that on $KO^q(S^1)$, and $E_2^{pq} = H^p(S^2; KO^q(S^1))$. The sequence converges to $KO^p(P_3)$. The only nonzero term in the diagonal giving $KO^{-6}(P_3)$ is the $(2, -8)$-entry $Z + Z_2$. The only differential which can hit $Z + Z_2$ comes from the $(0, -7)$-entry $0 + Z$. Evidently, this differential $d\colon 0 + Z \longrightarrow Z + Z_2$ is multiplication by 2 on the Z-term.

$E_2^{2,-8} = Z + Z_2$ is a subquotient of $E_1^{2,-8} = KO^{-6}((P_3)^2, (P_3)^1)$, where $(P_3)^r$ is π^{-1} of the r-skeleton of S^2. Then

$$E_1^{2,-8} = KO^{-6}(D^2 \times S^1, S^1 \times S^1) = \widetilde{KO}^{-6}(S^2 \wedge S^1_+)$$
$$= \widetilde{KO}^{-8}(S^1_+) = KO^{-8}(\text{point}) + \widetilde{KO}^{-8}(S^1) = Z + Z_2,$$

where S^1_+ is the union of S^1 and a base point. The ψ^m-action on $Z = KO^{-8}(\text{point})$ is multiplication by m^4, and the action on Z_2 is given by the identity action if m is odd, zero if m is even.

But d is a Ψ-map, so ψ^m preserves components in $E_2^{2,-8}$ and $E_\infty^{2,-8}$. In fact, ψ^m acts on $E_\infty^{2,-8} = \widetilde{KO}^{-6}(P_3)$ by

$$\psi^m(x) = \begin{cases} 0, & m \text{ even}, \\ x, & m \text{ odd}. \end{cases}$$

Thus, the claim has been established.

In all cases, then,

$$(\psi^m - 1)x = \begin{cases} -x, & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases}$$

In each case, $4x = 0$ for all $x \in \widetilde{KO}^{-1}(T(\gamma_k))$. Define the function $f\colon Z \longrightarrow Z^+$ to be the constant function $f = 2$. It follows that $m^{f(m)}(\psi^m - 1)x = 0$ for all m. Then Adams' $\widetilde{KO}_f(ST(\gamma_k))$ is 0, so $\widetilde{J}(ST(\gamma_k)) = \widetilde{J}''(ST(\gamma_k)) = \widetilde{KO}^{-1}(T(\gamma_k))$. It follows that $\widetilde{KO}^{-1}(P_4)$ is the entire kernel of j_4.

Now we are ready to calculate $\Lambda_{8,k}$. The stable homotopy groups of P_∞/P_r have been calculated by Mahowald [12].

If $k = 4$, then we have $0 = \pi^s_{13}(P_\infty/P_8) \longrightarrow \pi^s_{12}(P_8/P_3) \longrightarrow \pi^s_{12}(P_\infty/P_3)$, so $\pi^s_{12}(T(4\xi_4)) \longrightarrow \pi^s_{12}(T(4\xi))$ is a monomorphism. Hence,

$$\begin{tikzcd} \widetilde{KO}^{-1}(Q_0) \arrow[r] \arrow[d, equal] & \pi^s_{12}(T(4\xi)) \arrow[d, equal] \\ Z_2 + Z_2 + Z_2 \arrow[r] & Z_2 + Z_2 + Z_2 + Z_2 \end{tikzcd}$$

has kernel Z_2. Thus, $\Lambda_{8,4} = Z_2 + Z_2 + Z_2 + Z_2/Z_2 + Z_2 = Z_2 + Z_2$.

If $k = 6$, then from Mahowald's tables, we see $\pi^s_{15}(P_\infty/P_5) \longrightarrow \pi^s_{15}(P_\infty/P_{10})$ is onto. Hence, $\pi^s_{14}(P_{10}/P_5) \longrightarrow \pi^s_{14}(P_\infty/P_5)$ is a monomorphism, and

$$\begin{tikzcd} \widetilde{KO}^{-1}(Q_0) \arrow[r] \arrow[d, equal] & \pi^s_{14}(T(6\xi)) \arrow[d, equal] \\ Z_2 + Z_4 \arrow[r] & Z_2 + Z_4 \end{tikzcd}$$

is 0 on the Z_2 factor and an isomorphism on the Z_4. Thus, $\Lambda_{8,6} = Z_2 + Z_4/Z_4 = Z_2$.

If $k = 8$, then $0 = \pi^s_{17}(P_\infty/P_{12}) \longrightarrow \pi^s_{16}(P_{12}/P_7) \longrightarrow \pi^s_{16}(P_\infty/P_7)$, so the latter map is a monomorphism, and

$$\begin{tikzcd} \widetilde{KO}^{-1}(Q_0) \arrow[r] \arrow[d, equal] & \pi^s_{16}(T(8\xi)) \arrow[d, equal] \\ Z_2 + Z_2 + Z_2 \arrow[r] & Z_2 + Z_2 + Z_2 + Z_2 + Z_2 \end{tikzcd}$$

has kernel Z_2. Thus, $\Delta_{8,8} = Z_2 + Z_2 + Z_2 + Z_2 + Z_2/Z_2 + Z_2 = Z_2 + Z_2 + Z_2$.

If $k = 2$, we can calculate $\pi^s_r(P_6/P_1)$ from the Adams' spectral sequences for $\pi^s_r(P_6)$ and $\pi^s_r(P_1)$, and from the sequence

$$\cdots \longrightarrow \pi^s_r(P_1) \longrightarrow \pi^s_r(P_6) \longrightarrow \pi^s_r(P_6/P_1) \longrightarrow \pi^s_{r-1}(P_1) \longrightarrow \cdots.$$

The results are: $\pi^s_{10}(P_6/P_1) = Z_4 + Z_4$, $\pi^s_9(P_6/P_1) = Z_2 + Z_2 + Z_{16}$,

$\pi_8^s(P_6/P_1) = Z_2 + Z_2 + Z_2$. Putting these in the sequence

$$\cdots \longrightarrow \pi_r^s(P_6/P_1) \longrightarrow \pi_r^s(P_\infty/P_1) \longrightarrow \pi_r^s(P_\infty/P_6) \longrightarrow \pi_{r-1}^s(P_6/P_1) \longrightarrow \cdots,$$

we find that $\pi_{10}^s(P_6/P_1) \longrightarrow \pi_{10}^s(P_\infty/P_1)$ has image Z_4 and kernel Z_4. Thus, we have

$$\begin{array}{ccccc} KO^{-1}(Q_0) & \longrightarrow & \pi_{10}^s(T(2\xi_4)) & \longrightarrow & \pi_{10}^s(T(2\xi)) \\ \| & & \| & & \| \\ Z_2 + Z_4 & \longrightarrow & Z_4 + Z_4 & \longrightarrow & Z_4. \end{array}$$

The kernel of the first map is the Z_2 summand, and the kernel of the second map is Z_4. There are three possibilities for the composition. The image can be Z_4, Z_2, or 0. Thus, $\Lambda_{8,2}$ is $0, Z_2$, or Z_4.

Wells shows in [18] that $I_8(k)$ is isomorphic to $\Lambda_{8,k}$ for k even. Thus, we have found that I_8^+ is the disjoint union of groups

$$A \cup (Z_2 + Z_2) \cup Z_2 \cup (Z_2 + Z_2 + Z_2),$$

where A is $0, Z_2$, or Z_4.

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