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WILD SPHERES IN E^n THAT ARE LOCALLY FLAT MODULO TAME CANTOR SETS

BY

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ABSTRACT. Kirby has given an elementary geometric proof showing that if an $(n-1)$-sphere Σ in Euclidean n-space E^n is locally flat modulo a Cantor set that is tame relative to both Σ and E^n, then Σ *is* locally flat. In this paper we illustrate the sharpness of the result by describing a wild $(n-1)$-sphere Σ in E^n such that Σ is locally flat modulo a Cantor set C and C is tame relative to E^n. These examples then are used to contrast certain properties of embedded spheres in higher dimensions with related properties of spheres in E^3.

Rather obviously, as Kirby points out in [11], his result cannot be weakened by dismissing the restriction that the Cantor set be tame relative to E^n. It is well known that a sphere in E^n containing a wild (relative to E^n) Cantor set must be wild. Consequently the only variation on his work that merits consideration is the one mentioned above.

The phenomenon we intend to describe also occurs in 3-space. Alexander's horned sphere [1] is wild but is locally flat modulo a tame Cantor set. In fact, at one spot methods used here parallel those used to construct that example. However, other properties of 3-space are strikingly dissimilar to what can be derived from the higher dimensional examples constructed here, for, as discussed in §2, natural analogues to some important results concerning locally flat embeddings in E^3 are false.

Most of the terminology and notation is standard. We distinguish between the two senses of the term "boundary" by using ∂M to denote the boundary of a manifold M and $\operatorname{Bd} A$, for $A \subset X$, to denote the boundary of A in the space X. Our standard k-cell B^k is the set of points in E^k of norm ≤ 1. We use ρ to denote the standard complete metric on E^{n+1}, and for two maps f and f' of a space X into E^{n+1}, we use $\rho(f, f')$ to denote $\operatorname{lub}\{\rho(f(x), f'(x)) | x \in X\}$.

1. Construction of certain wild spheres. The somewhat intricate definition and lemma that follow are designed to slip naturally into the proof of Theorem 3

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and could readily have been incorporated there. By isolating the formation of special Cantor sets, however, we profit in two ways, the most obvious being a precise specification of those Cantor sets in the embedded sphere that can be employed; the other, a convenient description of locally flat embeddings that converge to the wild embedding desired in Theorem 3.

Let Q denote an n-manifold, $n \geqslant 3$, and C a Cantor set embedded in Q. A sequence $\{M_i\}_{i=1}^{\infty}$ of compact n-manifolds with boundary in Q is a *defining sequence for C* iff (1) $C = \bigcap M_i$, and (2) $M_{i+1} \subset \operatorname{Int} M_i$ for each i. The sequence $\{M_i\}$ is called a *doubly regular defining sequence for C* if, in addition, the following conditions are satisfied: (3) each component of each M_i is homeomorphic to $B^2 \times T^{n-2}$, where T^{n-2} denotes the Cartesian product of $n-2$ copies of S^1; (4) the inclusion of each boundary component of $Y_i = M_i - \operatorname{Int} M_{i+1}$ into the appropriate component of Y_i induces an injection of fundamental groups; (5) for every odd positive integer i and every component P of M_i, $P \cap M_{i+1}$ consists of exactly two components C_1 and C_2 determined by disjoint subdisks B_1 and B_2 of $\operatorname{Int} B^2$ such that, under some homeomorphism h of $B^2 \times T^{n-2}$ onto P, $C_e = h(B_e \times T^{n-2})$ ($e = 1, 2$), and, furthermore, there exists a homeomorphism g of B_1 onto B_2 reversing the induced orientations on these two disks such that

$$h(g \times 1_{T^{n-2}})h^{-1}(C_1 \cap M_k) = C_2 \cap M_k \quad \text{for all } k > i.$$

REMARK. It would be permissible in condition (3) above to require instead that to each component P of M_i there correspond an $(n-2)$-manifold N_P such that P and $B^2 \times N_P$ are homeomorphic; but we have found it more convenient to specify N_P as T^{n-2}.

LEMMA 1. *For $n \geqslant 3$ there exists a Cantor set C in S^n that has a doubly regular defining sequence.*

PROOF. Consider a sequence $\{N_i\}_{i=1}^{\infty}$ of compact n-manifolds with boundary in S^n such that (1) $\bigcap N_i$ is a Cantor set, (2) $N_{i+1} \subset \operatorname{Int} N_i$, (3) each component of N_i is homeomorphic to $B^2 \times T^{n-2}$, and (4) the inclusion of each boundary component of $Z_i = N_i - \operatorname{Int} N_{i+1}$ into the appropriate component of Z_i induces an injection of fundamental groups. We shall assume the reader is acquainted with Cantor sets in S^n having such defining sequences, for example, Antoine's necklace in S^3 [2] and Blankenship's generalizations in higher dimensional spheres [4]. (Explicit comments regarding why condition (4) applies in these cases can be found in [14].) We shall modify the sequence $\{N_i\}$ to obtain a doubly regular defining sequence for another Cantor set, which might be embedded differently than $\bigcap N_i$.

Let $M_1 = N_1$. Fix two disjoint disks B_1 and B_2 in $\operatorname{Int} B^2$. Then for each component R of M_1 there exists a homeomorphism h_R of $B^2 \times T^{n-2}$ onto R. Define M_2 as the union of the sets $h_R((B_1 \cup B_2) \times T^{n-2})$.

For M_3 we mimic a portion of $\{N_i\}$ in each component of M_2, attending to epsilonics to force $\bigcap M_i$ to be a Cantor set while exercising technical care in light of condition (5). Fix homeomorphisms g_e of B^2 onto B_e $(e = 1, 2)$ such that $g_2 g_1^{-1}: B_1 \longrightarrow B_2$ reverses the orientations induced from B^2, and define $G_e = g_e \times 1_{T^{n-2}}: B^2 \times T^{n-2} \longrightarrow B_e \times T^{n-2}$. For each component R of M_1 we have homeomorphisms $h_R G_e h_R^{-1}$ of R onto components of M_2; thus, there exists an integer $j(R) > 1$ such that for each component X of $R \cap N_{j(R)}$

$$\operatorname{diam} h_R G_e h_R^{-1}(X) < 1/3 \qquad (e = 1, 2).$$

We define M_3 to be the union of the sets $h_R G_e h_R^{-1}(R \cap N_{j(R)})$. Note that the components $C_e = h_R(B_e \times T^{n-2})$ of M_2 satisfy

$$h_R G_2 G_1^{-1} h_R^{-1}(C_1 \cap M_3) = C_2 \cap M_3.$$

Now M_4 is defined by choosing, for each component R' of M_3, two disjoint parallel copies of R' in its interior. Specifically, for each component R_1 of M_1 and each component R_3 of $R_1 \cap N_{j(R_1)}$, there exists a homeomorphism h_{R_3} of $B^2 \times T^{n-2}$ onto R_3. Define M_4 as the union of all such sets

$$h_{R_1} G_e h_{R_1}^{-1} h_{R_3}((B_1 \cup B_2) \times T^{n-2}) \qquad (e = 1, 2).$$

Note again that the components $C_e = h_{R_1}(B_e \times T^{n-2})$ of M_2 satisfy

$$h_{R_1} G_2 G_1^{-1} h_{R_1}^{-1}(C_1 \cap M_4) = C_2 \cap M_4.$$

Given a component R_1 of $N_1 = M_1$ and a component R_3 of $R_1 \cap N_{j(R_1)}$, we have homeomorphisms $h_{R_1} G_{e_1} h_{R_1}^{-1} h_{R_3} G_{e_3} h_{R_3}^{-1}$ $(e_1 = 1, 2; e_3 = 1, 2)$ of R_3 onto components of M_4. Thus, there exists an integer $j(R_3) > j(R_1)$ such that for each component X of $R_3 \cap N_{j(R_3)}$

$$\operatorname{diam} h_{R_1} G_{e_1} h_{R_1}^{-1} h_{R_3} G_{e_3} h_{R_3}^{-1}(X) < 1/5 \qquad (e_1 = 1, 2;\ e_3 = 1, 2),$$

and we define M_5 to be the union of such sets, emphasizing that this be done for each possible R_1 and R_3.

We continue this process, making certain that each component of M_{2i+1} has diameter less than $1/(2i + 1)$. Implicit in our procedure is the requirement that for each component R of M_j, $R \cap M_{j+1} \neq \varnothing$, which implies that $\bigcap M_i$ is a Cantor set. Furthermore, for each odd integer $i > 0$, a set P is a component of M_i iff (1) there exist sets $R_1, R_3, \cdots, R_i$ such that R_1 is a component of N_1

and R_{k+2} is a component of $R_k \cap N_{j(R_k)}$ $(k = 1, 3, \cdots, i-2)$ and (2) there exist homeomorphisms h_{R_k} of $B^2 \times T^{n-2}$ onto R_k $(k = 1, 3, \cdots, i-2)$ such that $P = H_1 H_3 \cdots H_{i-2}(R_i)$, where $H_k = h_{R_k} \cdot G_{e_k} \cdot h_{R_k}^{-1}$ $(e_k = 1, 2)$. In case i is odd each component of M_i contains exactly two components of M_{i+1}, which are obtained by a rule analogous to that given in defining M_2 and M_4. It follows that the resulting sequence $\{M_i\}$ is a doubly regular defining sequence for $\bigcap M_i$; condition (5) is verifiable in straightforward fashion in terms of the specific homeomorphisms concocted to determine components of the M_i's; condition (4) is obvious in case i is odd, and in case i is even it is a consequence of properties of $\{N_j\}$, since to each component Y_i of $M_i - \operatorname{Int} M_{i+1}$ there correspond integers j and k such that Y_i is homeomorphic to a component of $N_j - \operatorname{Int} N_k$.

The following lemma, which is used in proving Corollary 9, is not essential for the main results of the paper.

LEMMA 2. *For $n \geqslant 3$ there exists a Cantor set C^* in S^n such that C^* has a doubly regular defining sequence and $S^n - C^*$ is simply connected.*

PROOF. DeGryse and Osborne [9] have discovered a wild Cantor set A in S^n $(n \geqslant 3)$ having simply connected complement and having a defining sequence $\{N_i\}$ that satisfies conditions (3) and (4) in the definition of "doubly regular defining sequence." Relying largely on their techniques we shall suggest briefly how to establish that a Cantor set C^* constructed from $\{N_i\}$ according to the rules formulated in Lemma 1 also has simply connected complement. In [9] a defining sequence $\{A_i^n\}$ is prescribed for A, and we have set $N_1 = A_0^n$ and $N_j = A_{2j-3}^n$ $(j \geqslant 2)$ because this affords easy application of [9, Theorem 4.13]. Furthermore, following [9], throughout this proof T_i^n denotes an n-tube, which is a space homeomorphic to $B^2 \times T^{n-2}$. The algebraic manipulations of [9] focus on the following definition.

Let $\{T_i^n, i = 1, 2, \cdots, k\}$ be a collection of pairwise disjoint n-tubes in $\operatorname{Int} T_0^n$. Let P be a tree in $T_0^n - \bigcup_{i=1}^k \operatorname{Int} T_i^n$ such that $P \cap \partial T_i^n$ is a single point for $i = 0, 1, \cdots, k$. Let $K_i = \ker(\pi_1(\partial T_i^n) \longrightarrow \pi_1(T_i^n))$ and let G_i be a subgroup of $\pi_1(\partial T_i^n)$ such that $K_i \oplus G_i = \pi_1(\partial T_i^n)$. Denote by H_0 the smallest normal subgroup of $\pi_1(T_0^n - \bigcup_{i=1}^k \operatorname{Int} T_i^n)$ containing $\operatorname{im}(G_0 \longrightarrow \pi_1(T_0^n - \bigcup_{i=1}^k \operatorname{Int} T_i^n))$. The n-tubes $\{T_i^n, i = 1, 2, \cdots, k\}$ are *h-unlinkable in* T_0^n if for every $i = 1, 2, \cdots, k$

$$G_i \subset \ker\left[\pi_1(\partial T_i^n) \longrightarrow \pi_1\left(T_0^n - \bigcup_{i=1}^k \operatorname{Int} T_i^n\right)\Big/ H_0\right]$$

and

$$\pi_1\left(P \cup \bigcup_{i=1}^{k} \partial T_i^n\right) \longrightarrow \pi_1\left(T_0^n - \bigcup_{i=1}^{k} \operatorname{Int} T_i^n\right)\Big/ H_0$$

is an epimorphism.

Using the notation $T_0^n = B^2 \times T^{n-2}$ and $T_e^n = B_e^2 \times T^{n-2}$ $(e = 1, 2)$, we claim that $\{T_1^n, T_2^n\}$ are h-unlinkable in T_0^n. To prove it we can define an appropriate tree P in $B^2 \times t$ $(t \in T^{n-2})$ as the union of two arcs $\alpha_e \times t$, each joining a point of ∂T_0^n to a point of ∂T_e^m. Note that G_0 is generated by loops in $p_0 \times T^{n-2}$, where $p_0 \in \partial B^2$, and that G_e similarly is generated by loops in $p_e \times T^{n-2}$, where $p_e \in \partial B_e$. It follows, by deforming loops in $p_e \times T^{n-2}$ across $\alpha_e \times T^{n-2}$ into $p_0 \times T^{n-2}$, that each loop representing an element of G_e is homotopic in $T_0^n - \operatorname{Int}(T_1^n \cup T_2^n)$ to a loop representing an element of G_0. Furthermore, we see that

$$\pi_1(T_0^n - \operatorname{Int}(T_1^n \cup T_2^n)) \cong \pi_1(B^2 - \operatorname{Int}(B_1 \cup B_2)) \times \pi_1(T^{n-2})$$

and H_0 corresponds to the $\pi_1(T^{n-2})$ factor. Since $B^2 - \operatorname{Int}(B_1 \cup B_2)$ collapses to $(\alpha_1 \cup \alpha_2) \cup (\partial B_1 \cup \partial B_2)$, it follows that

$$\pi_1(P \cup (\partial T_1^n \cup \partial T_2^n)) \longrightarrow \pi_1(T_0^n - \operatorname{Int}(T_1^n \cup T_2^n))/H_0$$

is an epimorphism. This establishes the claim.

We can assume that $A_0^n = N_1$ consists of a single component such that

$$G_0 \subset \ker(\pi_1(\partial A_0^n) \longrightarrow \pi_1(S^n - \operatorname{Int} A_0^n))$$

and that $\{M_i\}$ is the doubly regular defining sequence determined from $\{N_i\}$ by applying Lemma 1. It follows from the claim and repeated applications of [9, Lemma 4.9] that the components of each M_i are h-unlinkable in the n-tube A_0^n. Theorem 4.13 of [9] then implies that the inclusion induced homomorphism $\pi_1(S^n - M_k) \longrightarrow \pi_1(S^n - M_{k+1})$ is trivial for even indices k. This proves that $S^n - C^* = S^n - \bigcap M_i$ is simply connected.

THEOREM 3. *If the Cantor set C in S^n $(n \geqslant 3)$ has a doubly regular defining sequence $\{M_i\}$, then there exists a wild embedding $f\colon S^n \longrightarrow E^{n+1}$ such that f is locally flat at each point of $S^n - C$ and $f(C)$ is tame relative to E^{n+1}.*

PROOF. *Construction of f.* We shall obtain f as the limit of a sequence of embeddings of S^n in E^{n+1}. Throughout the proof all embeddings, excepting f, of manifolds and of manifolds with boundary will be locally flat.

Let f_1 denote an embedding of S^n onto the boundary of a round $(n + 1)$-ball D in E^{n+1}. Appealing to the definition of "doubly regular defining sequence," we can easily obtain an embedding $g_1: I \times M_1 \longrightarrow E^{n+1}$ such that

(6) $g_1(I \times M_1) \cap D = g_1(\partial I \times M_1) = f_1(M_2)$.

To do this, for each component R of M_1, we use the collar structure on S in $E^{n+1} - \operatorname{Int} D$ to "extend" $f_1 h_R: B^2 \times T^{n-2} \longrightarrow f_1(R)$ to an embedding F_R of $I \times B^2 \times T^{n-2}$ into $E^{n+1} - \operatorname{Int} D$ such that

$$F_R(I \times B^2 \times T^{n-2}) \cap S = F_R(\{0\} \times B^2 \times T^{n-2}) = f_1(R)$$

where $F_R((0, b, t)) = f_1 h_R((b, t))$ for $(b, t) \in B^2 \times T^{n-2}$. We require, in addition, that the images of the various F_R be pairwise disjoint. The orientation reversing homeomorphism $g: B_1 \longrightarrow B_2$ (the subdisks of Int B^2) prescribed in the definition of "doubly regular" can be realized in terms of a locally flat embedding ψ of $I \times B^2$ in $I \times B^2$ such that

$$\psi(I \times B^2) \cap \partial(I \times B^2) = \psi(\partial I \times B^2),$$
$$\psi(\{0\} \times B^2) = \{0\} \times B_1,$$
$$\psi(\{1\} \times B^2) = \{0\} \times B_2,$$

where ψ is related to g in the following sense: for each $b \in B^2$, $\psi((0, b)) = (0, b_1)$ and $\psi((1, b)) = (0, g(b_1))$. Define $g_1: I \times M_1 \longrightarrow E^{n+1}$ on each component R of M_1 as $g_1 = F_R(\psi \times 1_{T^{n-2}})(1_I \times h_R^{-1})$. Now condition (5) in the pertinent definition permits us to assert that there is a subset L_1 of M_2, namely that "half" of M_2 corresponding to the images of $B_1 \times T^{n-2}$ under certain homeomorphisms of $B^2 \times T^{n-2}$ onto components of M_1, such that for $k > 2$,

(7) $g_1(\partial I \times (L_1 \cap M_k)) = f_1(M_k)$.

Let $A_1 = g_1(I \times M_1)$. Without loss of generality we may assume

(8) $\operatorname{diam} A_1 < 1$.

There exists an odd positive integer $j(2)$ such that for each component $X_{2,e}$ $(e = 1, \cdots, n_2)$ of $M_{j(2)}$

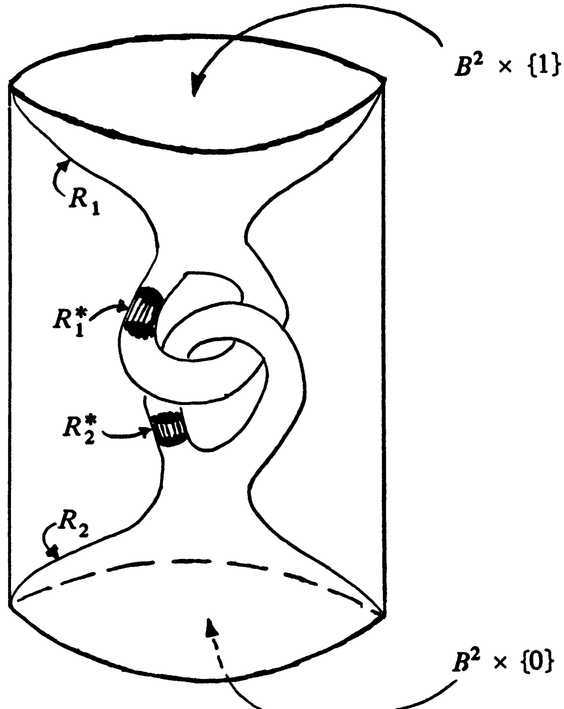
(9) $\operatorname{diam} g_1(t \times X_{2,e}) < ½$ for each $t \in I$.

There exists a homeomorphism $H_{2,e}$ of $I \times B^2 \times T^{n-2}$ onto $g_1(I \times X_{2,e})$ such that

(10) $H_{2,e}(t \times B^2 \times T^{n-2}) = g_1(t \times X_{2,e})$ for each $t \in I$.

Represent $I \times B^2$ as a solid cylinder as shown in the figure, and let R_1 and R_2 denote the cubes-with-one-handle as indicated. Note that the handles in R_e can be made arbitrarily thin, and consequently the solid cylinder R_δ^* can be constructed with arbitrarily small preassigned diameter ($\delta = 1, 2$). In particular, using (9) and (10) we require that

(11) $\operatorname{diam} H_{2,e}(R_\delta^* \times T^{n-2}) < ½$ $(\delta = 1, 2; e = 1, \cdots, n_2)$.



FIGURE

Define

$$A_2 = \bigcup_{e=1}^{n_2} H_{2,e}((R_1^* \cup R_2^*) \times T^{n-2}),$$

$$Q = [(\text{Int}\, I \times B^2)] \cap [\partial\, \text{Cl}\,(R_1 - R_1^*) \cup \partial\, \text{Cl}\,(R_2 - R_2^*)],$$

$$Q^* = [R_1^* \cap \text{Cl}\,(R_1 - R_1^*)] \cup [R_2^* \cap \text{Cl}\,(R_2 - R_2^*)].$$

Here Q is the union of the interiors of two disks while Q^* is the union of four disks. We see that there exists a homeomorphism f_2 of S^n onto

$$f_1(S^n - \text{Int}\, M_{j(2)}) \cup \left(\bigcup_{e=1}^{n_2} H_{2,e}(Q \times T^{n-2})\right)$$

such that

(12) $f_2 | S^n - \text{Int}\, M_{j(2)} = f_1 | S^n - \text{Int}\, M_{j(2)}$,

(13) $f_2(M_{j(2)+1}) = \bigcup_{e=1}^{n_2} H_{2,e}(Q^* \times T^{n-2})$.

It follows from (8) that $\rho(f_2, f_1) < 1$.

To continue, note that by (13) and by the definitions of A_2, Q^* and the R_δ^*'s we can define a homeomorphism g_2 of $I \times M_{j(2)}$ onto A_2 such that

(14) $f_2(M_{j(2)+1}) = g_2(\partial I \times M_{j(2)})$.

Specifically, using condition (5) in the definition of "doubly regular defining sequence" we find a subset $L_{j(2)}$ of $M_{j(2)+1}$, the union of half of the components of $M_{j(2)+1}$, chosen as before, such that for $k > j(2) + 1$

(15) $g_2(\partial I \times (L_{j(2)} \cap M_k)) = f_2(M_k)$.

There exists an odd integer $j(3) > j(2)$ such that for each component $X_{3,e}$ ($e = 1, \cdots, n_3$) of $M_{j(3)}$

(16) $\operatorname{diam} g_3(t \times X_{3,e}) < 1/3$ for each $t \in I$.

There also exists a homeomorphism $H_{3,e}$ of $I \times B^2 \times T^{n-2}$ onto $g_2(I \times X_{3,e})$ such that

(17) $H_{3,e}(t \times B^2 \times T^{n-2}) = g_2(t \times X_{3,e})$ for each $t \in I$.

In particular, we can suppose now that R_1^* and R_2^* are so constructed that

(18) $\operatorname{diam} H_{3,e}(R_\delta^* \times T^{n-2}) < 1/3$ ($\delta = 1, 2; e = 1, \cdots, n_3$).

It follows that there exists a homeomorphism f_3 of S^n onto

$$f_2(S^n - \operatorname{Int} M_{j(3)}) \cup \left(\bigcup_{e=1}^{n_3} H_{3,e}(Q \times T^{n-2}) \right)$$

such that

(19) $f_3 | S^n - \operatorname{Int} M_{j(3)} = f_2 | S^n - \operatorname{Int} M_{j(3)}$,

(20) $f_3(M_{j(3)+1}) = \bigcup_{e=1}^{n_3} H_{3,e}(Q^* \times M_{j(3)})$.

It follows from (11) that $\rho(f_3, f_2) < 1/2$.

By repeating this process we can establish the existence of an increasing sequence $\{j(i)\}_{i=2}^\infty$ of odd positive integers, a sequence $\{f_i\}_{i=1}^\infty$ of locally flat imbeddings of S^n into E^{n+1}, and a decreasing sequence $\{A_i\}_{i=1}^\infty$ of compact subsets of E^{n+1} such that, for $i = 1, 2, \cdots$,

(21) $\rho(f_{i+1}, f_i) < 1/i$,

(22) $f_{i+1} | S^n - \operatorname{Int} M_{j(i+1)} = f_i | S^n - \operatorname{Int} M_{j(i+1)}$,

(23) $f_{i+1}(M_{j(i+1)}) \subset A_{i+1}$,

(24) diam (largest component of A_i) $\longrightarrow 0$ as $i \longrightarrow \infty$,

(25) if x and y belong to distinct components of $M_{j(i+1)}$, then $f_{i+1}(x)$ and $f_{i+1}(y)$ belong to distinct components of A_{i+1},

(26) A_{i+1} is homeomorphic to $I \times M_{j(i+1)}$.

One can show in routine fashion that $f = \lim f_i$ is an embedding, and (22) implies that f is locally flat at each point of $S^n - C$.

Proof that $f(C)$ is tame. By (23) $f(C) \subset \bigcap A_i$. It follows from (26) that $f(C)$ has a defining sequence in E^{n+1} such that each component at each stage has an $(n-2)$-spine. Because these spines have codimension 3 relative to E^{n+1}, $f(C)$ is defined by cells and must be tame (see the proof of [16, Corollary 1]).

Proof that f is a wild embedding. Let $h(B^2 \times T^{n-2})$ denote one of the components of M_2, and let U denote the bounded component of $E^{n+1} - f(S^n)$. Define a homeomorphism m of B^2 onto $fh(B^2 \times p) \subset A_1$, for some $p \in T^{n-2}$. If f were locally flat at each point, locally $m(B^2)$ could be pushed slightly into $E^{n+1} - \mathrm{Cl}\, U$. Consequently we shall have proved that f is wild once we establish the following: *if $m'\colon B^2 \longrightarrow A_1$ is a map such that $m'|\partial B^2 = m|\partial B^2$, then $m'(\mathrm{Int}\, B^2) \cap f(S^n) \neq \varnothing$.* An equivalent statement, based on the construction of f, is the following: *if $m'\colon B^2 \longrightarrow A_1$ is a map such that $m'|\partial B^2 = m|\partial B^2$ and if $k \geq 2$ is an integer, then $m'(\mathrm{Int}\, B^2) \cap (f_k(S^n) \cup A_k) \neq \varnothing$.*

To prove the latter of these two statements, we decompose a set slightly larger than the "best" component of $A_1 - (f_k(S^n) \cup A_k)$. For $i = 1, \cdots, k-1$ define U_{i+1} as the bounded component of $E^{n+1} - f_{i+1}(S^n)$ and

$$Z_i = A_i - (A_{i+1} \cup U_{i+1} \cup (f_{i+1}(S^n) \cap \mathrm{Int}\, A_i)).$$

Retracing our earlier constructions, we find that

$$Z_i = g_i(I \times [M_{j(i)} - (L_{j(i)} \cap \mathrm{Int}\, M_{j(i+1)+1})])$$
$$\cup \left(\bigcup_{e=1}^{n_{i+1}} H_{i+1,e}([(\mathrm{Int}\, I \times B^2) - (R_1 \cup R_2)] \times T^{n-2}) \right),$$

where we have set $j(1) = 1$. Recall that $M_{j(i)} - \mathrm{Int}\, L_{j(i)}$ is homeomorphic to $I \times \partial M_{j(i)}$ because $L_{j(i)}$ was defined so as to contain exactly one component of $M_{j(i)+1}$ in each component of $M_{j(i)}$ and because condition (5) in the definition of "doubly regular defining sequence" implies that $L_{j(i)}$ is situated nicely in $M_{j(i)}$. Consequently $M_{j(i)} - (L_{j(i)} \cap \mathrm{Int}\, M_{j(i+1)})$ is homeomorphic to $L_{j(i)} - \mathrm{Int}\, M_{j(i+1)}$. It follows from several applications of condition (4) in the definition of "doubly regular ..." that the inclusion of each $H_{i+1,e}(\mathrm{Int}\, I \times \partial B^2 \times T^{n-2})$ into its intersection with

$$g_i(I \times [M_{j(i)} - (L_{j(i)} \cap \mathrm{Int}\, M_{j(i+1)+1})])$$

induces an injection of fundamental groups. It follows from Theorem 9 of [3] that the inclusion of each $H_{i+1,e}(\mathrm{Int}\, I \times \partial B^2 \times T^{n-2})$ into

$$H_{i+1,e}([(\mathrm{Int}\, I \times B^2) - (R_1 \cup R_2)] \times T^{n-2})$$

also induces a fundamental group injection. For similar reasons the inclusions of each component S of $Z_{i+1} \cap \mathrm{Cl}\, Z_i$ into Z_{i+1} and $S \cup Z_i$, respectively (S is homeomorphic to Int $I \times S^1 \times T^{n-2}$ and corresponds to the product of Int I with the boundary of a component of $M_{j(i+1)+1}$ or to the image under some $H_{i,e}$ of the product of an open annulus in ∂R_δ^* with T^{n-2}), induce injections. In short, $\pi_1(\bigcup Z_i)$ is a generalized free product with amalgamation. According to the definition of m, $m(\partial B^2) = fh(\partial B^2 \times p) \subset \partial A_1$, which means that $m(\partial B^2)$ is not contractible in $fh(\partial B^2 \times T^{n-2})$ nor in $g_1(I \times \partial M_1)$. From the above remarks we find first that $m|\partial B^2$ is not null homotopic in

$$g_1(I \times [M_1 - (L_1 \cap \operatorname{Int} M_{j(2)})])$$

and second that $m|\partial B^2$ is not null homotopic in $\bigcup Z_i$. Since the unique component of $A_1 - (f_k(S^n) \cup A_k)$ whose closure contains $m(\partial B^2)$ is a subset of $\bigcup Z_i$, the statement at the end of the preceding paragraph holds, and the proof is complete.

Addendum. The closure B of the bounded component of $E^{n+1} - f(S^n)$ is an $(n+1)$-cell. This can be established in elementary fashion by carefully extending the locally flat embeddings f_i to embeddings F_i of B^{n+1} in such a way that the F_i's obviously converge to an embedding. Alternatively, for $n + 1 \geqslant 5$ this follows from Theorem 6 of [15], because f can be approximated arbitrarily closely by locally flat embeddings in $(B - f(S^n))$.

COROLLARY 4. *There exists a wild n-cell B in E^n $(n \geqslant 4)$ such that ∂B is locally flat modulo a Cantor set that is tame relative to E^n.*

2. Generalizations of certain 3-space theorems in higher dimensions. The embeddings described in §1 signify that in dimensions greater than three $(n-1)$-spheres and n-cells in E^n can have properties contradictory to results concerning the comparable properties in 3-space. This section is devoted to contrasting some prominent 3-dimensional tameness theorems with higher dimensional versions.

COROLLARY 5. *For $1 \leqslant k < n$ and $n \geqslant 4$ there exists a wild $(n-1)$-sphere Σ in E^n that is locally flat modulo a flat (relative to E^n) k-cell.*

PROOF. Let $f\colon S^{n-1} \longrightarrow E^n$ be a wild embedding promised by Theorem 3 such that, for some Cantor set C in S^{n-1}, f is locally flat modulo C and $f(C)$ is tame. In S^{n-1} there exists an $(n-1)$-cell B such that ∂B contains C, ∂B is locally flat in S^{n-1} modulo C, and C is a tame subset of ∂B (the technique for forming B is due to Alexander [1], was generalized by Blankenship [4, Theorem 3F], and has been formalized by Osborne [13, Theorem 3]). Obviously $f(S^{n-1})$ is locally flat modulo $f(B)$, and it follows from [11] that $f(B)$ is flat relative to E^n. Now for $1 \leqslant k < n-1$ it is a simple matter to identify a k-cell K in $f(B)$ that is flat relative to E^n and that contains $f(C)$, completing the proof.

Compare Corollary 5 with Theorem 2 of [10].

COROLLARY 6. *There exist two n-cells D and D' in E^n $(n \geqslant 4)$ such that ∂D is wildly embedded at each point of a Cantor set C, $\partial D'$ is locally flat at each point, and $C \subset D' \subset D$.*

PROOF. Using the proof and terminology of Corollary 5, we let D denote the closure of the bounded component of $f(S^{n-1})$, and we thicken the (flat) $(n-1)$-cell $f(B)$ to form a flat n-cell D' in D.

Corollary 6, which should be compared with Theorem 5 of [5], reveals that the theory of *-taming sets developed in [8] does not expand to rich generalizations in high dimensions, because, as Cannon points out in [8], Burgess' work in [5] can be regarded as an initial result about *-taming sets. Should one attempt to extend the definition of *-taming set (see [8]) without additional restrictions, Corollary 7 indicates how limited a theory would result.

A *crumpled n-cube* is a space homeomorphic to the closure of a component of $S^n - \Sigma$, where Σ denotes an $(n-1)$-sphere topologically embedded in S^n.

COROLLARY 7. *Suppose X is a compact proper subset of S^n $(n \geqslant 4)$ having the following property: if K is a crumpled n-cube in S^n such that $K \cap X \subset \mathrm{Bd}\, K$ and $\mathrm{Bd}\, K$ is locally flat at each point of $\mathrm{Bd}\, K - X$, then K is an n-cell. Then X is a countable set.*

PROOF. Suppose to the contrary that X is uncountable. Starting with a flat n-cell B in $S^n - X$, we can pull out arms from B towards X, as was done in Theorem 3 of [13], to obtain an embedding g of B into S^n such that $g^{-1}(g(B) \cap X)$ is a Cantor set C' that is tame relative to ∂B and $g|\partial B$ is locally flat modulo C'. Using the notation of Corollary 6, we then can define an embedding f of $E^n -$ Int D' into $g(B)$ such that $f(\partial D') = g(\partial B)$ and $f(C) = g(C')$. Define K to be the closure of the component of $S^n - f(\partial D)$ contained in B. Since K satisfies the property in the hypothesis, it must be an n-cell. This leads to a contradiction, however, because $K - \{p\}$ (for $p \in K - f(\partial D)$) is homeomorphic to $E^n -$ Int D.

A similar argument may be given for one implication in the following corollary, which provides a characterization of taming sets for $(n-1)$-spheres in E^n $(n \geqslant 4)$ that should be compared with the more positive results collected in [7]. The other implication can be derived from [11].

COROLLARY 8. *Let X be a compact, proper subset of an $(n-1)$-sphere in E^n $(n \geqslant 4)$. Then X is countable if and only if, for each $(n-1)$-sphere Σ in E^n that contains X and that is locally flat modulo X, Σ is locally flat.*

COROLLARY 9. *There exists a wild n-cell B in E^n ($n \geqslant 5$) such that B is cellular and the set of points at which ∂B is wildly embedded is a Cantor set.*

PROOF. By Lemma 2 and Theorem 3 there exists an n-cell B in E^n such that (i) ∂B is wildly embedded, (ii) ∂B is locally flat modulo a Cantor set C, (iii) $\partial B - C$ is simply connected, and (iv) C is tame relative to E^n. To show that B is cellular we sketch a proof that B satisfies McMillan's cellularity criterion [12, Theorem 1]. Given a neighborhood U of B we choose a neighborhood V of B such that any loop in V is contractible in U. Any map f: $B^2 \longrightarrow U$ for which $f(\partial B^2) \subset V$ can be approximated, since C is tame relative to E^n, by a map g: $B^2 \longrightarrow U - C$ such that $g|\partial B^2 = f|\partial B^2$. Furthermore, since ∂B is locally flat at each point of $\partial B - C$, g can be obtained so that $g^{-1}(\partial B)$ consists of a finite number of simple closed curves. The outermost such curves bound pairwise disjoint disks $F_1, \cdots, F_k$ in B^2. Now $g|\partial F_i$ can be extended to a map of F_i into $\partial B - C$, from which we can piece together a map h: $B^2 \longrightarrow U - (C \cup \operatorname{Int} B)$ such that $h|\partial B^2 = f|\partial B^2$. Finally, $h(B^2)$ can be pushed off B to obtain the desired map.

It follows from [11] that the set of points at which ∂B is wild contains no isolated point and hence must be a Cantor set.

Compare Corollary 9 with Corollary 1 of [6].

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