TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 207, 1975

GENERALIZATION OF RIGHT ALTERNATIVE RINGS

BY

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ABSTRACT. We study nonassociative rings R satisfying the conditions (1) $(ab, c, d) + (a, b, [c, d]) = a(b, c, d) + (a, c, d)b$ for all $a, b, c, d \in R$, and (2) $(x, x, x) = 0$ for all $x \in R$. We furthermore assume weakly characteristic not 2 and weakly characteristic not 3. As both (1) and (2) are consequences of the right alternative law, our rings are generalizations of right alternative rings. We show that rings satisfying (1) and (2) which are simple and have an idempotent $\neq 0, \neq 1$, are right alternative rings.

We show by example that (x, e, e) may be nonzero. In general, $R = R' + (R, e, e)$ (additive direct sum) where R' is a subring and (R, e, e) is a nilpotent ideal which commutes elementwise with R.

We examine R' under the added assumption of Lie admissibility: (3) $(a, b, c) + (b, c, a) + (c, a, b) = 0$ for all $a, b, c \in R$. We generate the Peirce decomposition. If R' has no trivial ideals contained in its center, the table for the multiplication of the summands is associative, and the nucleus of R' contains $R'_{10} + R'_{01}$. Without the assumption on ideals, the table for the multiplication need not be associative; however, if the multiplication is defined in the most obvious way to force an associative table, the new ring will still satisfy (1), (2), (3).

1. Introduction. We study nonassociative rings which satisfy the generalized right alternative law:

$$(1)\quad 0 \equiv \bar{A}(a, b, c, d) = (ab, c, d) + (a, b, [c, d]) - a(b, c, d) - (a, c, d)b$$

and

$$(2)\qquad 0 \equiv (x, x, x).$$

We furthermore assume all rings are weakly characteristic $\neq 2, \neq 3$, where weakly characteristic $\neq N$, N a positive integer, means the map $x \longrightarrow Nx$ is either one-to-one or onto (or both). Both conditions (1) and (2) are consequences of the right alternative law for nonassociative rings and so we are examining a class of

Received by the editors March 22, 1974.

AMS (MOS) subject classifications (1970). Primary 17A30; Secondary 17D05.

Key words and phrases. Right alternative, Peirce decomposition.

(1) Partially supported by the Iowa State University Science and Humanities Research Institute.

(2) Written while the second author was at Iowa State University with a fellowship from the Consiglio Nazionale delle Ricerche, Italy.

rings larger than the class of right alternative rings. We shall call rings that satisfy (1) and (2), as well as the assumptions on the characteristic, "GRA rings" (for generalized right alternative rings). GRA rings are power associative, and applying the Albert decomposition approach [1], we show

LEMMA 2. *If R is a GRA ring with an idempotent $e \neq 0, \neq 1$, then $R = R' + H$ (additive direct sum) where $H = (R, e, e)$ is an ideal of R, $H^2 = 0$, and each element of H commutes with all elements of R. R' is a subring of R containing e.*

THEOREM 2. *If R is a simple GRA ring with an idempotent $e \neq 0, \neq 1$, then R is a right alternative ring.*

Lemma 2 is a generalization of Kleinfeld's work where he proved the corresponding result with additional hypotheses. In [6], in addition to (1) and (2), Kleinfeld assumed the generalized left alternative law.

$$0 \equiv (a, b, cd) + ([a, b], c, d) - c(a, b, d) - (a, b, c)d \tag{3}$$

for all elements a, b, c, d. In [7], he assumed (1), (2) and the flexible law

$$0 \equiv (a, b, a) \tag{4}$$

for all elements a, b.

We offer, as a corollary, a short and illuminating proof of H. F. Smith's result [10] that (1), (2) and (3), or (1), (2) and (4) is enough to show that prime rings with an idempotent are alternative.

E. Kleinfeld [6], [7] and H. F. Smith [10] have published Peirce decompositions for GRA rings satisfying either (3) or (4) as additional hypotheses. The generalized left alternative law and the flexible law are two natural conditions to study. A further criterion to study is Lie admissibility or

$$0 \equiv \bar{B}(a, b, c) = (a, b, c) + (b, c, a) + (c, a, b) \tag{5}$$

for all elements a, b, c. We call these rings generalized $(-1, 1)$ rings. We construct the Peirce decomposition of the subring R' and show that the multiplication table for the summands is not an associative table. We show that if the product in the ring is defined in the most straightforward way to get an associative table, the new ring still satisfies (1), (2) and (5). If R' has no trivial ideals in its center, then again the multiplication table of the summands will be associative, and, in fact, R'_{10} and R'_{01} are contained in the nucleus of R'. As an important consequence, we have

COROLLARY. *If R is a prime generalized $(-1, 1)$ ring with idempotent $e \neq 0, \neq 1$, then R is associative.*

2. **Preliminaries.** The associator (a, b, c) is defined to mean $(a, b, c) = (ab)c - a(bc)$. Similarly, the commutator $[a, b]$ means $[a, b] = ab - ba$. Both "$\cdot$" and juxtaposition will be used to express multiplication. In such expressions, the juxtaposition product is performed first. For example, $(a, b, c) = ab \cdot c - a \cdot bc$.

In many expressions where an element should appear, a set of elements will appear. In such a case, we mean the additive span of those elements computed as the variables run through the indicated sets. For example, $[a, R]$ = the additive group generated by $\{[a, b] \mid b \in R\}$.

A ring R is called simple if $R^2 \neq 0$ and R has no proper ideals. A ring R is called prime if $IJ \neq 0$ whenever I and J are nonzero ideals of R. An ideal $I \neq 0$ is called trivial if $I^2 = 0$, and a ring is called semiprime if it contains no trivial ideals.

The nucleus of a nonassociative ring $R = \{x \in R \mid (x, R, R) = (R, x, R) = (R, R, x) = 0\}$. The center of a nonassociative ring $R = \{x \in R \mid x \in$ nucleus of R and $[x, R] = 0\}$.

We shall use the following identities:

$$\begin{aligned} 0 &\equiv \bar{C}(a, b, c, d) \\ &= (ab, c, d) - (a, bc, d) + (a, b, cd) - a(b, c, d) - (a, b, c)d, \end{aligned} \tag{6}$$

$$0 \equiv \bar{D}(a, b, c, d) = (a, b, cd) - (a, bd, c) - (a, b, d)c + (a, d, c)b, \tag{7}$$

$$\begin{aligned} 0 &\equiv \bar{E}(a, b, c, x, y) \\ &= ((a, b, c), x, y) - ((a, x, y), b, c) - (a, (b, x, y), c) \\ &\quad - (a, b, (c, x, y)) + (a, b, c)[x, y] - (a, b, c[x, y]) + (a, b, [x, y])c, \end{aligned} \tag{8}$$

$$0 \equiv \bar{F}(a, b, c) = [ab, c] - a[b, c] - [a, c]b - (a, b, c) + (a, c, b) - (c, a, b), \tag{9}$$

$$0 \equiv \bar{G}(a, x) = (a, x, x) + (x, a, x) + (x, x, a), \tag{10}$$

$$\begin{aligned} 0 &\equiv \bar{H}(a, b, c, x) \\ &= [a, ([b, c], x, x)] + [x, ([b, c], a, x)] + [x, ([b, c], x, a)], \end{aligned} \tag{11}$$

$$\begin{aligned} 0 &\equiv \bar{I}(a, b, c, x) \\ &= ([a, [b, c]], x, x) + ([x, [b, c]], a, x) + ([x, [b, c]], x, a). \end{aligned} \tag{12}$$

PROOF. Item (6) is the Teichmüller identity which holds in any nonassociative ring. Item (7) is equivalent to (1) since $\bar{C}(a, b, c, d) = \bar{A}(a, b, c, d) + \bar{D}(a, b, d, c)$. Item (8) is given in [11, equation 6)]. It is a consequence of (1) since

$$\bar{E}(a, b, c, x, y) = \bar{A}(ab, c, x, y) + \bar{A}(a, b, x, y) \cdot c - \bar{A}(a, b, c, [x, y]) - \bar{A}(a, bc, x, y) - a \cdot \bar{A}(b, c, x, y) - \bar{D}(a, b, [x, y], c).$$

Item (9) holds in any nonassociative ring. Item (10) is a linearization of (2). Items (11) and (12) are linearizations of $0 \equiv [x, ([a, b], x, x)]$ and $0 \equiv ([x, [a, b]], x, x)$, respectively, which are proved next. $0 \equiv \bar{E}(x, x, x, a, b) + \bar{G}((x, a, b), x) = (x, x, [a, b])x - (x, x, x[a, b])$. Continuing, $0 \equiv (x, x, [a, b])x - (x, x, x[a, b]) + \bar{D}(x, x, x, [a, b]) = (x, [a, b], x)x - (x, x[a, b], x)$. Now, using the two preceding equations,

$$0 \equiv \bar{G}(x[a, b], x) - \bar{G}([a, b], x) \cdot x = (x[a, b], x, x) - ([a, b], x, x)x.$$

From (1) we have $x([a, b], x, x) = ([a, b], x, x)x$, which is linearized to give (11). We get $[x, ([a, b], x, x)] = ([x, [a, b]], x, x)$ from (1), and this is linearized to give (12).

THEOREM 1. *GRA rings are power associative.*

PROOF. For $a \in R$, define $a^1 = a$ and $a^{t+1} = a^t a$. The proof is by induction on the hypothesis $a^r a^s = a^{r+s}$ for all $r + s \leqslant N$. The hypothesis holds by definition for $N = 2$, and by (2) for $N = 3$. Assume $N \geqslant 4$ and $a^r a^s = a^{r+s}$ for all positive integers r and s where $r + s \leqslant N - 1$. This means $(a^i, a^j, a^k) = 0$ if $i + j + k < N$, and $[a^i, a^j] = 0$ if $i + j < N$. Furthermore, from (1),

$$(a^i a, a^j, a^k) = -(a^i, a, [a^j, a^k]) + a^i(a, a^j, a^k) + (a^i, a^j, a^k)a.$$

So $(a^i, a^j, a^k) = 0$ if $i + j + k = N$, provided that $i \geqslant 2$. Thus

$$a^i a^{N-i} = a^i(a^{N-i-1}a) = (a^i a^{N-i-1})a - (a^i, a^{N-i-1}, a) = a^N \qquad \text{for } 2 \leqslant i \leqslant N-2.$$

We have shown $a^i a^{N-i} = a^N$ for $2 \leqslant i \leqslant N - 2$. The case $i = N - 1$ is by definition. The remaining case, $i = 1$, is given next. From (10)

$$(a, a, a^{N-2}) + (a, a^{N-2}, a) + (a^{N-2}, a, a) = 0.$$

Expanding, remembering $N - 2 \geqslant 2$ implies $(a^{N-2}, a, a) = 0$, we have $a^2 a^{N-2} - 2aa^{N-1} + a^N = 0$. Since we have already shown that $a^2 a^{N-2} = a^N$, we have $2aa^{N-1} = 2a^N$. Weakly characteristic not two implies $aa^{N-1} = a^N$. This completes the proof by induction.

LEMMA 1. *The additive group I spanned by all associators of the form (a, b, b) where a and b range over a GRA ring R is a two-sided ideal of R.*

PROOF. We first show $IR \subseteq I$ by showing $(a, b, b)x \in I$ for all $a, b, x \in R$.

$$\begin{aligned}0 &\equiv \bar{D}(a, b, x, b) - \bar{D}(a, x, b, b) + \bar{A}(a, b, x, b) + \bar{A}(a, b, b, x)\\ &\quad + \bar{A}(a, x, b, b) + a \cdot \bar{G}(x, b)\\ &= (a, b, xb) + (a, xb, b) - (a, b^2, x) - (a, x, b^2)\\ &\quad + (ab, x, b) + (ab, b, x) + (ax, b, b) - 3(a, b, b)x.\end{aligned}$$

We have shown $3(a, b, b)x$ is a sum of associators which are in I; by weakly characteristic $\neq 3$, $(a, b, b)x \in I$. This proves I is a right ideal, and (1) shows that I is also a left ideal. It is possible to show that if c is any element satisfying $0 \equiv ([x, c], x, x)$ for all elements x in R (see (12)), then the additive subgroup generated by all associators of the form (c, x, x) as x varies over R is a right ideal. Furthermore $([R, R], x, x)(c, x, x) = 0$. We do not need this result and we do not prove it here.

3. GRA rings with idempotent. We shall now consider GRA rings with an idempotent $e \neq 0, \neq 1$. We give examples of GRA rings with an idempotent e which do not satisfy $(x, e, e) = 0$ for all elements x in the ring. However, if R is a GRA ring, the set of elements (R, e, e) can be isolated from the ring. They form an ideal of R whose square is zero, and they commute with every element of R. Furthermore, $R = R' + (R, e, e)$ (additive direct sum) and R' is a subring of R. After we show this, we proceed and examine the structure of R'.

GRA rings are power associative and have an Albert decomposition [1] with respect to an idempotent e. Thus if e is an idempotent of a GRA ring R, then $R = R_1 + R_{1/2} + R_0$ (additive direct sum) where $x_i \in R_i \Longleftrightarrow ex_i + x_ie = 2ix_i$. One has the symmetric products

$$x_1y_1 + y_1x_1 \in R_1, \quad x_0y_0 + y_0x_0 \in R_0, \quad x_{1/2}y_{1/2} + y_{1/2}x_{1/2} \in R_1 + R_0.$$

One also has $ex_1 = x_1e = x_1$, $ex_0 = x_0e = 0$. In [1], the hypotheses on the characteristic are stronger than weakly characteristic $\neq 2, \neq 3$, but the proofs for the decomposition properties listed above are valid under these weaker hypotheses. For readers verifying this, let R^+ be the ring with the same additive structure as R and with the symmetric product $x \circ y = xy + yx$ for multiplication. $(R^+, +, \circ)$ is a commutative power associative ring. If $2t = e$, then $2t^2$ is an idempotent of R^+. The decomposition of R^+ with respect to the idempotent $2t^2$ has the required properties with respect to e. The proof given in [1] for $R_1R_0 = R_0R_1 = 0$ requires characteristic not 5. We shall prove this property in the remark following Lemma 4 for GRA rings without this extra hypothesis.

Let $H = \{h \in R | [e, h] = 0 \text{ and } 2eh = 2he = h\}$. It is immediate that $H \subseteq R_{1/2}$ and that $4(e, e, h) = -4(h, e, e) = h$ for all $h \in H$. We claim $(R, e, e) = H$. From $\bar{D}(e, e, x_{1/2}, e)$ we get $(e, e, x_{1/2})e = (e, e, ex_{1/2})$. By $\bar{C}(e, e, e, x_{1/2})$ we have $e(e, e, x_{1/2}) = (e, e, ex_{1/2})$. This shows that $[e, (e, e, x_{1/2})] = 0$. As e is the identity on R_1 and annihilates R_0, we have actually shown

$$(13) \qquad (e, e, x)e = e(e, e, x) = (e, e, ex) \text{ for all } x \in R.$$

From $\bar{D}(e, x_{1/2}, e, e)$ we get $(e, x_{1/2}, e)e = (e, ex_{1/2}, e)$. We use $\bar{G}(ex_{1/2}, e) = 0 = \bar{G}(x_{1/2}, e) \cdot e$ to show $(ex_{1/2}, e, e) = (x_{1/2}, e, e)e$. From (1) we have $e(x_{1/2}, e, e) = (ex_{1/2}, e, e)$ and so $[e, (x_{1/2}, e, e)] = 0$. Using (1),

$$(x_{1/2}, e, e) = (ex_{1/2} + x_{1/2}e, e, e) = e(x_{1/2}, e, e) + (x_{1/2}, e, e)e = 2(x_{1/2}, e, e)e.$$

This shows $(R, e, e) \subseteq H$. Since we initially showed $H \subseteq (R, e, e)$, we have $H = (R, e, e)$.

LEMMA 2. *Let R be a GRA ring with idempotent e. Then H is an ideal of R, $H^2 = 0$, $[H, R] = 0$, and $R = R' + H$ (additive direct sum) where R' is a subring of R and $e \in R'$.*

PROOF. Let $R' = \{r \in R | (r, e, e) = 0\}$. Now $e \in R'$, and by (1) R' is a subring of R. If r is any element of R, the summand of r in R' is $r + 4(r, e, e)$; the summand of r in H is $-4(r, e, e)$. The proof that this decomposition is additively direct is left to the reader. We will now show $H^2 = 0$. Let x and y be any elements of R. Let $\widetilde{x} = (x, e, e)$ and $\widetilde{y} = (y, e, e)$. We shall expand $8(\widetilde{x}\widetilde{y}, e, e)$ in two different ways. First, by (1), $4(\widetilde{x}\widetilde{y}, e, e) = 4\widetilde{x}(\widetilde{y}, e, e) + 4(\widetilde{x}, e, e)\widetilde{y} = -2\widetilde{x}\widetilde{y}$. Here we used that $\widetilde{x}$ and $\widetilde{y}$ are elements of H and $4(h, e, e) = -h$ for all $h \in H$. This shows $8(\widetilde{x}\widetilde{y}, e, e) = -4\widetilde{x}\widetilde{y}$, and it also shows that $2\widetilde{x}\widetilde{y} \in (R, e, e) \subseteq H$. But $2\widetilde{x}\widetilde{y} \in H$ implies $8(\widetilde{x}\widetilde{y}, e, e) = 4(2\widetilde{x}\widetilde{y}, e, e) = -2\widetilde{x}\widetilde{y}$. Comparison of these two expressions for $8(\widetilde{x}\widetilde{y}, e, e)$ gives $2\widetilde{x}\widetilde{y} = 0$, and so $2(x, e, e)(y, e, e) = 0$. Weakly characteristic $\neq 2$ implies $(x, e, e)(y, e, e) = 0$ for all x and y elements of R. Since $H = (R, e, e)$, then $H^2 = 0$.

To show H is an ideal, let $r \in R$ and $h \in H$. Then

$$rh = -4r(h, e, e) = -4(rh, e, e) + 4(r, e, e)h + 4\bar{A}(r, h, e, e) \subseteq (R, e, e)$$

since $(r, e, e)h \subseteq H^2 = 0$. The proof that $hr \in H$ is similar. The proof that $[H, R] = 0$ makes use of the fact that H is already known to be an ideal and so $(e, h, r) \in H$.

$$[h, r] = 4(e, e, [h, r]) = 4e(e, h, r) + 4(e, h, r)e - 4(e, h, r) + 4\bar{A}(e, e, h, r) = 0.$$

We now give an example of a GRA ring possessing an idempotent e for which $(R, e, e) \neq 0$. Let A be any commutative associative ring with identity 1 and element ½. Let M be any module over A. Our GRA ring R consists of the set $A \times M$ under coordinatewise addition, but multiplication in R is given by

$(a, m)(a', m') = (aa', \frac{1}{2}am' + \frac{1}{2}a'm)$. We leave it to the reader to verify this. The element $(1, 0)$ of R is the idempotent, and $(R, e, e) = \{0\} \times M$.

This example is characteristic of the relationship in any GRA ring between H and the rest of the ring. This paragraph gives a summary of these results. The proofs, which are often quite involved, are not given. If $R = R' + H = R'_1 + R'_{1/2} + R'_0 + H$ (Lemma 2), then $R_{1/2}H = 0$. One can verify that if $\mathcal{A}$ is the subring of R' generated by R'_1, then for all $\alpha, \beta \in \mathcal{A}$, $h \in H$, $(\alpha\beta)h = 2\alpha(\beta h)$. This means that if a new product $*$ is defined between $\mathcal{A}$ and H, $\alpha * h = 2\alpha h$, H becomes a module over $\mathcal{A}$ with all the properties of an associative module except, of course, that the ring $\mathcal{A}$ itself is not associative. It may be further verified that if Q is the ideal of $\mathcal{A}$ generated by all associators and commutators of $\mathcal{A}$, then $QH = 0$, and H is a module over the associative and commutative ring $\mathcal{A}/Q$. Identical results can be obtained for the subring $\mathcal{B}$ generated by R'_0 in R'.

LEMMA 3. *If R is a GRA ring with idempotent e, then $(e, x, x) \subseteq H$ for all $x \in R$.*

PROOF. If $a_{1/2} \in R_{1/2}$, then $[e, [e, a_{1/2}]] = a_{1/2} + 2(a_{1/2}, e, e) - 2(e, e, a_{1/2})$. This equality becomes evident by expansion as

$$[e, [e, a_{1/2}]] - 2(a_{1/2}, e, e) + 2(e, e, a_{1/2}) = -e \circ (e \circ a_{1/2}) + 2e \circ a_{1/2} = a_{1/2}.$$

We next show $[e, [e, (e, x, x)]] = 0$.

$$\begin{aligned} 0 &= [e, \bar{A}(e, x, e, x) + \bar{A}(e, x, x, e) - \bar{A}(x, e, e, x) - \bar{A}(x, e, x, e)] \\ &\quad - \bar{H}(x, e, x, e) - [e, [x, \bar{G}(x, e)]] + [e, [e, \bar{G}(e, x)]] \\ &= [e, [e, (e, x, x)]] - [x, ([e, x], e, e)] - [e, [x, (x, e, e)]] \end{aligned}$$

and, by Lemma 2, $0 = [e, [e, (e, x, x)]]$.

From (1), $(e, x, x) \in R_{1/2}$, and combining the above results,

$$0 = [e, [e, (e, x, x)]] = (e, x, x) + 2((e, x, x), e, e) - 2(e, e, (e, x, x)).$$

Using Lemma 2 and (13), e commutes with (e, x, x) and since $(e, x, x) \in R_{1/2}$, we must have $(e, x, x) \in H$.

LEMMA 4. *If R is a GRA ring with idempotent e, then $(R_{1/2}, x, x) \subseteq H$ for all $x \in R$.*

PROOF. From (1), for all $a_{1/2} \in R_{1/2}$ and $x \in R$, we have

$$(a_{1/2}, x, x) = (ea_{1/2} + a_{1/2}e, x, x) = e \circ (a_{1/2}, x, x) + a_{1/2} \circ (e, x, x).$$

Since $a_{1/2} \circ (e, x, x) \in (R_1 + R_0) \cap H$, we have $a_{1/2} \circ (e, x, x) = 0$, and so

$(a_{1/2}, x, x) \in R_{1/2}$. Continuing, $0 \equiv \bar{H}(e, e, [e, a_{1/2}], x) - [x, \bar{I}(x, e, a_{1/2}, e)] = [e, ([e, [e, a_{1/2}]], x, x)]$. We have shown $([e, [e, a_{1/2}]], x, x)$ commutes with e. From the first sentence in the proof of Lemma 3,

$$([e, [e, a_{1/2}]], x, x) = (a_{1/2}, x, x) + 2((a_{1/2}, e, e), x, x) - 2((e, e, a_{1/2}), x, x).$$

From Lemma 2, Lemma 3, $0 = \bar{E}(e, e, a_{1/2}, x, x)$, and (13), we have that $(a_{1/2}, x, x)$ commutes with e. This means $(a_{1/2}, x, x) \in H$, and so $(R_{1/2}, x, x) \subseteq H$ for all $x \in R$.

REMARK. $R_1 R_0 = R_0 R_1 = 0$.

PROOF. By (1), $(x_1, e, y_0) + (x_1, y_0, e) \in R_1$, $(y_0, e, x_1) + (y_0, x_1, e) \in R_0$, $(e, x_1, y_0) + (e, y_0, x_1) \in R_{1/2}$. Yet (2) implies the sum of all six associators is zero. Therefore, all three pairs are zero. This gives

$$x_1 y_0 + (x_1 y_0)e = 0, \quad 2y_0 x_1 = (y_0 x_1)e, \quad x_1 y_0 - e(x_1 y_0 + y_0 x_1) = 0.$$

From $2y_0 x_1 = (y_0 x_1)e$, we get $(y_0 x_1, e, e) = 2y_0 x_1$. By (1), $(y_0 x_1, e, e) = 0$. Thus $y_0 x_1 = 0$. We know $x_1 y_0 = e(x_1 y_0 + y_0 x_1) = e(x_1 y_0)$; so $e(x_1 y_0) + (x_1 y_0)e = x_1 y_0 - x_1 y_0 = 0$. Therefore $x_1 y_0 \in R_0$, and yet $(x_1 y_0)e = -x_1 y_0$; this implies $x_1 y_0 = 0$.

THEOREM 2. *If R is a simple GRA ring with an idempotent $e \neq 0, \neq 1$, then R is a right alternative ring.*

PROOF. If R is simple, then $H = 0$. It follows from Lemma 2, Lemma 3, and Lemma 4 that $(b, e, e) = (e, b, b) = (R_{1/2}, b, b) = 0$ for all $b \in R$. Using (1) we find $(a, b, b) = (a_1, b, b) + (a_0, b, b) \in R_1 + R_0$ for all $a, b \in R$. If R is not a right alternative ring, by Lemma 1, $R \subseteq R_1 + R_0$. By (1) $(x_0, y_0, e) \in R_0$. This implies $(x_0 y_0)e \in R_0$, and since we already know $R_{1/2} = 0$, it follows that $(x_0 y_0)e = 0$. Therefore R_0 is closed under multiplication. By the previous remark, R_0 is an ideal, and since $e \notin R_0$, $R_0 = 0$. Since $R = R_1$, e is an identity; contradiction.

COROLLARY (KLEINFELD). *If R is a simple flexible GRA ring with idempotent $e \neq 0, \neq 1$, then R is an alternative ring.*

COROLLARY (KLEINFELD). *If R is a simple GRA ring with an idempotent $e \neq 0, \neq 1$ which also satisfies the weakly left alternative law, then R is alternative.*

COROLLARY. *If R is a simple, Lie admissible GRA ring, then R is associative.*

PROOFS. The proofs are immediate. In the first corollary, R is a right alternative and flexible ring; thus R is alternative. In the second case, we have the mirror form of Theorem 2 which says that if R satisfies (3) and (2) and R is simple with idempotent $e \neq 0, \neq 1$, then R is left alternative. R is then both left alternative and right alternative and consequently alternative. It is well known that simple alternative rings are either associative, or Cayley-Dickson algebras [5], [9]. The first two corollaries were proved respectively in [7] and [6]. In both papers the ideas we have presented were used, but stronger hypotheses were used throughout so that the generality of the theorems was obscured.

If R is a simple, Lie admissible GRA ring with idempotent $e \neq 0, \neq 1$, by Theorem 2, R is a simple, Lie admissible, right alternative ring. Such rings are simple $(-1, 1)$ rings, and by [8], they are associative.

4. The structure of R'. If R is any GRA ring with an idempotent $e \neq 0$, $\neq 1$, we may write $R = R' + H$ (additive direct sum) as in Lemma 2. We have studied the ideal H, and now we will examine the subring R'. R' has the idempotent e, and so R' has the Albert decomposition: $R' = R'_1 + R'_{1/2} + R'_0$. From Lemma 4 and Lemma 3 and the fact that R' is a subring, we have

$$(14) \qquad (x, e, e) = (e, x, x) = (R'_{1/2}, x, x) = 0 \quad \text{for all } x \in R'.$$

As a consequence of (1) and Lemma 1, we have the following lemma.

LEMMA 5. *If $a, b \in R'$, then $(a_1, b, b) \in R_1$, $(a_{1/2}, b, b) = 0$, $(a_0, b, b) \in R_0$. Furthermore $\{(a, b, b) | a, b \in R'\}$ is an ideal of R' contained in $R_1 + R_0$.*

Since the summands of R' are determined by the multiplication by e, it is immediate that $R_1 = R'_1$, $R_0 = R'_0$, $R_{1/2} = R'_{1/2} + H$ and $R = R_1 + R'_{1/2} + R_0 + H$ (additive direct sum).

The remainder of this paper will be concerned with the structure of $R' = R_1 + R'_{1/2} + R_0$. It will be assumed that when we write elements, as for example in the associator (R', x, x), the element x comes from the ring under discussion, namely R'. The only exceptions to this are Lemma 9, the first part of the proof of Theorem 3, the corollary to Lemma 13, Theorem 6, Theorem 7, and Theorem 8. All these exceptions are clearly indicated in the proofs.

LEMMA 6. *The subring generated by e and $R'_{1/2}$ is a right alternative ring.*

PROOF. The result follows immediately from (1) and (14).

We are interested in the linearized form of the right alternative law: $(a, b, c) + (a, c, b) = 0$. We can show that $(a, b, c) + (a, c, b) = 0$ if any of the arguments is in $R'_{1/2}$.

LEMMA 7. $(x, e, y) + (x, y, e) = 0$ *for all $x, y \in R'$.*

PROOF. We will consider the summands of the elements x and y. By (2),

$$0 = (x_i, y_j, e) + (x_i, e, y_j) + (e, x_i, y_j) + (e, y_j, x_i) + (y_j, e, x_i) + (y_j, x_i, e).$$

By Lemma 5, $(x_i, y_j, e) + (x_i, e, y_j) \in R_i$ and $(y_j, x_i, e) + (y_j, e, x_i) \in R_j$ where $i, j \in \{1, ½, 0\}$. If $i = j = ½$ or $i \neq j$, then, using (14), $(x_i, y_j, e) + (x_i, e, y_j) = 0 = (y_j, x_i, e) + (y_j, e, x_i)$. The only remaining case is for $i = j = 1$ or $i = j = 0$. For $i = 0$ or $i = 1$, $(x_i, e, y_i) = 0$, and it remains to show that $(x_i, y_i, e) = 0$. We see this from $0 = \bar{A}(x_i, e, y_i, e) - \bar{D}(x_i, y_i, e, e) = \pm(x_i, y_i, e)$.

LEMMA 8. *If $(a_i, b_j, c_k) + (a_i, c_k, b_j) \neq 0$, then $i = j = k = 1$ or $i = j = k = 0$.*

PROOF. Let us assume that $(a_i, b_j, c_k) + (a_i, c_k, b_j) \neq 0$. By Lemma 5, $(a_i, b_j, c_k) + (a_i, c_k, b_j) \neq 0$ implies $i \neq ½$. Using $i \neq ½$, Lemma 5 and Lemma 7, we calculate the following:

$$\begin{aligned} 0 &= \bar{D}(a_i, b_j, e, c_k) + \bar{D}(a_i, c_k, e, b_j) - \bar{D}(a_i, e, b_j, c_k) - \bar{D}(a_i, e, c_k, b_j) \\ &\quad - \bar{D}(a_i, c_k, b_j, e) + \bar{D}(a_i, b_j, c_k, e) \\ &= 2(k - i)\{(a_i, b_j, c_k) + (a_i, c_k, b_j)\} + (a_i, c_k, [e, b_j]) + (a_i, [e, b_j], c_k). \end{aligned}$$

If we interchange the roles of b_j and c_k we get

$$0 = 2(j - i)\{(a_i, b_j, c_k) + (a_i, c_k, b_j)\} + (a_i, b_j, [e, c_k]) + (a_i, [e, c_k], b_j).$$

If $j \neq ½$, then from the first formula $i = k$, and consequently $k \neq ½$. Now, from the second formula $j = i$. This shows that $j \neq ½ \Rightarrow i = j = k$. If $k \neq ½$, working first from the second formula, we also would have gotten $i = j = k$. We know that $(a_i, b_j, c_k) + (a_i, c_k, b_j) = 0$ unless $i = j = k = 0$ or $i = j = k = 1$, except for the single remaining case $(a_i, b_{½}, c_{½}) + (a_i, c_{½}, b_{½})$. This is zero by the linearization of (2) and the result that $(R'_{½}, x, x) = 0\ \forall x \in R'$.

We now give the multiplication table of the summands of the Albert decomposition of R'. The table is given with respect to the symmetric product $x \circ y = xy + yx$.

(15)

$\circ$	R_1	$R'_{½}$	R_0
R_1	R_1	$R'_{½}$	0
$R'_{½}$	$R'_{½}$	$R_1 + R_0$	$R'_{½}$
R_0	0	$R'_{½}$	R_0

The entries except for $R_{½} \circ R_i$ where $i = 0$ or 1 are discussed in the section "GRA rings with idempotent". We shall show $R_1 \circ R'_{½} \subseteq R'_{½}$; the others can be proved similarly. We shall first show

(16) $$(R', R_1 + R_0, e) = (R', e, R_1 + R_0) = 0.$$

Now $0 = \bar{D}(x, y_1, e, e)$ implies $(x, y_1, e)e = 0$; $0 = \bar{D}(x, e, y_1, e)$ implies $(x, e, y_1)e = 0$. Also, $0 = \bar{D}(x, e, e, y_1)$ implies $(x, e, y_1) = (x, y_1, e)$. From Lemma 7, $(x, e, y_1) + (x, y_1, e) = 0$, and by the assumption on the characteristic, $(x, e, y_1) = (x, y_1, e) = 0$. The proof $(R', R_0, e) = (R', e, R_0) = 0$ is similar. Using (14), (16), and Lemma 7, it is easy to verify that $e(x_1 \circ x_{1/2}) = x_1x_{1/2}$ and $(x_1 \circ x_{1/2})e = x_{1/2}x_1$. Thus $e \circ (x_1 \circ x_{1/2}) = x_1 \circ x_{1/2}$ and $x_1 \circ x_{1/2} \in R_{1/2}$. The proof that $e(x_0 \circ x_{1/2}) = x_{1/2}x_0$ and $(x_0 \circ x_{1/2})e = x_0x_{1/2}$ is similar.

H. F. Smith generalized Kleinfeld's work [6] and [7] by replacing the condition of simplicity with prime. If R is a GRA ring with an idempotent $e \neq 0, \neq 1$ and R is prime (or even semiprime), then $H = 0$ and $R = R'$. Lemma 6 says that e and $R_{1/2} = R'_{1/2}$ generate a right alternative ring. We thus find that the right alternative law holds in a large portion of the ring. Since the half space is really the only portion of the ring that an idempotent lets one manipulate, the products in R_1 and R_0 are hard to explore. Lemma 5 shows that $I = \{(a, b, b) | a, b \in R\}$ is an ideal of R contained in $R_1 + R_0$. Lemma 8 says that $(a, b, b) = (a_1, b_1, b_1) + (a_0, b_0, b_0)$. This means that the only places where R fails to be right alternative are "internal" products of R_1 and "internal" products of R_0. The word "internal" is put in quotations because, although $R_1R_0 = R_0R_1 = 0$, we do not necessarily have $R_1R_1 \subseteq R_1$ and $R_0R_0 \subseteq R_0$. It would seem reasonable that the condition of prime would force R to be right alternative, especially since we can show that $IR_{1/2} = R_{1/2}I = 0$. The difficulty is that in GRA rings, the annihilator of an ideal might not be an ideal.

LEMMA 9. *Let R be a GRA ring with idempotent e. Let I be any ideal contained in $R_1 + R_0$. Then $IR_{1/2} = R_{1/2}I = (R_{1/2}, I, R) = (R_{1/2}, R, I) = 0$.*

PROOF. We prove this lemma without assuming $H = 0$. The method of the proof is to examine various associators. We show that each associator must be an element of $R_{1/2}$. We then quote the hypothesis that I is an ideal contained in $R_1 + R_0$ and this implies the associator is in $R_{1/2} \cap I = 0$. We use (1) and the described technique to show successively that (e, I, R), (e, R, I), $(R_{1/2}, e, I)$, and $(R_{1/2}, I, e)$ are all zero. From these four associators being zero and by linearizing (2) we get $(i, e, x_{1/2}) + (i, x_{1/2}, e) = 0$ for all $i \in I$ and $x_{1/2} \in R_{1/2}$. Continuing, $0 = \bar{F}(e, x_{1/2}, i) + \bar{F}(x_{1/2}, e, i) = [x_{1/2}, i] - e \circ [x_{1/2}, i]$. We now have $[R_{1/2}, I] \subseteq R_{1/2} \cap I = 0$. Thus $[R_{1/2}, I] = 0$. We can apply the table (15) to the quotient ring R/H to deduce that $ix_{1/2} + x_{1/2}i \subseteq R'_{1/2} + H = R_{1/2}$. Since $[R_{1/2}, I] = 0$, we have $x_{1/2}i = ix_{1/2} \subseteq R_{1/2} \cap I = 0$. We have shown $R_{1/2}I = IR_{1/2} = 0$. Now

$(R_{1/2}, I, R) \subseteq (R_{1/2}I)R + R_{1/2}I = 0$ because $R_{1/2}I = 0$. By Lemma 4, $(R_{1/2}, R, I) \subseteq (R_{1/2}, I, R) + H = 0 + H$. Therefore $(R_{1/2}, R, I) \subseteq R_{1/2} \cap I = 0$.

THEOREM 3. *The right ideal P generated by $R'_{1/2}$ in R' is a right alternative ring, and $P + H$ is a right ideal of R.*

PROOF. Let $I = \{(R', y, y) | y \in R'\}$. I is an ideal of R' contained in $R_1 + R_0$. Let $P_1 = \{x \in R' | xI = (x, R', I) = (x, y, y) = 0$ for all $y \in R'\}$. By (1) P_1 is a right ideal of R'. By Lemma 9 and (14), $R'_{1/2} \subseteq P_1$. Since $P \subseteq P_1$, it will suffice to show that P_1 is a right alternative ring. But the condition $(P_1, y, y) = 0$ for all $y \in R'$ makes this obvious. In fact, we have actually showed that P_1 is in the right alternative nucleus of $R' = \{x \in R' | (x, y, y) = 0$ for all $y \in R'\}$. The last statement of the theorem is immediate.

We can use Lemma 9 to give an easy and illuminating proof of H. F. Smith's result, by showing that by adding either of the additional hypotheses of flexibility or of generalized left alternativity, annihilators of ideals become ideals.

THEOREM 4 (H. F. SMITH). *If R is a prime GRA ring with an idempotent $e \neq 0, \neq 1$ and R satisfies either* (3) *or* (4), *then R is alternative.*

PROOF. Let I be any ideal of R. Then $J = \{x | xI = Ix = (x, I, R) = (x, R, I) = (I, x, R) = (I, R, x) = (R, x, I) = (R, I, x) = 0\}$ is an ideal of R. We must show $x \in J \Rightarrow xR, Rx \subseteq J$. The proofs of $xR \cdot I = Rx \cdot I = I \cdot xR = I \cdot Rx = 0$ are immediate from assumptions on associators. The proofs $(xR, I, R) = (Rx, I, R) = (xR, R, I) = (Rx, R, I) = 0$ are from (1). In addition, from (1) we also have $I(x, R, R) = (x, R, R)I = 0$. The proofs of $(R, I, xR) = (R, I, Rx) = (I, R, xR) = (I, R, Rx) = 0$ are consequences of either (3) or (4). In addition, $(R, x, R)I = I(R, x, R) = (R, R, x)I = I(R, R, x) = 0$ from (1) and the previous calculations. The remaining cases, $(R, xR, I) = (R, Rx, I) = (I, Rx, R) = (I, xR, R) = 0$, follow from (6) and the previous calculations.

At this part of the proof we use the requirement that R is prime. If R is prime, $H = 0$ and $R = R'$. Thus $I = \{(R, x, x) | x \in R\}$ is an ideal contained in $R_1 + R_0$. By Lemma 9, $R_{1/2}I = IR_{1/2} = (R_{1/2}, I, R) = (R_{1/2}, R, I) = 0$. By (3) or (4) we get $(R, I, R_{1/2}) = (I, R, R_{1/2}) = 0$. By Lemma 8, $(R, R_{1/2}, I) \subseteq (R, I, R_{1/2}) = 0$ and $(I, R_{1/2}, R) \subseteq (I, R, R_{1/2}) = 0$.

By assuming R is prime, we conclude either $R_{1/2} = 0$ or $I = 0$. If $R_{1/2} = 0$, then, by (16), $R = R_1 \oplus R_0$ (direct sum); thus $R = R_1$, and e is an identity; contradiction. If $I = 0$ and R is flexible, then R is right alternative and flexible and clearly alternative. If $I = 0$ and R is generalized left alternative, by the mirror form of this proof, R is left alternative, and left alternative and right alternative imply R is alternative.

We have shown that a simple GRA ring with an idempotent $e \neq 0, \neq 1$ is right alternative. If the idempotent e also satisfies $(e, R, R) = 0$, then Thedy's result [11] for right alternative rings implies that the ring is alternative. The structure of simple GRA rings not possessing an idempotent, as well as the structure of simple right alternative rings, is still an open question.

The multiplication between the summands has been studied by [12] and others for the right alternative ring, and unless some further condition on the idempotent is made, usually $(e, e, R) = 0$, even with full right alternativity, the multiplication for the summands is unmanageable. The tables for GRA rings satisfying (3) and (4) have been constructed in [6] and [7}. We now construct the tables for Lie admissible GRA rings; i.e. GRA rings with idempotent $\neq 0$, $\neq 1$ satisfying (5). Such rings are called generalized $(-1, 1)$ rings.

5. **Generalized $(-1, 1)$ rings.** We have studied the structure of rings satisfying identities (1) and (2). We shall now study rings satisfying identities (1) and (5). In this section, we assume characteristic $\neq 2$ and characteristic $\neq 3$, so that we can apply the results of [3]. We shall call such rings generalized $(-1, 1)$ rings. Since identity (5) implies identity (2), all the results obtained so far will apply to generalized $(-1, 1)$ rings. In particular, if a generalized $(-1, 1)$ ring R has an idempotent, then $R' = \{x \in R | (x, e, e) = 0\}$ is a subring of R. We show that R' has a decomposition $R' = R_{11} + R'_{10} + R'_{01} + R_{00}$ and if there are no trivial R' ideals contained in the center of R', then the multiplication table for the summands will be the same as if R' were associative.

$$(R', R', R') \subseteq R_1 + R_0, \tag{17}$$

$$[e, (R', R', R')] = 0 = (e, e, (R', R', R')), \tag{18}$$

$$(e, R', R') \subseteq (e, e, R'). \tag{19}$$

PROOF. By Lemma 5 in [3], (17) is true if R' were a $(-1, 1)$ ring; consequently, $(R', R', R') \subseteq R_1 + R_0 + I$ where $I = \{(x, y, y) | x, y \in R'\}$ is an ideal by Lemma 1. By Lemma 5, $I \subseteq R_1 + R_0$. Statement (18) follows from the Albert decomposition since $ex_i = x_i e = ix_i$ for $i = 1$ or $i = 0$. Using the same argument as in Lemma 4 of [3] (last four lines) and also (1), (17), (13), (16) of this paper, we can prove that $(e, x, y) = (e, e, [e, [x, y]])$, which tells us that $(e, R', R') \subseteq (e, e, R')$.

Let us consider an element $x_{1/2} \in R'_{1/2}$. Then

$$x_{1/2} = ex_{1/2} + (e, e, [e, x_{1/2}]) + x_{1/2}e + (e, [e, x_{1/2}], e).$$

If we set $x_{10} = ex_{1/2} + (e, e, [e, x_{1/2}])$ and $x_{01} = x_{1/2}e + (e, [e, x_{1/2}], e)$, it is easily seen that $x_{10}, x_{01} \in R'_{1/2}$, that $x_{10}e \in R_1$, and that $ex_{01} \in R_0$. If we call $R'_{10} = \{y \in R' | y = ex_{1/2} + (e, e, [e, x_{1/2}])$ for some $x_{1/2} \in R'_{1/2}\}$, $R'_{01} = \{y \in R' | y = x_{1/2}e + (e, [e, x_{1/2}], e)$ for some $x_{1/2} \in R'_{1/2}\}$, then $R' = R_1 + R'_{10} + R'_{01} + R_0$ where the sum on the right-hand side is an additive direct sum.

Before proving the next lemma, we shall need the following:

(20) $\qquad 0 \equiv [a, (b, c, d)] - [b, (c, d, a)] + [c, (d, a, b)] - [d, (a, b, c)],$

(21) $\qquad R_1R_1 \subseteq R_1, \quad R_0R_0 \subseteq R_0.$

PROOF. The expression on the right-hand side of (20) equals $\bar{C}(a, b, c, d) - \bar{C}(b, c, d, a) + \bar{C}(c, d, a, b) - \bar{C}(d, a, b, c) - \bar{B}(ab, c, d) + \bar{B}(bc, d, a) - \bar{B}(cd, a, b) + \bar{B}(da, b, c)$. (21) is immediate from $0 \equiv \bar{B}$, (16) and Lemma 7.

LEMMA 10. *(e, e, R') is a trivial ideal in the center of R'. Furthermore $R'_{1/2}(e, e, R') = (e, e, R')R'_{1/2} = 0$.*

The proof is divided into five stages. In (a) we show $[R', (e, e, R')] = 0$. In (b), we show $(e, e, R'_{1/2})R'_{1/2} = R'_{1/2}(e, e, R'_{1/2}) = 0$. In (c), we show that (e, e, R') is an ideal by showing $(e, e, x)a = (e, e, (a_1 + a_0)x)$. To show (e, e, R') is in the nucleus, in (d) we show $(R', R', (e, e, R')) = 0$ and in (e) we show $((e, e, R'), R', R') = 0$. From $0 \equiv \bar{B}$, it is clear that $(R', (e, e, R'), R') = 0$. This will complete the proof that (e, e, R') is an ideal in the center of R'. The fact that (e, e, R') is an ideal in the center implies $(e, e, R')(e, e, R') = 0$.

PROOF OF (a). By (20), $[x, (e, e, y)] - [e, (e, y, x)] + [e, (y, x, e)] - [y, (x, e, e)] = 0$ from which it easily follows from (17) that $[x, (e, e, y)] = 0$ for every $x, y \in R'$.

PROOF OF (b). By (15) and (17) we derive $(e, e, R') \circ R'_{1/2} \subseteq R'_{1/2}$, whereas by Lemma 2 in [3], part (a), and Lemma 5, we have $(e, e, R') \circ R'_{1/2} \subseteq R_1 + R_0$. So $(e, e, R') \circ R'_{1/2} = 0$. By part (a) again we have $R'_{1/2}(e, e, R') = (e, e, R')R'_{1/2} = 0$.

PROOF OF (c). We shall first show $(e, x_{1/2}, a_1 + a_0)e = (e, ex_{1/2}, a_1 + a_0)$. By $0 = \bar{C}(a_1 + a_0, x_{1/2}, e, e)$ it is easily seen that $(a_1 + a_0, ex_{1/2}, e) = (a_1 + a_0, x_{1/2}, e)e$. But then $\bar{B}(a_1 + a_0, ex_{1/2}, e) = 0 = \bar{B}(a_1 + a_0, x_{1/2}, e)$ and (16) tell us that $(e, a_1 + a_0, x_{1/2})e = (e, a_1 + a_0, ex_{1/2})$. Equation (14) gives us $(e, x_{1/2}, a_1 + a_0)e = (e, ex_{1/2}, a_1 + a_0)$. From $0 = \bar{D}(e, e, a_1 + a_0, x_{1/2})$ we get $(e, e, (a_1 + a_0)x_{1/2}) = (e, e, x_{1/2})(a_1 + a_0)$. By part (b), (16) and (21) we can say

(22) $\qquad (e, e, x)a = (e, e, (a_1 + a_0)x_{1/2}) = (e, e, (a_1 + a_0)x).$

PROOF OF (d). We will show $(b, c, (e, e, x)) = 0$ when b, c, and x are in the summands. If b or c are elements of $R'_{1/2}$, then by Lemma 8, $(b, c, (e, e, x)) = -(b, (e, e, x), c) = 0$ by parts (b) and (c). Let us suppose that both b and c

are elements of R_1 or R_0. By (21), (a), (22), and (18), $(b, c, (e, e, x)) = (e, e, (b, c, x)) = 0$.

PROOF OF (e). We will show $((e, e, x), b, c) = 0$ for all $x, b, c \in R'$. From (16), $(e, e, x) = (e, e, x_{1/2}) = (e, e, x_{10} + x_{01}) = x_{10}e + ex_{01}$, where $(e, e, x_{10}) = x_{10}e \in R_1$ and $(e, e, x_{01}) = ex_{01} \in R_0$. We use $0 \equiv \bar{B}$ and part (d), followed by part (b) and part (c), to get

$$\begin{aligned}((e, e, x), b, c) &= -(c, (e, e, x), b) = -(c_1 + c_0, (e, e, x), b_1 + b_0)\\ &= -(c_1, (e, e, x_{10}), b_1) - (c_0, (e, e, x_{01}), b_0).\end{aligned}$$

It will suffice to show that $((e, e, x_{10}), b_1, c_1) = 0$ and $((e, e, x_{01}), b_0, c_0) = 0$. We will prove the first. The second follows by reversing subscripts. From part (d), Lemma 8, (21) and (16) we get

$$\begin{aligned}0 &= \bar{A}(b_1, c_1, x_{10}, e) - \bar{A}(c_1, b_1, x_{10}, e) - \bar{A}(e, x_{10}, b_1, c_1)\\ &\quad + \bar{B}(e, x_{10}, [b_1, c_1]) + \bar{B}(b_1, c_1, x_{10})\\ &= e(x_{10}, b_1, c_1) + (x_{10}e, b_1, c_1) + [c_1, (b_1, x_{10}, e)] - [b_1, (c_1, x_{10}, e)].\end{aligned}$$

Using $0 \equiv \bar{B}$ and the results (16), (19), and part (a), both $[c_1, (b_1, x_{10}, e)] = 0$ and $[b_1, (c_1, x_{10}, e)] = 0$. From $\bar{A}(x_{10}, e, b_1, c_1)$, (21), (16), and (18), we have $(x_{10}e, b_1, c_1) = (x_{10}, b_1, c_1)e = e(x_{10}, b_1, c_1)$. We have shown $2(x_{10}e, b_1, c_1) = 0$ and therefore $0 = (x_{10}e, b_1, c_1) = ((e, e, x_{10}), b_1, c_1)$.

This completes the proof of Lemma 10.

THEOREM 5. *The multiplication table for* $R' = R_{11} + R'_{10} + R'_{01} + R_{00}$ *is*:

	R_{11}	R'_{10}	R'_{01}	R_{00}
R_{11}	R_{11}	$R'_{10} + R_{11}$	R_{00}	0
R'_{10}	R_{11}	R_{11}	R_{11}	$R'_{10} + R_{11}$
R'_{01}	$R'_{01} + R_{00}$	R_{00}	R_{00}	R_{00}
R_{00}	0	R_{11}	$R'_{01} + R_{00}$	R_{00}

PROOF. The summands themselves have been previously defined. We write R_{11} for R_1 and R_{00} for R_0. We will prove the table summand by summand; after one product is calculated, it will be used to compute subsequent results. It is best to remember that $R'_{10}e \subseteq R_{11}$ and $eR'_{01} \subseteq R_{00}$. We have already seen in the observation following Lemma 4 that $R_{11}R_{00} = R_{00}R_{11} = 0$. By (21) $R_{ii}R_{ii} \subseteq R_{ii}$. By (16) $(R', e, R_{11} + R_{00}) = 0$ and we consequently have $R'_{01}R_{00} \subseteq R_{00}$ and $R'_{10}R_{11} \subseteq R_{11}$. By Lemma 7 and $\bar{B}$, $(x_{11}, e, x_{01}) =$

$-(x_{11}, x_{01}, e) = (e, x_{11}, x_{01})$, from which $(x_{11}x_{01})e = 0 = e(x_{11}x_{01})$ and so $R_{11}R'_{01} \subseteq R_{00}$. Reversing subscripts gives $R_{00}R'_{10} \subseteq R_{11}$. From $(x_{01}, x_{11}, e) = 0$ and $(e, x_{01}, x_{11}) = -(e, x_{11}, x_{01})$, we obtain $(x_{01}x_{11})e = x_{01}x_{11}$ and $e(x_{01}x_{11}) = x_{11}x_{01} \in R_{00}$. This gives us $x_{01}x_{11} = a_{01} - ea_{01}$ and so $R'_{01}R_{11} \subseteq R'_{01} + R_{00}$. Reversing the subscripts we shall have $R'_{10}R_{00} \subseteq R'_{10} + R_{11}$. From Lemma 8, $(x_{00}, e, x_{01}) = -(x_{00}, x_{01}, e)$ and $(e, x_{00}, x_{01}) = -(e, x_{01}, x_{00})$. We thus obtain $(x_{00}x_{01})e = x_{00}x_{01}$ and $e(x_{00}x_{01}) = (ex_{01})x_{00} \in R_{00}$. So $x_{00}x_{01} = a_{01} - ea_{01}$ and $R_{00}R'_{01} \subseteq R'_{01} + R_{00}$, and by reversing subscripts, $R_{11}R'_{10} \subseteq R'_{10} + R_{11}$.

Let $x_{01}, y_{01} \in R'_{01}$. By Lemma 10

$$(ex_{01})y_{01} = 0 = x_{01}(ey_{01}). \tag{23}$$

From $\bar{C}(x_{01}, y_{01}, e, e)$ and $(R'_{01}, R_{00}, e) = 0$, we have $(x_{01}, y_{01}, e)e = 0 = -(x_{01}, e, y_{01})e$, which together with (23) gives $(x_{01}y_{01})e = 0$. Since $x_{01}y_{01} = (x_{01}, e, y_{01}) \in R_1 + R_0$, we have $x_{01}y_{01} \in R_{00}$. Reversing subscripts gives $R'_{10}R'_{10} \subseteq R_{11}$.

Let $x_{01} \in R'_{01}, y_{10} \in R'_{10}$. By Lemma 7 and Lemma 10 we derive $0 = (x_{01}, e, y_{10}) = -(x_{01}, y_{10}, e)$. Thus $(x_{01}y_{10})e = 0$. Similarly $0 = (y_{10}, e, x_{01}) = -(y_{10}, x_{01}, e)$. So $(y_{10}x_{01})e = y_{10}x_{01}$. By (15), $x_{01}y_{10} + y_{10}x_{01} \in R_{11} + R_{00}$. It is seen then that $x_{01}y_{10} \in R_{00}$, and $y_{10}x_{01} \in R_{11}$.

LEMMA 11. $[R', (R', R', e)] = 0$.

PROOF. From (20) we have $0 = [a, (b, c, e)] - [b, (c, e, a)] + [c, (e, a, b)] - [e, (a, b, c)]$. Lemma 10, (19), and (18) imply $[a, (b, c, e)] = -[b, (c, a, e)]$. Iterating this three times gives $[a, (b, c, e)] = -[a, (b, c, e)]$, and weakly characteristic $\neq 2$ implies $[a, (b, c, e)] = 0$.

LEMMA 12. *If R' has no trivial R' ideals contained in its center, then the table for $R' = R_{11} + R'_{10} + R'_{01} + R_{00}$ is an associative table. I.e. $R_{ij}R_{kl} \subseteq \delta_{jk}R_{il}$ *.*

PROOF. * is a compact way of expressing the product. It is to be understood that R_{10} and R_{01} are to be primed, R'_{10} and R'_{01}, when they appear. The Kronecker δ_{ij} is defined by $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$. The table is called an associative table because any associative ring has such a table. We have shown $(e, R', R') \subseteq (e, e, R')$ is a trivial ideal in the center by Lemma 10 and (19). This implies * holds except for the cases $R'_{10}R'_{10}$ and $R'_{01}R'_{01}$. We will show $R'_{10}R'_{10}$ is a trivial ideal in the center; the case $R'_{01}R'_{01}$ is obtained by reversing subscripts. Since $x_{10}y_{10} = (x_{10}, y_{10}, e)$, Lemma 11 tells us that $[(R'_{10})^2, R'] = 0$. Since $(R'_{10})^2R'_{1/2} = R'_{1/2}(R'_{10})^2 \subseteq R_{11}R'_{01} + R'_{10}R_{11} = 0$, we have $(R'_{10})^2R'_{1/2} =$

$R'_{1/2}(R'_{10})^2 = 0$. To show $(R'_{10})^2$ is an ideal, it suffices to show $R_{11}(R'_{10})^2 \subseteq (R'_{10})^2$. Now

$$a_{11}(b_{10}c_{10}) = (b_{10}c_{10})a_{11} = b_{10}(c_{10}a_{11}) - (b_{10}, a_{11}, c_{10})$$
$$= b_{10}(a_{11}c_{10}) \in R'_{10}R'_{10}.$$

$$a_{11}(b_{10}c_{10}) = (b_{10}c_{10})a_{11} = b_{10}(a_{11}c_{10}). \tag{24}$$

We now will show that $(R'_{10})^2$ is contained in the nucleus of R'. If x and y are not both in R_{11}, we have just proven that $(x, (R'_{10})^2, y) = 0$. By Lemma 8 and (21), $(x, y, (R'_{10})^2) = 0$, and by $0 \equiv \bar{B}$, $((R'_{10})^2, x, y) = 0$. To show $(R'_{10})^2$ is in the nucleus, it is sufficient to show

$$(R_{11}, R_{11}, (R'_{10})^2) = (R_{11}, (R'_{10})^2, R_{11}) = ((R'_{10})^2, R_{11}, R_{11}) = 0.$$

Using (24) we have $(a_{11}, b_{11}, c_{10}d_{10}) = c_{10}(a_{11}, b_{11}, d_{10}) = -c_{10}(a_{11}, d_{10}, b_{11}) = 0$. Thus $(R'_{10})^2$ is in the right nucleus. By $0 = \bar{B}(e, c_{10}, d_{10})$ we get $c_{10}d_{10} = d_{10}c_{10}$, and by (24) $a_{11}(c_{10}d_{10}) = (a_{11}c_{10})d_{10}$. This means

$$(a_{11}, c_{10}d_{10}, b_{11}) = (a_{11}c_{10})(b_{11}d_{10}) - (a_{11}c_{10})(b_{11}d_{10}) = 0,$$

and $(R'_{10})^2$ is in the middle nucleus. By $0 \equiv \bar{B}$, $(R'_{10})^2$ is in the center, and consequently $(R'_{10})^2(R'_{10})^2 = 0$.

LEMMA 13. *If R' is a generalized* $(-1, 1)$ *ring with no trivial ideals in the center, then $R'_{10}R'_{01} + R'_{10} + R'_{01} + R'_{01}R'_{10}$ is an ideal in the nucleus.*

PROOF. The proof is the same as Theorem 2 in [3].

COROLLARY. *If R is a prime generalized* $(-1, 1)$ *ring with idempotent $e \neq 0, \neq 1$, then R is associative.*

PROOF. From Lemma 2, $R = R'$. The proof follows because if I is an R ideal contained in the nucleus of R, then the annihilator of I is an ideal containing all associators. Thus if R is not associative, then $R_{1/2} = 0$ and e is an identity.

One would hope that the results of Lemma 12 and Lemma 13 would be true for R itself and not just for R'. We would like to say that if R has no trivial R ideals in its center, then R has an associative table, or equivalently, that e is in the nucleus of R. In finite dimensional $(-1, 1)$ algebras, this result is true, and, in fact, finite dimensional $(-1, 1)$ algebras without trivial ideals contained in their centers are associative [4]. In generalized $(-1, 1)$ rings, showing that e is in the nucleus is impossible. $(H, e, e) = H$ shows that, while H is a trivial ideal and $[H, R] = 0$, H is not contained in the center and so will not be forced to be zero by the hypotheses. At the same time, since $H = (H, e, e)$, it will be impossible to show that e is in the nucleus of R. The example following Lemma 2 is actually a generalized $(-1, 1)$ ring, so there is no hope of proving that $H = 0$ from the basic generalized $(-1, 1)$ ring hypotheses. We can see that the existence of H means that Lemma 12 cannot be automatically generalized. But as the next theorem shows, H is the only obstacle, and in fact Lemma 12 and Lemma 13 can be extended except for the obvious impossibilities involving H.

THEOREM 6. *If R is a generalized $(-1, 1)$ ring without trivial R ideals contained in the center of R, then e is in the nucleus of R' and $R'_{10}R'_{01} + R'_{10} + R'_{01} + R'_{01}R'_{10}$ is an ideal in the nucleus of R.*

PROOF. The same proof given for Lemma 12 will apply here if we can show that the appropriate ideals in the center of R' are also in the center of R. Please refer to the proof of Lemma 12. By expansion, $(e, H, R') = 0$. Applying this to three of the four terms of $\bar{D}(e, H, R', e)$, we get $(e, e, R')H = 0$. Thus (e, e, R') is a trivial ideal in the center of R and is zero by hypothesis. Let $h_{1/2} \in H$, $r_{1/2} \in R'_{1/2}$. From Lemma 2 and the Albert decomposition for power associative rings, $2h_{1/2}r_{1/2} = h_{1/2}r_{1/2} + r_{1/2}h_{1/2} \subseteq (R_1 + R_0) \cap H = 0$. Therefore $R'_{1/2}H = HR'_{1/2} = 0$. Using (1) we have $(R'_{10}R'_{10} + R'_{01}R'_{01} + R'_{10}R'_{01} + R'_{01}R'_{10}, e, H) = 0$ and hence $(R'_{10}R'_{10})H = (R'_{01}R'_{01})H = (R'_{10}R'_{01})H = (R'_{01}R'_{10})H = 0$. This implies that $(R'_{10})^2$ and $(R'_{01})^2$ are trivial ideals in the center of R. It also implies $R'_{10}R'_{01} + R'_{10} + R'_{01} + R'_{01}R'_{10}$ is an ideal in the nucleus of R.

Except for the ideal H, generalized $(-1, 1)$ rings have a Peirce decomposition, and again, very minor hypotheses will bring this out.

THEOREM 7. *Let R be a generalized $(-1, 1)$ ring with an idempotent $e \neq 0, \neq 1$. Furthermore, assume that R has no trivial ideals I such that $[I, R] = 0$. Then R has a Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ and $R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10}$ is an ideal in the nucleus of R.*

PROOF. From Lemma 2, $H = 0$. Thus $R = R'$. The theorem follows from Lemma 12 and Lemma 13.

6. **Truncation.** We now add a result which is of a different nature than the preceding results. In those, we showed that if R were a generalized $(-1, 1)$ ring, then a homomorphic image of R has a Peirce decomposition. The kernel of this homomorphism was, among other things, a nilpotent ideal. We now show how to get a Peirce decomposition in another way, not by considering a quotient ring, but by changing the multiplication in R slightly. We let $H * R = R * H = 0$, and for the products in R', where $x_{ij}y_{kl} = a_{11} + a_{10} + a_{01} + a_{00}$, let $x_{ij} * y_{kl} = \delta_{jk}a_{il}$. We call $*$ the truncation product. Such a product will obviously give R' an associative table, but the interesting result is that $R + *$ remains a generalized $(-1, 1)$ ring. Furthermore, the variation of the product from the original

product is contained in the ideal $K = (R', R', e) + H$ which satisfies $[K, R] = 0$, $K^2 \subseteq (e, e, R') \subseteq$ center of R and $K^3 = 0$.

THEOREM 8. *If R is a generalized $(-1, 1)$ ring with idempotent $e \neq 0, \neq 1$, then R under the truncation product is still a generalized $(-1, 1)$ ring.*

PROOF. The proof that R remains a generalized $(-1, 1)$ ring is a case-by-case procedure. We will not present it here.

REFERENCES

1. A. A. Albert, *Power-associative rings*, Trans. Amer. Math. Soc. **64** (1948), 552–593. MR **10**, 349.

2. ———, *The structure of right alternative algebras*, Ann. of Math. (2) **59** (1954), 408–417. MR **15**, 774.

3. I. R. Hentzel, *(−1, 1) rings*, Proc. Amer. Math. Soc. **22** (1969), 367–374. MR **39** #4236.

4. ———, *(−1, 1) algebras*, Proc. Amer. Math. Soc. **24** (1970), 24–28. MR **40** #1440.

5. E. Kleinfeld, *Alternative nil rings*, Ann. of Math. (2) **66** (1957), 395–399. MR **19**, 383.

6. ———, *Generalization of alternative rings*. I, J. Algebra **18** (1971), 304–325. MR **43** #308.

7. ———, *Generalization of alternative rings*. II, J. Algebra **18** (1971), 326–339. MR **43** #308.

8. C. Maneri, *Simple (−1, 1) rings with an idempotent*, Proc. Amer. Math. Soc. **14** (1963), 110–117. MR **26** #165.

9. M. Slater, *The open case for simple alternative rings*, Proc. Amer. Math. Soc. **19** (1968), 712–715. MR **37** #267.

10. H. F. Smith, *Prime generalized alternative rings with nontrivial idempotent*, Proc. Amer. Math. Soc. **39** (1973), 242–246. MR **47** #1903.

11. A. Thedy, *Right alternative rings*, Preprint Series 11, Matematisk Institut, Aarhus Universitet, 1972 (unpublished).

12. ———, *Right alternative rings with Peirce decomposition*, Preprint Series 17, Matematisk Institut, Aarhus Universitet, 1972 (unpublished).

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