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A GENERALIZED TOPOLOGICAL MEASURE THEORY

BY

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ABSTRACT. The theory of measures in a topological space, as developed by V. S. Varadarajan for the algebra C^b of bounded continuous functions on a completely regular topological space, is extended to the context of an arbitrary uniformly closed algebra A of bounded real-valued functions. Necessary and sufficient conditions are given for A^* to be represented in the natural way by a space of regular finitely-additive set functions. The concepts of additivity and tightness for these set functions are considered and some remarks about weak convergence are made.

Introduction. It is now more than three decades since the appearance in 1940 and 1941 of the pioneering investigations of A. D. Alexandrov [1] into finitely-additive set functions in topological spaces, and more than a decade since the definitive study of V. S. Varadarajan in 1961 of measures in topological spaces [21]. From these beginnings, there has developed in recent years a rather substantial topological measure theory through the efforts of many investigators. Among them should be mentioned Fremlin, Garling and Haydon [4], E. Granirer [6], J. D. Knowles [10], W. Moran [11], [12], F. D. Sentilles [15], [16], [17], R. F. Wheeler [17], [20] and S. Mosiman [13]. Almost all of this work has been done in the context of a completely regular Hausdorff space, and the studies have been concerned with the algebra C^b of bounded, continuous real-valued functions and its dual space. The first author of the present note has believed for some time now that there should be an analogous theory, for an arbitrary uniformly closed algebra of bounded, real-valued functions and its dual space. Besides being of interest in its own right, such a theory should provide new insight into topological measure theory by allowing one to study C^b and its dual by means of its proper subalgebras and their duals. It should also be of interest in probability theory where a proper subalgebra of C^b may be of more relevance in a particular problem than C^b itself. It is our purpose in the present paper to develop the beginnings of such a theory.

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From the outset of our study, we adopted the principle that a theory of representations of duals of algebras by spaces of measures will only be useful if analogs of the major results in topological measure theory can be shown to hold. Subject to this principle, we have tried to develop our results in as general a setting as possible. As the expert will see, the ideas for the proofs of some of the theorems are similar to the proofs in the topological case but often require nontrivial technical modifications. A number of the deeper results and questions (in particular metrization problems and certain questions about weak compactness) require further investigation and are not treated here. We believe that the theory developed is quite comprehensive and that it should serve well for future developments.

In §1, we introduce the notion of a paving $\mathcal{W}$ and discuss the space $M(\mathcal{W})$ of $\mathcal{W}$-regular set functions. There are no proofs here, and the reader is referred to [8] for further details. In §§2 and 3 a uniformly closed algebra A of bounded real-valued functions which contains the constants is considered. Theorem 3.8 gives necessary and sufficient conditions on the paving $\mathcal{W}$ in order that $M(\mathcal{W})$ represent the dual of A in the usual way. (This provides a generalization of Alexandrov's representation theorem.) As an application of this theorem, it is shown in Theorem 3.12 that if $\mathcal{W}$ is a normal base for the weak topology generated by A on the underlying set X and if A is the set of restrictions to X of $C(X_{\mathfrak{W}})$ (where $X_{\mathfrak{W}}$ is the Wallman compactification of X for $\mathcal{W}$), then $M(\mathcal{W})$ represents the dual of A. This means that if the Frink conjecture from general topology holds (that is, if every compactification is a Wallman compactification), then the dual of every algebra can be represented by $M(\mathcal{W})$ for some paving $\mathcal{W}$. In §4 various notions of additivity for elements of $M(\mathcal{W})$ are discussed and they are related to corresponding conditions on the functionals on A. In §5, tightness for the set functions and the corresponding functionals is discussed. In §6 some results about weak convergence are obtained. Finally in an appendix, counterexamples to several natural conjectures are given.

1. The space $M(\mathcal{W})$.

DEFINITION 1.1. Let X be a set. A family $\mathcal{W}$ of subsets of X is a *paving* if the following hold.

1. $\emptyset \in \mathcal{W}$.
2. $\bigcup \mathcal{W} = X$.
3. If $W_1, W_2 \in \mathcal{W}$, then $W_1 \cap W_2, W_1 \cup W_2 \in \mathcal{W}$.

If $X \in \mathcal{W}$, then $\mathcal{W}$ is a *full paving.*

If $\mathcal{W}$ is a paving on X, then $\mathcal{F}(\mathcal{W})$ and $\Sigma(\mathcal{W})$ (or just $\mathcal{F}$ and Σ) denote the ring and the σ-ring respectively of subsets of X generated by $\mathcal{W}$. The following is a list of some important pavings.

EXAMPLES. 1. Any ring $\mathcal{W}$ of subsets of X with $X = \bigcup \mathcal{W}$.

2. The family $\mathcal{G}$ of closed sets in a topological space.

3. The family $\mathcal{Z}$ of zero sets in a topological space.

4. The family $\mathcal{K}$ of compact sets in a topological space.

5. The family of bounded sets in a uniform space. (A set is bounded if it is bounded for every pseudometric in the gauge of the uniformity.)

PROPOSITION 1.2. *Let $\mathcal{W}$ be a paving on X. If $F \subset X$, then $F \in \mathcal{F}$ if and only if there are sets $W_i, V_i \in \mathcal{W}$ $(i = 1, \cdots, n)$ such that the following hold.*

1. $V_i \subset W_i$ *for* $i = 1, \cdots, n$.

2. $(W_i - V_i) \cap (W_j - V_j) = \emptyset$ *for* $i \neq j$.

3. $F = \bigcup_{i=1}^n (W_i - V_i)$.

DEFINITION 1.3. Let $\mathcal{W}$ be a paving on X. A finitely-additive, real-valued function m on $\mathcal{F}$ is *regular* (*$\mathcal{W}$-regular*) if for each $F \in \mathcal{F}$ and each $\epsilon > 0$, there is $W \in \mathcal{W}$ with $W \subset F$ and $|m(G)| \leq \epsilon$ whenever $G \in \mathcal{F}(\mathcal{W})$ with $G \subset F - W$. It is called *finite* if $\sup\{|m(G)|: G \in \mathcal{F} \text{ and } G \subset F\} < \infty$ for each $F \in \mathcal{F}$.

PROPOSITION 1.4. *Let m be a nonnegative finitely-additive, real-valued function on $\mathcal{F}$. These are equivalent.*

1. *m is regular.*

2. *For* $F \in \mathcal{F}$, $m(F) = \sup\{m(W): W \in \mathcal{W}, W \subset F\}$.

3. *Let* $W_1, W_2 \in \mathcal{W}$ *with* $W_1 \subset W_2$. *Then* $m(W_2) = \sup\{m(W_1 \cup W): W \in \mathcal{W}, W \subset W_2, W_1 \cap W = \emptyset\}$.

Let $M(\mathcal{W})$ (or M) denote the set of all finite, finitely-additive, regular real-valued functions on $\mathcal{F}$. With respect to the usual operations of pointwise addition, multiplication and order, $M(\mathcal{W})$ is an ordered vector space.

DEFINITION 1.5. Let $m \in M$ and $F \in \mathcal{F}$. Define

$$m^+(F) = \sup\{m(G): G \in \mathcal{F} \text{ and } G \subset F\},$$

$$m^-(F) = \sup\{-m(G): G \in \mathcal{F} \text{ and } G \subset F\},$$

$$|m|(F) = \sup\{m(G_1) - m(G_2): G_1, G_2 \in \mathcal{F}, G_1 \cup G_2 \subset F \text{ and } G_1 \cap G_2 = \emptyset\}.$$

PROPOSITION 1.6. *If $m \in M$, then $m^+, m^-, |m| \in M$. Furthermore, $m^+ = \sup(m, 0)$, $m^- = -\inf(m, 0)$ and $|m| = \sup(m, -m)$.*

PROPOSITION 1.7. *M is a Dedekind complete Riesz space.*

In the case that $\mathcal{W}$ is a full paving, define $\|m\| = |m|(X)$ for each $m \in M$. Then $\|\cdot\|$ is a Riesz norm on M. (That is, $\|m_1\| \leq \|m_2\|$ whenever $0 \leq m_1 \leq m_2$.) Furthermore, for $0 \leq m_1, m_2$ it follows that $\|m_1 + m_2\| = \|m_1\| + \|m_2\|$. As a consequence of the following proposition M is an L-space.

PROPOSITION 1.8. *Let $\mathcal{W}$ be a full paving. Then M is norm complete.*

The proofs of the above propositions appear in [9, pp. 452–454].

2. **Standard representations.** Throughout the remainder of the paper, A will denote a uniformly closed algebra of bounded, real-valued functions on X which contains the constants and separates points. Let A^* denote the Banach dual of A for the sup norm. It is an immediate consequence of the Weierstrass approximation theorem that A is a Riesz space under the usual ordering. Furthermore, since $1 \in A$, A^* is also the order dual of A and, hence, a Dedekind complete Riesz space.

DEFINITION 2.1. Let $\mathcal{W}$ be a full paving on X. A *standard representation* of A^* in $M(\mathcal{W})$ is a linear map I of A^* into $M(\mathcal{W})$ with the property that if $0 \leqslant \varphi \in A^*$, then $I_\varphi(W) = \inf\{\varphi(f)\colon f \in A, \chi_W \leqslant f\}$ for all $W \in \mathcal{W}$.

REMARKS. 1. If there is a standard representation of A^* in $M(\mathcal{W})$, then it is unique.

2. If I is a standard representation of A^* in $M(\mathcal{W})$, then I is order preserving.

3. If I is a standard representation of A^* in $M(\mathcal{W})$, then I is bounded (for the norm topologies) and $\|I\| \leqslant 1$.

PROPOSITION 2.2. *Let I be a standard representation of A^* in $M(\mathcal{W})$. If $m \in M(\mathcal{W})^+$, then there is $0 \leqslant \varphi \in A^*$ with $I_\varphi = m$.*

PROOF. Let S denote the Riesz space of simple functions over $\mathcal{F}$. For $s = \Sigma a_i \chi_{F_i} \in S$ in canonical form, define $\psi(s) = \Sigma a_i m(F_i)$. Then $\psi(s) \leqslant p(s)$ for all $s \in S$ where p is the subadditive functional defined on $B(X)$ (all bounded real-valued functions) by $p(f) = \|f^+\|_X \|m\|$. By the Hahn-Banach theorem, there is a linear functional Ψ on $B(X)$ which extends ψ and satisfies $\Psi(f) \leqslant p(f)$ for all $f \in B(X)$. Since $f \leqslant 0$ implies $p(f) = 0$ and so $\Psi(f) \leqslant 0$, it is clear that Ψ is a nonnegative linear functional.

Let φ be the restriction of Ψ to A. Since $0 \leqslant \varphi$, $\varphi \in A^*$. If $I_\varphi = m'$, then for each $W \in \mathcal{W}$, $m'(W) = \inf\{\varphi(f)\colon f \in A, \chi_W \leqslant f\} \geqslant \psi(\chi_W) = m(W)$. The $\mathcal{W}$-regularity now implies that $m \leqslant m'$. Since $m(X) = \varphi(1) = m'(X)$ it follows that $m = m'$. Thus $I\varphi = m' = m$, and the proof is complete.

COROLLARY 2.3. *Let I be a standard representation of A^* in M. Then I is onto M.*

PROPOSITION 2.4. *Let I be a standard representation of A^* in M. The following are then equivalent.*

1. *I is lattice preserving.*
2. *I is norm preserving.*
3. *I is one-one.*

PROOF. ($1 \Rightarrow 2$) $\|I\varphi\| \leqslant \|\varphi\| \leqslant \||\varphi|\| = |\varphi|(1) = I|\varphi|(X) = |I\varphi|(X) = \|I\varphi\|$. Thus $\|I\varphi\| = \|\varphi\|$ as claimed.

($2 \Rightarrow 3$) This is obvious.

($3 \Rightarrow 1$) Let $\varphi \in A^*$. It is sufficient to show that $(I\varphi)^+ = I(\varphi^+)$. Since $0 \leqslant (I\varphi)^+ \in M$, there is by Proposition 2.2 a $\psi \in A^*$ with $0 \leqslant \psi$ and $I\psi = (I\varphi)^+$. Since $(I\varphi)^+ \leqslant I(\varphi^+)$, the fact that $0 \leqslant I(\varphi^+ - \psi)$ implies that $0 \leqslant \varphi^+ - \psi$. (Indeed, by Proposition 2.2, there is $0 \leqslant \xi \in A^*$ with $I\xi = I(\varphi^+ - \psi)$. Since I is one-one, $\xi = \varphi^+ - \psi$.) Since $0 \leqslant (I\varphi)^+ - I\varphi = I(\psi - \varphi)$, the same reasoning gives that $0 \leqslant \psi - \varphi$. But $0 \leqslant \psi$ and $\varphi \leqslant \psi$ imply that $\varphi^+ \leqslant \psi$. But $\psi \leqslant \varphi^+$ was shown above so that $\varphi^+ = \psi$. Thus $(I\varphi)^+ = I\psi = I(\varphi^+)$ as claimed.

DEFINITION 2.5. *$M(\mathfrak{W})$ represents A^** if there is a standard representation of A^* onto $M(\mathfrak{W})$ which is a Banach lattice isomorphism.

REMARK. In view of Propositions 2.2 and 2.4, $M(\mathfrak{W})$ represents A^* if and only if there is a one-one standard representation of A^* in $M(\mathfrak{W})$.

Let $S(\mathfrak{W})$ (or S) denote the Riesz space of simple functions over the algebra $\mathfrak{F} = \mathfrak{F}(\mathfrak{W})$. For $m \in M^+$ and $s \in S$, let $\int_X s\,dm$ be defined in the usual way. An $f \in B(X)$ is *m-integrable* (Riemann sense) if

$$(*) \qquad \sup\left\{\int_X s\,dm\colon\ s \in S,\ s \leqslant f\right\} = \inf\left\{\int_X t\,dm\colon\ t \in S,\ f \leqslant t\right\}.$$

If f is m-integrable, the number defined by $(*)$ is denoted by $\int_X f\,dm$. The function $f \in B(X)$ is *$\mathfrak{W}$-integrable* if f is m-integrable for all $m \in M^+$.

The following observation is an immediate corollary of the usual proof of the analytic form of the Hahn-Banach theorem. We state it here without proof.

PROPOSITION 2.6. *Let E be a real linear space, p a subadditive and positive homogeneous functional on E, F a subspace of E and $\varphi(x) \leqslant p(x)$ for all $x \in F$. Then φ has a unique extension to a linear functional Φ on E satisfying $\Phi(y) \leqslant p(y)$ for all $y \in E$ if and only if, for each $x \in E$,*

$$\sup_{y \in F}\,[\varphi(y) - p(y - x)] = \inf_{z \in F}\,[p(x + z) - \varphi(z)].$$

THEOREM 2.7. *Let M represent A^*. Then every $f \in A$ is $\mathfrak{W}$-integrable. Furthermore, if $m = I\varphi$ for some $\varphi \in A^*$, then $\varphi(f) = \int_X f\,dm$ for all $f \in A$.*

PROOF. Fix $0 \leqslant m \in M$ and without loss of generality assume that $m(X) = 1$. For $s \in S$, define $\varphi(s) = \int_X s\,dm$. Let E denote the linear hull of $A \cup S$ in $B(X)$; and for $f \in E$, let $p(f) = \|f^+\|_X$. Then p is a subadditive and positive homogeneous functional on E and $\varphi(s) \leqslant p(s)$ for all $s \in S$. The proof hinges on the following fact.

(*) *There is only one linear functional* Φ *on* E *which extends* φ *and satisfies* $\Phi(f) \leq p(f)$ *for all* $f \in E$.

Indeed, let Φ be any such functional. Then $\Phi \leq p$ implies that Φ is nonnegative. Hence if ψ is the restriction of Φ to A, $0 \leq \psi$ so that $\psi \in A^*$. Let $0 \leq m' = I\psi$, where I is the given standard representation of A^* in $M(\mathcal{W})$. Then for all $W \in \mathcal{W}$, $m'(W) = \inf\{\psi(f): f \in A, \chi_W \leq f\}$. But for $f \in A$ and $W \in \mathcal{W}$, if $\chi_W \leq f$, then $m(W) = \varphi(\chi_W) \leq \Phi(f) = \psi(f)$ so that $m(W) \leq m'(W)$. By $\mathcal{W}$-regularity, it follows that $m \leq m'$. On the other hand, $m(X) = \varphi(1) = \psi(1) = m'(X)$ so that again by $\mathcal{W}$-regularity, $m' \leq m$. Thus $m = m'$. That is $I\psi = m$.

In order to complete the verification of (*), let Φ_1 and Φ_2 be two linear functionals on E of the kind specified in (*). If ψ_1 and ψ_2 denote the restrictions to A of Φ_1 and Φ_2 respectively, then $I\psi_1 = m = I\psi_2$ as shown above. Since I is one-one, it follows that $\psi_1 = \psi_2$. If $h \in E$, then $h = s + f$ for some choice of $s \in S$ and $f \in A$. Hence $\Phi_1(h) = \varphi(s) + \psi_1(f) = \varphi(s) + \psi_2(f) = \Phi_2(h)$. That is, $\Phi_1 = \Phi_2$ as was to be shown.

By combining (*) with Proposition 2.6, we obtain for each $f \in A$,

$$\psi(f) = \Phi(f) = \sup_{s \in S} [\varphi(s) - p(s - f)] = \inf_{t \in S} [p(t + f) - \varphi(t)] \tag{1}$$

where ψ, Φ are as above. Let (s_n^*) and (t_n^*) be sequences in S such that $\varphi(s_n^* + t_n^*) - [p(s_n^* - f) + p(t_n^* + f)] \to 0$ as $n \to \infty$. (This is possible by (1).) For each $n \in N$, define $s_n = s_n^* - p(s_n^* - f)1$ and $t_n = p(f + t_n^*)1 - t_n^*$. Then $s_n, t_n \in S$ for all $n \in N$. Since $s_n^* - f \leq p(s_n^* - f)$, it follows that $s_n \leq f$. Similarly, it follows that $f \leq t_n$. Since $0 \leq \int_X (t_n - s_n)\,dm = \varphi(t_n - s_n) = -\varphi(s_n^* + t_n^*) + [p(f + t_n^*) + p(s_n^* - f)] \to 0$ as $n \to \infty$ and since $s_n \leq f \leq t_n$ for all $n \in N$, it follows that f is m-integrable. Furthermore, from (1), it follows that $\psi(f) = \lim_{n\to\infty} \varphi(s_n) = \int_X f\,dm$ where $m = I\psi$. The proof is complete.

COROLLARY 2.8. *Let* M *represent* A^* *under the standard representation* I. *Then for* $m \in M$, $I^{-1}m(f) = \int_X f\,dm$, *for all* $f \in A$.

3. Representation theorems. As above A will denote a uniformly closed algebra of bounded real-valued functions on X which contains the constants and separates points. The weakest topology on X for which all the functions in A are continuous is a completely regular Hausdorff topology which will be denoted by τ_A. A proof of the following may be found in [9, p. 444].

THEOREM 3.1. *There is a pair* (X_A, h) *where* X_A *is a compact Hausdorff space and* h *is a homeomorphism of* (X, τ_A) *onto a dense subspace of* X_A *such that if* $f \in A$, *then* $f \circ h^{-1}$ *has a (unique) extension to an element of* $C(X_A)$ *and such that every element of* $C(X_A)$ *arises in this way. Furthermore,* X_A *is unique up to homeomorphism.*

In what follows, we will identify X with $h[X]$. That is, we will assume that (X, τ_A) is a subspace of X_A. For $Y \subset X$, $\bar{Y}$ will be used to denote the closure of Y in X_A; and if $\mathcal{W}$ is any family of subsets of X, then $\bar{\mathcal{W}} = \{\bar{W}\colon W \in \mathcal{W}\}$. Finally, if $f \in A$, then $\bar{f}$ will denote the unique extension of f to X_A. We will make the blanket assumption that *all pavings $\mathcal{W}$ considered are full pavings of τ_A-closed subsets of X.*

DEFINITION 3.2. The algebra A *separates* the paving $\mathcal{W}$ if whenever $W_1, W_2 \in \mathcal{W}$ with $W_1 \cap W_2 = \varnothing$, then there is an $f \in A$ with $f = 0$ on W_1 and $f = 1$ on W_2. The algebra A *strongly separates* $\mathcal{W}$ if for all $W_1, W_2 \in \mathcal{W}$, it follows that $\overline{W_1 \cap W_2} = \bar{W}_1 \cap \bar{W}_2$.

REMARKS. 1. A separates $\mathcal{W}$ if and only if $\bar{W}_1 \cap \bar{W}_2 = \varnothing$ whenever $W_1, W_2 \in \mathcal{W}$ and $W_1 \cap W_2 = \varnothing$.

2. If A strongly separates $\mathcal{W}$, then A separates $\mathcal{W}$. (The converse is false in general.) For example, let $X = N$, A the algebra of all convergent sequences and $\mathcal{W} = \{W \subset N\colon 1 \in W\} \cup \{\varnothing\}$. Then A separates $\mathcal{W}$ even though strong separation fails for $W_1 = \{1, 2, 4, 6, \cdots\}$ and $W_2 = \{1, 3, 5, 7, \cdots\}$.(1)

PROPOSITION 3.3. 1. *A strongly separates $\mathcal{W}$ if and only if $\bar{\mathcal{W}}$ is a paving on X_A.*

2. *If A strongly separates $\mathcal{W}$, then the map $W \rightsquigarrow \bar{W}$ extends (uniquely) to a Boolean isomorphism of $\mathcal{F}(\mathcal{W})$ onto $\mathcal{F}(\bar{\mathcal{W}})$.*

PROOF. 1. Let $W_1, W_2 \in \mathcal{W}$. Since $\bar{\mathcal{W}}$ is a paving, there is $W \in \mathcal{W}$ with $\bar{W} = \bar{W}_1 \cap \bar{W}_2$. Hence $\overline{W_1 \cap W_2} \subset \bar{W}$ so that $W_1 \cap W_2 \subset W$, since W_1, W_2 and W are τ_A-closed. For the same reason, $W \subset W_1$ and $W \subset W_2$. Thus $W = W_1 \cap W_2$.

2. The proof is essentially the same as in Proposition 2.5 in [9], and so we omit it here.

REMARK. If A strongly separates $\mathcal{W}$, the Boolean algebra isomorphism of the above proposition induces a Riesz space isomorphism σ of $S(\mathcal{W})$ onto $S(\bar{\mathcal{W}})$ and a Banach lattice isomorphism ν of $M(\mathcal{W})$ onto $M(\bar{\mathcal{W}})$ in the obvious way.

COROLLARY 3.4. *Assume that A strongly separates $\mathcal{W}$. Then an $f \in A$ is $\mathcal{W}$-integrable if and only if $\bar{f}$ is $\bar{\mathcal{W}}$-integrable.*

PROOF. Let σ and ν be the maps of the above remark. Then for $f \in A$ and $s, t \in S(\mathcal{W})$, it is easily seen that $s \leqslant f \leqslant t$ if and only if $\sigma s \leqslant \bar{f} \leqslant \sigma t$ and that $\int_X s\,dm = \int_{X_A} \sigma s\,d\nu(m)$. The result is now immediate.

PROPOSITION 3.5. *Let $M(\mathcal{W})$ represent A^*. Then A strongly separates $\mathcal{W}$.*

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(1) The authors would like to express their thanks to the referee for suggesting this example and for the other helpful suggestions which he made regarding the paper.

PROOF. Let $W_1, W_2 \in \mathcal{W}$, and fix $x \in \bar{W}_1 \cap \bar{W}_2$. For $f \in A$, define $\varphi(f) = \bar{f}(x)$. Then $0 \leqslant \varphi$ so $\varphi \in A^*$. If $I\varphi = m$, then for all $W \in \mathcal{W}$, $m(W) = 1$ if $x \in \bar{W}$ and $m(W) = 0$ if $x \notin \bar{W}$. (Indeed, in view of the natural identification of A with $C(X_A)$, $m(W) = \inf\{\varphi(f)\colon f \in A, \chi_W \leqslant f\} = \inf\{g(x)\colon g \in C(X_A), \chi_W \leqslant g\}$. The result is now immediate by Urysohn's lemma.) Since $m(W_1) = m(W_2) = 1$, $m(W_1 \cap W_2) = 1$ which implies that $x \in \overline{W_1 \cap W_2}$. Hence $\bar{W}_1 \cap \bar{W}_2 \subset \overline{W_1 \cap W_2}$. Since $\overline{W_1 \cap W_2} \subset \bar{W}_1 \cap \bar{W}_2$ is always valid, it follows that $\bar{W}_1 \cap \bar{W}_2 = \overline{W_1 \cap W_2}$.

PROPOSITION 3.6. *Let A strongly separate $\mathcal{W}$. The space $M(\mathcal{W})$ represents A^* if and only if $M(\bar{\mathcal{W}})$ represents $C(X_A)^*$.*

PROOF. ($\Rightarrow$) Let T be the Banach lattice isomorphism of A onto $C(X_A)$ defined by $Tf = \bar{f}$. Then the adjoint T^* is a Banach lattice isomorphism of $C(X_A)^*$ onto A^*. Since A strongly separates $\mathcal{W}$, there is a natural Banach lattice isomorphism ν of $M(\mathcal{W})$ onto $M(\bar{\mathcal{W}})$. (See the remark following Proposition 3.3.) Let I be the standard representation of A onto $M(\mathcal{W})$. Define $\bar{I} = \nu \circ I \circ T^*$. It is a simple matter to verify that $\bar{I}$ is a one-one standard representation of $C(X_A)^*$ in $M(\bar{\mathcal{W}})$.

($\Leftarrow$) Taking T^* and ν as above, if $\bar{I}$ is the standard representation of $C(X_A)^*$ in $M(\bar{\mathcal{W}})$, define $I = \nu^{-1} \circ \bar{I} \circ (T^*)^{-1}$. Then it is easy to verify that I is a one-one, standard representation of A^* in $M(\mathcal{W})$.

LEMMA 3.7. *Let X be a compact Hausdorff space, let $\mathcal{W}$ be a full paving of closed subsets of X and let $M(\mathcal{W})$ represent $C(X)^*$. If $0 \leqslant \mu \in M(\mathcal{G})$, where $\mathcal{G}$ is the paving of closed sets of X, then for each $\epsilon > 0$ and each $G \in \mathcal{G}$, there is a $W \in \mathcal{W}$ with $W \subset X - G = G^c$ and $\mu(G^c - W) \leqslant \epsilon$.*

PROOF. First note that $M(\mathcal{G})$ represents $C(X)^*$. By $\mathcal{G}$-regularity, there is $G_0 \in \mathcal{G}$ with $G_0 \subset G^c$ and $\mu(G^c - G_0) \leqslant \epsilon$. Take $f \in C(X)$ with $\chi_{G_0} \leqslant f \leqslant \chi_{G^c}$. Let m denote the restriction of μ to $\mathcal{F}(\mathcal{W})$. Since $M(\mathcal{W})$ represents $C(X)^*$, it follows that $m \in M(\mathcal{W})$. (Indeed, let $\varphi(f) = \int_X f\,d\mu$, and let $m' \in M(\mathcal{W})$ represent φ under the given standard representation of $C(X)^*$ in $M(\mathcal{W})$. Then for each $W \in \mathcal{W}$, $m'(W) = \inf\{\varphi(f)\colon f \in C(X), \chi_W \leqslant f\} = \mu(W) = m(W)$.) By Theorem 2.7 there is $0 \leqslant s \in S(\mathcal{W})$ with $s \leqslant f$ and $\int_X (f - s)\,dm \leqslant \epsilon$. Let $F = \{x \in X\colon 0 < s(x)\}$. Then $F \in \mathcal{F}(\mathcal{W})$ and $F \subset G^c$. Now take $W \in \mathcal{W}$ with $W \subset F$ and $m(F - W) \leqslant \epsilon$. Then $W \subset G^c$ and

$$\begin{aligned} 0 &\leqslant \mu(G^c - W) \\ &= \mu(G^c - G_0) + [\mu(G_0) - \textstyle\int_X f\,d\mu] \\ &\qquad + \textstyle\int_X (f - s)\,d\mu + [\int_X s\,d\mu - \mu(F)] + [\mu(F) - \mu(W)] \\ &\leqslant 3\epsilon, \end{aligned}$$

since $[\mu(G_0) - \int_X f d\mu] \leq 0$ and $[\int_X s\, d\mu - \mu(F)] \leq 0$. Since $\epsilon > 0$ was arbitrary, the proof is complete.

Recall that a set $Z \subset X$ is an *A-zero set* if there is an $f \in A$ such that $Z = Z(f) = \{x \in X: f(x) = 0\}$. The family of all A-zero sets is denoted by $\mathcal{Z}(A)$. If $Z \in \mathcal{Z}(A)$, then $\mathcal{W}(Z)$ will denote the set of all $W \in \mathcal{W}$ such that there is an $f \in A$ with $f = 0$ on Z and $f = 1$ on W. We then have the following.

THEOREM 3.8. *Let A be a uniformly closed algebra of bounded real-valued functions on X which contains the constants and separates points, and let $\mathcal{W}$ be a paving of τ_A-closed subsets of X. Then $M(\mathcal{W})$ represents A^* if and only if the following two conditions hold.*

1. *A strongly separates $\mathcal{W}$.*
2. *Let $0 \leq \varphi \in A^*$ and $Z \in \mathcal{Z}(A)$. Then* $\sup_{W \in \mathcal{W}(Z)} \inf \{\varphi(f): f \in A, \chi_W \leq f\} = \sup \{\varphi(g): g \in A, g \leq \chi_{Z^c}\}$.

PROOF. ($\Rightarrow$) Assume that $M(\mathcal{W})$ represents A^* under the standard representation I. Then A strongly separates $\mathcal{W}$ by Proposition 3.5. Fix $0 \leq \varphi \in A^*$ and let $I\varphi = m$. Thus for $W \in \mathcal{W}$, $m(W) = \inf \{\varphi(f): f \in A$ and $\chi_W \leq f\}$. Fix $Z_0 \in \mathcal{Z}(A)$, and let $a = \sup \{m(W): W \in \mathcal{W}(Z_0)\}$ and $b = \sup \{\varphi(g): g \in A, g \leq \chi_{Z_0^c}\}$. We must show that $a = b$.

Take $\epsilon > 0$ and choose $W_0 \in \mathcal{W}(Z_0)$ with $a - \epsilon \leq m(W_0)$. Since $W_0 \in \mathcal{W}(Z_0)$, there is an $f \in A$ with $0 \leq f \leq 1$, $f = 1$ on W_0 and $f = 0$ on Z_0. Hence, it follows that $a - \epsilon \leq m(W_0) \leq \varphi(f) \leq b$. Since ϵ was arbitrary, $a \leq b$.

Now define $\bar{\varphi}$ on $C(X_A)$ by $\bar{\varphi}(\bar{f}) = \varphi(f)$ for all $f \in A$. Since $M(\mathcal{G})$ represents $C(X_A)^*$ (where $\mathcal{G}$ is the paving of closed sets in X_A), there is $0 \leq \mu \in M(\mathcal{G})$ with $\varphi(f) = \int_{X_A} \bar{f}\, d\mu$ for all $f \in A$. Since $M(\mathcal{W})$ represents A^*, it follows by Proposition 3.6 that $M(\overline{\mathcal{W}})$ represents $C(X_A)^*$. Hence by Lemma 3.7, if $\epsilon > 0$ is given, there is $\overline{W}_0 \in \overline{\mathcal{W}}$ with $\overline{W}_0 \subset \overline{Z}_0^c$ and $\mu(\overline{Z}_0^c - \overline{W}_0) \leq \epsilon$. Since $\overline{W}_0 \cap \overline{Z}_0 = \emptyset$, it is immediate from Urysohn's lemma that $W_0 \in \mathcal{W}(Z_0)$. Now let $g \in A$ with $g \leq \chi_{Z_0^c}$. Then $\bar{g} \leq \chi_{\overline{Z}_0^c}$ so that

$$\varphi(g) = \bar{\varphi}(\bar{g}) \leq \mu(\overline{Z}_0^c) \leq \epsilon + \mu(\overline{W}_0) = \epsilon + m(W_0) \leq \epsilon + a.$$

Since g and $\epsilon > 0$ were arbitrary, we obtain $b \leq a$.

($\Leftarrow$) Since A strongly separates $\mathcal{W}$, it is sufficient by Proposition 3.6 to show that $M(\overline{\mathcal{W}})$ represents $C(X_A)^*$. Let $\mathcal{G}$ be the paving of closed sets of X_A. Then $M(\mathcal{G})$ represents $C(X_A)$. Hence in order to show that $M(\overline{\mathcal{W}})$ represents $C(X_A)^*$, it is enough (by the remark following Definition 2.5) to show that if $0 \leq \mu \in M(\mathcal{G})$ and if m is the restriction of μ to $\mathcal{F}(\overline{\mathcal{W}})$, then $m \in M(\overline{\mathcal{W}})$ and the $\mu \rightsquigarrow m$ is one-one.

In order to verify that $m \in M(\overline{\mathcal{W}})$, it is enough to verify the $\overline{\mathcal{W}}$-regularity of m. Hence fix $W_0 \in \mathcal{W}$ and $\epsilon > 0$. Since $\{\overline{Z}: Z \in \mathcal{Z}(A)\}$ forms a basis for the closed sets in X_A, the fact that $M(\mathcal{G}) = M_\tau(\mathcal{G})$ implies that there is a $Z_0 \in \mathcal{Z}(A)$ with $\overline{W}_0 \subset \overline{Z}_0$ and $\mu(\overline{Z}_0 - \overline{W}_0) \leqslant \epsilon$. It follows from condition 2 that there is $W_1 \in \mathcal{W}(Z_0)$ with $\mu(\overline{Z}_0^c - \overline{W}_1) \leqslant \epsilon$. Hence $\overline{W}_1 \subset \overline{Z}_0^c \subset \overline{W}_0^c$ and $m(\overline{W}_0^c - \overline{W}_1) = \mu(\overline{Z}_0 - \overline{W}_0) + \mu(\overline{Z}_0^c - \overline{W}_1) \leqslant 2\epsilon$. Since $\epsilon > 0$ was arbitrary, the $\overline{\mathcal{W}}$-regularity of m follows from Proposition 1.2.

Now assume that $0 \leqslant \mu_1, \mu_2 \in M(\mathcal{G})$ have the same restriction m to $\mathcal{F}(\overline{\mathcal{W}})$. In order to show $\mu_1 = \mu_2$, it is enough to show $\mu_1(\overline{Z}) = \mu_2(\overline{Z})$ for every $Z \in \mathcal{Z}(A)$. (This is a consequence of the $\mathcal{G}$-regularity of μ_1 and μ_2, the fact that $\{\overline{Z}: Z \in \mathcal{Z}(A)\}$ is a basis for the closed sets in X_A and the fact that $M(\mathcal{G}) = M_\tau(\mathcal{G})$.) Hence fix $Z_0 \in \mathcal{Z}(A)$. From condition 2, it follows that

$$\mu_1(\overline{Z}_0^c) = \sup\{m(\overline{W}): W \in \mathcal{W}(Z_0)\} = \mu_2(\overline{Z}_0^c).$$

Thus $\mu_1(\overline{Z}_0) = \mu_2(\overline{Z}_0)$. The proof of the theorem is complete.

REMARK. Condition 2 in Theorem 3.8 is equivalent to the following.

(2′) Let $0 \leqslant \mu \in M(\mathcal{G})$ where $\mathcal{G}$ is the family of closed sets in X_A. Then for all $G \in \mathcal{G}$, $\mu(G^c) = \sup\{\mu(\overline{W}): W \in \mathcal{W}$ and $\overline{W} \subset G^c\}$.

It would be interesting to know if the condition that A strongly separates $\mathcal{W}$ in Theorem 3.8 can be replaced with the weaker condition that A separates $\mathcal{W}$. The present authors believe that this is unlikely although they have no counterexample. The following example shows that it can happen that A strongly separates $\mathcal{W}$ and that each element of A is $\mathcal{W}$-integrable even though $M(\mathcal{W})$ does not represent A^*.

EXAMPLE. Let $X = [0, 1]$, let $\mathcal{W}$ consist of X together with the finite subsets of X. Then $A = C([0, 1])$ strongly separates $\mathcal{W}$ and each $f \in A$ is $\mathcal{W}$-integrable. However, $M(\mathcal{W}) = l'([0, 1])$ clearly does not represent A^*.

We will now show that for a rather large class of algebras there is a paving to which Theorem 3.8 can be applied. First we recall the following. (See [5] or [14].)

DEFINITION 3.9. Let X be a completely regular Hausdorff topological space. A family $\mathcal{W}$ of closed subsets of X is a *normal base* (or *Wallman base*) if the following hold.

1. $\mathcal{W}$ is a paving on X.
2. $\mathcal{W}$ is a base for the closed sets of X.
3. If G is a closed set in X and if $x \in G^c$, then there are $W_1, W_2 \in \mathcal{W}$ with $G \subset W_1$, $x \in W_2$ and $W_1 \cap W_2 = \varnothing$.
4. Let $W_1, W_2 \in \mathcal{W}$ with $W_1 \cap W_2 = \varnothing$. Then there are $V_1, V_2 \in \mathcal{W}$ with $V_1 \cup V_2 = X$, $W_1 \subset V_1^c$ and $W_2 \subset V_2^c$.

With each normal base $\mathcal{W}$ on a completely regular Hausdorff space X is associated a compactification $X_{\mathcal{W}}$ called the *$\mathcal{W}$-compactification* of X. (See [5].) We will describe briefly how this is done. Let $X_{\mathcal{W}}$ be the set of all maximal $\mathcal{W}$-filters. For $W \in \mathcal{W}$, define $\mathrm{cl}(W) = \{\xi \in X_{\mathcal{W}} : W \in \xi\}$. The topology on $X_{\mathcal{W}}$ is defined by taking $\{\mathrm{cl}(W): W \in \mathcal{W}\}$ as a base for the closed sets. Finally, for $x \in X$, let $\xi_x = \{W \in \mathcal{W}: x \in W\}$. Then $\xi_x \in X_{\mathcal{W}}$. Let h be the map from X into $X_{\mathcal{W}}$ defined by $h(x) = \xi_x$ for each $x \in X$. Then the following holds. (For the proof see [5].)

THEOREM. *Let X be a completely regular Hausdorff space, and let $\mathcal{W}$ be a normal base on X. Then $X_{\mathcal{W}}$ is a compact Hausdorff space and h is a homeomorphism of X onto a dense subspace of $X_{\mathcal{W}}$.*

DEFINITION 3.10. Let X be a completely regular Hausdorff space. A compactification of X is a *Wallman compactification* if it is homeomorphic to $X_{\mathcal{W}}$ for some normal base $\mathcal{W}$ on X under a homeomorphism which keeps X pointwise fixed.

PROPOSITION 3.11. *Let A be an algebra on a point set X, and let X be given the completely regular Hausdorff topology τ_A. Then X_A is a Wallman compactification of (X, τ_A) if and only if there is a paving $\mathcal{W}$ of τ_A-closed sets on X such that the following hold.*

1. *A strongly separates $\mathcal{W}$.*
2. *$\overline{\mathcal{W}}$ forms a base for the closed sets of X_A.*

PROOF. ($\Rightarrow$) Let $\mathcal{W}$ be a normal base on X with $X_A = X_{\mathcal{W}}$. (We identify these two homeomorphic spaces.) Then condition 2 follows immediately from the definition of $X_{\mathcal{W}}$. Let $W_1, W_2 \in \mathcal{W}$ if $x \in \overline{W}_1 \cap \overline{W}_2$, then $W_1, W_2 \in x$ (by definition of $X_{\mathcal{W}}$). Hence $W_1 \cap W_2 \in x$ so that $x \in \overline{W_1 \cap W_2}$. Hence $\overline{W}_1 \cap \overline{W}_2 = \overline{W_1 \cap W_2}$.

($\Leftarrow$) We will show that $\mathcal{W}$ is a normal base for (X, τ_A) and that X is homeomorphic to X_A. It is given that $\mathcal{W}$ is a paving of τ_A-closed sets, and it is immediate from condition 2 that $\mathcal{W}$ is a base for the τ_A-closed sets. Let $W_1, W_2 \in \mathcal{W}$ with $W_1 \cap W_2 = \emptyset$. By condition 1 it follows that $\overline{W}_1 \cap \overline{W}_2 = \emptyset$. By condition 2 and the fact that a compact Hausdorff space is normal, there are $V_1, V_2 \in \mathcal{W}$ with $\overline{V_1 \cup V_2} = X_A$, $\overline{W}_1 \subset \overline{V}_1^c$ and $\overline{W}_2 \subset \overline{V}_2^c$. Hence $V_1 \cup V_2 = X$, $W_1 \subset V_1^c$ and $W_2 \subset V_2^c$. Now let $G \subset X$ be τ_A-closed and $x \notin G$. By condition 1, $x \notin \overline{G}$. Again condition 2 and the normality of X_A imply the existence of $W_1, W_2 \in \mathcal{W}$ with $x \in \overline{W}_1$, $\overline{G} \subset \overline{W}_2$ and $\overline{W}_1 \cap \overline{W}_2 = \emptyset$. Hence $\mathcal{W}$ is a normal base.

For $x \in X_A$, define $\xi_x = \{W \in \mathcal{W}: x \in \overline{W}\}$. It is clear from conditions 1 and 2 that $\xi_x \in X_{\mathcal{W}}$. Define h from X_A into $X_{\mathcal{W}}$ by $h(x) = \xi_x$. It is clear that h is one-one; and, since X_A is compact, condition 2 implies that h is onto. In order to show that h is a homeomorphism, it is enough to verify that h is continuous.

Let (x_i) be a net in X_A with $x_i \to x$, and take $W \in \mathcal{W}$ with $\xi_x \notin \mathrm{cl}(W)$. Thus $W \notin \xi_x$ which is equivalent to $x \notin \overline{W}$. Hence there is an index i_0 such that $x_i \notin \overline{W}$ for all $i \geqslant i_0$. Hence, $\xi_{x_i} \notin \mathrm{cl}(W)$ for all $i \geqslant i_0$. Thus $\xi_{x_i} \to \xi_x$. The proof is complete.

THEOREM 3.12. *Let A be a uniformly closed algebra of bounded real-valued functions on X which contains the constants and separates points. If X_A is a Wallman compactification, then there is a paving $\mathcal{W}$ of τ_A-closed sets such that $M(\mathcal{W})$ represents A^*. In fact, $\mathcal{W}$ may be taken to be any normal base for τ_A with $X_{\mathcal{W}} = X_A$.*

PROOF. Let $\mathcal{W}$ be a normal base for τ_A with $X_{\mathcal{W}} = X_A$. Then A strongly separates $\mathcal{W}$ by Proposition 3.11. Furthermore, by the same proposition, $\overline{\mathcal{W}}$ is a base for the closed sets in X_A. We will now show that condition (2′) of the remark following Theorem 3.8 holds. The result will then follow from Theorem 3.8.

Hence let $0 \leqslant \mu \in M(\mathcal{G})$ where $\mathcal{G}$ is the family of closed sets in X_A, and let $G_0 \in \mathcal{G}$. Then given $\epsilon > 0$, there is a $G_1 \in \mathcal{G}$ with $G_1 \subset G_0^c$ and $\mu(G_0^c - G_1) \leqslant \epsilon$. Since $\overline{\mathcal{W}}$ is a base for $\mathcal{G}$ and since X_A is compact, there is $\overline{W} \in \overline{\mathcal{W}}$ with $G_1 \subset \overline{W} \subset G_0^c$. Hence, $\mu(G_0^c) \leqslant \mu(\overline{W}) + \epsilon$. Since $\epsilon > 0$ was arbitrary, the result holds.

PROBLEM. Given an algebra A, is there always a paving $\mathcal{W}$ of τ-closed sets such that $M(\mathcal{W})$ represents A^*?

Theorem 3.12 states that the problem has a positive answer for any algebra A for which the compactification X_A is a Wallman compactification. There is a conjecture in topology known as *Frink's conjecture* which asserts that every compactification is a Wallman compactification. There has been a considerable effort in recent years to solve Frink's conjecture; and, as a result, many types of compactifications are known to be Wallman. (For instance, the reader is referred to [5], [14], [18].) If Frink's conjecture is true, then by Theorem 3.12 the problem has a positive answer. It may be that the problem is equivalent to Frink's conjecture, although the authors believe that this is probably not the case. It should be noted in this regard that $M(\mathcal{W})$ may represent A^* even though $\mathcal{W}$ is not a τ_A-normal base. (See (a) of appendix.) Finally, we remark that if A is z-separating (that is, if A separates $\mathcal{Z}(A)$) then $\mathcal{Z}(A)$ is a normal base whose Wallman compactification is X_A. Hence Theorem 3.11 of [9] is an immediate consequence of Theorem 3.12 above. As a result Theorem 3.12 contains all known representations of the Alexandrov type.

4. **Additivity.** Throughout this section α will denote an infinite cardinal number and $\mathcal{W}$ will be a paving. A nonempty set $I \subset \mathcal{W}$ is an *α-system* if I is directed downward, $\bigcap I = \varnothing$ and $\mathrm{card}(I) \leqslant \alpha$. (Of course, I is directed downward if whenever $W_1, W_2 \in \mathcal{W}$, then there is a $W_3 \in I$ with $W_3 \subset W_1 \cap W_2$.)

DEFINITION 4.1. An $m \in M$ is *α-additive* if $\inf\{|m|(W)\colon W \in I\} = 0$ for every α-system I in $\mathcal{W}$. The set of all α-additive elements in M is denoted by M_α (or $M_\alpha(\mathcal{W})$).

The following examples show that for any regular infinite cardinal α, there are set functions m which are β-additive for all cardinals $\beta < \alpha$ but which are not α-additive. For the purpose of the example, we recall that a cardinal number α is the set of all ordinals β with $\operatorname{card}(\beta) < \alpha$. A cardinal number α is *regular* if there is no smaller cardinal number which is order isomorphic to a cofinal subset of α. (For the properties of regular cardinals, the reader should consult [19].)

EXAMPLE 1. Let α be a regular infinite cardinal and let $X = \alpha$. For each $\beta \in \alpha$, let $W_\beta = \{\gamma \in X\colon \beta < \gamma < \alpha\}$, and define $\mathcal{W} = \{\varnothing, X\} \cup \{W_\beta\colon \beta \in \alpha\}$. It is clear that $\mathcal{W}$ is a paving on X. Define m on $\mathcal{F}(\mathcal{W})$ by $m(F) = 1$ if there is $\varnothing \neq W \in \mathcal{W}$ with $W \subset F$ and $m(F) = 0$ otherwise. It is trivial to check that $m \in M(\mathcal{W})$. If $\beta < \alpha$ is a cardinal number and if $I \subset \mathcal{W}$ is any β-system, the fact that α is regular implies that $\varnothing \in I$. Hence $\inf\{m(W)\colon W \in I\} = 0$ so that m is β-additive. However, $I = \mathcal{W} - \{\varnothing\}$ is an α-system and $\inf\{m(W)\colon W \in I\} = 1$ so that m is not α-additive.

PROPOSITION 4.2. *M_α is a band in M.*

PROOF. It is clear that M_α is an ideal in M. Let (m_i) be a net in M_α^+ with $m_i \uparrow m \in M$, and let I be an α-system. Without loss of generality, assume that there is $W_0 \in \mathcal{W}$ with $W \subset W_0$ for all $W \in I$. Given $\epsilon > 0$, choose i_0 such that $m(W_0) < m_{i_0}(W_0) + \epsilon$. (It is not difficult to show that $0 \leqslant m_i \uparrow m$ if and only if $m_i(W) \uparrow m(W)$ for all $W \in \mathcal{W}$.) Then, for arbitrary $W \in I$, $0 \leqslant (m - m_{i_0})(W) \leqslant (m - m_{i_0})(W_0) \leqslant \epsilon$. It then follows that

$$\inf\{m(W)\colon W \in I\} \leqslant \epsilon + \inf\{m_{i_0}(W)\colon W \in I\} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the proof is complete.

For $\alpha \leqslant \beta$ it is clear that $M_\beta \subset M_\alpha$. For $\alpha = \aleph_0$, it is customary to denote M_α by M_σ. The elements of M_σ are called *σ-additive.* Obviously, there is a cardinal α with the property that $M_\alpha = M_\beta$ whenever $\alpha \leqslant \beta$. (For example, take $\alpha = \operatorname{card}(2^X)$.) The least cardinal with this property is denoted by τ (or $\tau(\mathcal{W})$). It is called the *additivity index* of $\mathcal{W}$.

As a consequence of Propositions 1.6 and 4.2 there is a Riesz decomposition $M = M_\alpha \oplus M_\alpha^\perp$, where $M_\alpha^\perp = \{m \in M\colon \inf(|m|, |m'|) = 0 \text{ for all } m' \in M_\alpha\}$. The next proposition describes the elements of $M_\alpha^\perp$.

DEFINITION 4.3. An element $m \in M$ is *α-singular* if there is an α-system I in $\mathcal{W}$ with $|m|(V) = |m|(V \cap W)$ for all $V \in \mathcal{W}$ and $W \in I$.

It is clear that the α-singular elements form an ideal in the Riesz space M.

PROPOSITION 4.4. *$M_\alpha^\perp$ is the band generated by the ideal of α-singular elements.*

PROOF. Let $0 < m \in M_\alpha^\perp$. Then there is an α-singular element m' with $0 < m' \leqslant m$. Indeed, since m is not α-additive, there is an α-system I in $\mathcal{W}$ with $\inf\{m(W)\colon W \in I\} = a > 0$. For each $F \in \mathcal{F}$ define $m'(F) = \inf\{m(F \cap W)\colon W \in I\}$. It is easy to check that $m' \in M$, $0 < m' \leqslant m$ and $m'(W) = m'(W \cap V)$ for all $W \in \mathcal{W}$ and all $V \in I$.

Let M_0 denote the band in M generated by the α-singular elements. Since it is clear that the α-singular elements belong to $M_\alpha^\perp$, it follows that $M_0 \subset M_\alpha^\perp$. Since $M_\alpha^\perp$ is a Dedekind complete Riesz space, there is a Riesz decomposition $M_\alpha^\perp = M_0 + M_0^\perp$ where $M_0^\perp = \{m \in M_\alpha^\perp\colon \inf(|m|, |m'|) = 0 \text{ for all } m' \in M_0\}$. If $0 < m_0 \in M_0^\perp$, then by the previous paragraph there is an α-singular element $0 < m'$ with $m' \leqslant m_0$. But $m' \in M_0$ so that $0 = \inf(m', m_0) = m' > 0$. This is a contradiction. Hence $M_0^\perp = \{0\}$ so that $M_\alpha = M_0$. The proof is complete.

EXAMPLE. The following example shows that, in general, $M_\alpha^\perp$ is bigger than the ideal of α-singular elements. Let X be the set of all irrationals in $[0, 1]$ and let $\mathcal{W}$ denote the set of all finite unions of sets of the form $X \cap J$ where $J = [a, b]$ with a and b rational together with the empty set. Then every element in $\mathcal{W}$ has a unique representation of the form (1) $W = X \cap ([a_1, b_1] \cup \cdots \cup [a_n, b_n])$ where a_i, b_i are rational for $i = 1, \cdots, n$ and $0 \leqslant a_1 < b_1 < a_2 < \cdots < a_n < b_n \leqslant 1$. Since $\mathcal{W}$ is closed with respect to complementation, $\mathcal{W}$ is an algebra so that $\mathcal{W} = \mathcal{F}$.

Let $\{r_1, r_2, \cdots\}$ be an enumeration of the rationals in $[0, 1]$. For $W \in \mathcal{W}$ and $p \in N$, define $m_p(W) = 1/2^p$ if in the representation (1) of W, r_p satisfies $a_i \leqslant r_p < b_i$ for some $i = 1, \cdots, n$, and $m_p(W) = 0$ otherwise. It is not difficult to check that $m_p \in M(\mathcal{W})$ and is σ-singular for all $p \in N$.

Now define $m(W) = \Sigma_1^\infty m_p(W)$ for all $W \in \mathcal{W}$. Then $m \in M$, and, in fact, since $m = \sup\{\Sigma_{i=1}^p m_i\colon p \in N\}$, $m \in M_\sigma^\perp$ by Proposition 4.4. However, m is not σ-singular. Indeed, if m is σ-singular, there is a decreasing sequence $\{W_k\} \subset \mathcal{W}$ with $\bigcap^\infty W_i = \varnothing$ and $1 = m(X) = m(X \cap W_k)$ for all $k \in N$. But if x is any irrational in $[0, 1]$, there is a k with $x \notin W_k$. Hence there are irrationals s, t with $s < x < t$ and $(s, t) \cap W_k = \varnothing$. Let r be a rational with $s < r < t$. Then $r = r_p$ for some $p \in N$, and $m_p(W_k) = 0$. Thus $m(X \cap W_k) < 1$. This is a contradiction.

Again let A be a uniformly closed algebra of real-valued functions on X which separates points and contains constants. For $\varphi \in A^*$, let φ^+, φ^- and $|\varphi|$ denote the positive, negative and total variations of φ respectively.

DEFINITION 4.5. Let $\varphi \in A^*$ and let α be an infinite cardinal. Then φ is *α-additive* if for every α-system $I \subset A^+$, $\inf\{|\varphi|(f)\colon f \in I\} = 0$. (Of course,

$I \subset A^+$ is an α-*system* if I is a downward directed system with $\operatorname{card}(I) \leqslant \alpha$ and $\inf\{f(x)\colon f \in I\} = 0$ for all $x \in X$.) Let A_α^* denote the α-additive functionals in A^*.

It is clear that A_α^* is an ideal in A^*. Let A_σ^* denote $A_{\aleph_0}^*$. Furthermore, there is an α such that $A_\alpha^* = A_\beta^*$ for all cardinals $\beta \geqslant \alpha$. Let $\tau = \tau(A)$ denote the smallest such cardinal. Then τ is the *additivity index* of A. (Using the same notation as for the additivity index for a paving should not cause confusion since it will be clear from context which is meant.) The proof of the following is straightforward.

PROPOSITION 4.6. *A_α^* is a band in A^* with $A_\tau^* \subset A_\alpha^* \subset A_\sigma^*$ for $\sigma \leqslant \alpha \leqslant \tau$.*

It is now natural to ask the following question. If $M(\mathcal{W})$ represents A^*, does A_α^* correspond to $M_\alpha(\mathcal{W})$ under the standard representation. Unfortunately, the answer is no even if $\mathcal{W}$ is a normal base. (See (b) of appendix.) However, we do have the following.

PROPOSITION 4.7. *Let $M(\mathcal{W})$ represent A^*. If $m \in M_\sigma(\mathcal{W})$ and if m is the image of $\varphi \in A^*$ under the standard representation, then $\varphi \in A_\sigma^*$.*

PROOF. Without loss of generality, assume that $0 \leqslant m \in M_\sigma(\mathcal{W})$. Let (f_n) be a sequence in A^+ which decreases pointwise to 0. From Theorem 2.7 and the monotone convergence theorem, it follows that $\varphi(f_n) = \int_X fn\, dm \to 0$. Hence $\varphi \in A_\sigma$ as claimed.

PROPOSITION 4.8. *Let $\mathcal{W}$ be a paving of τ_A-closed sets with $M(\mathcal{W})$ representing A^*. If $\varphi \in A_\tau^*$ and if $m \in M(\mathcal{W})$ represents φ, then $m \in M_\tau(\mathcal{W})$. If $\mathcal{W}$ is a base for the closed sets in τ_A, the converse holds.*

PROOF. Let I be a τ-system in $\mathcal{W}$ and without loss of generality, assume that $0 \leqslant \varphi$ (so also $0 \leqslant m$). Let $J = \{f \in A\colon \chi_W \leqslant f \leqslant 1$ for some $W \in I\}$. Then J is a τ-system in A^+. Since $0 \leqslant \varphi \in A_\tau^*$, $0 = \inf\{\varphi(f)\colon f \in J\}$. But, by Theorem 2.7, if $\chi_W \leqslant f$, it follows that $0 \leqslant m(W) \leqslant \int_X f\, dm = \varphi(f)$. Hence $0 \leqslant \inf\{m(W)\colon W \in I\} \leqslant \inf\{\varphi(f)\colon f \in J\} = 0$. Thus $m \in M_\tau$ as claimed.

Now assume that $\mathcal{W}$ is a base for the τ_A-closed sets and that $m \in M_\tau(\mathcal{W})$. Without loss of generality, assume that $0 \leqslant m$ and that $m(X) = 1$. Let J be a τ-system in A and assume that $\inf\{\varphi(f)\colon f \in J\} = 3a > 0$. Assume $f \leqslant 1$ for all $f \in J$. For $f \in J$, define $Z_f = \{x \in X\colon f(x) \geqslant a\}$, and define $I = \{W \in \mathcal{W}\colon$ there exists $f \in J$, $Z_f \subset W\}$. Since $\mathcal{W}$ is a base for the τ_A-closed sets, I is a τ-system in $\mathcal{W}$ so that $\inf\{m(W)\colon W \in I\} = 0$. Let $W_0 \in I$ with $m(W_0) \leqslant a$ and let $f_0 \in J$ with $Z_{f_0} \subset W_0$. Then we have that

$$3a \leqslant \varphi(f_0) = \int_X f_0\, dm = \int_{W_0} f_0\, dm + \int_{X-W_0} f_0\, dm \leqslant m(W_0) + a \leqslant 2a.$$

This is a contradiction, and the proof is complete.

COROLLARY 4.9. *Let* $\mathcal{W}$ *be a paving of* τ_A*-closed sets and assume that* $M(\mathcal{W})$ *represents* A^*. *If* $A_\tau^* = A_\sigma^*$, *then* $M_\sigma = M_\tau$.

PROOF. This is immediate from Propositions 4.7 and 4.8.

PROBLEM. Assume $M(\mathcal{W})$ represents A^*. If α is an uncountable cardinal number and if $m \in M_\alpha(\mathcal{W})$ represents $\varphi \in A^*$, is it then necessary that $\varphi \in A_\alpha^*$? In general, what can be said of the relation of $M_\alpha(\mathcal{W})$ and A_α^*?

Let $\mathcal{Z} = \mathcal{Z}(A)$ denote the family of A-zero sets. In general, $M(\mathcal{Z})$ does not represent A^*. (In fact, from [9], $M(\mathcal{Z})$ represents A^* if and only if A^* separates $\mathcal{Z}$.) However, A_σ^* is always represented in the standard way by $M_\sigma(\mathcal{Z})$ as we will now proceed to show.

PROPOSITION 4.10. *Let* $0 \leqslant \varphi \in A_\sigma^*$. *Then there is a unique* $m \in M_\sigma(\mathcal{Z})^+$ *with* $\varphi(f) = \int_X f\,dm$ *for all* $f \in A$. *Furthermore,* $m(Z) = \inf\{\varphi(f)\colon f \in A, \chi_Z \leqslant f\}$, *for all* $Z \in \mathcal{Z}$.

PROOF. By assumption φ is a simple integral on A. Hence, via the Daniell extension process, φ may be extended (uniquely) to an integral $\hat{\varphi}$ on a Riesz space $\hat{A}$ which contains all suprema and infima of uniformly bounded, countable subsets of A. If $Z \in \mathcal{Z}(A)$, χ_Z is the infimum over a bounded, countable subset of A and so $\chi_Z \in \hat{A}$. Hence, it follows that $\chi_F \in \hat{A}$ for all $F \in \mathcal{F}(\mathcal{Z})$. Define m on $\mathcal{F}(\mathcal{Z})$ by $m(F) = \hat{\varphi}(\chi_F)$ for all $F \in \mathcal{F}(\mathcal{Z})$. Then m is a finite, countably-additive set function on $\mathcal{F}(\mathcal{Z})$. In order to see that m is $\mathcal{Z}$-regular, fix $Z_0 \in \mathcal{Z}$. Then there is an increasing sequence (Z_n) in $\mathcal{Z}$ with $Z_0^c = \bigcup\{Z_n\colon n \in N\}$. Since m is countably-additive, $m(Z_0^c) = \sup\{m(Z_n)\colon n \in N\}$. Proposition 1.2 now gives that m is $\mathcal{Z}$-regular. Since, if $f \in A$, f can be uniformly approximated by simple functions over $\mathcal{F}(\mathcal{Z})$, it is clear that $\varphi(f) = \int_X f\,dm$. If $Z \in \mathcal{Z}$, there is $f \in A$ with $0 \leqslant f \leqslant 1$ and $Z = \{x \in X\colon f(x) = 1\}$. Since $f^n \downarrow \chi_Z$, $\varphi(f^n) \downarrow m(Z)$. Thus $m(Z) = \inf\{\varphi(f)\colon f \in A, \chi_Z \leqslant f\}$. The uniqueness of m is obvious. The proof is complete.

For $\varphi \in A_\sigma^*$, let m_1 and m_2 be the unique representatives of φ^+ and φ^- in $M_\sigma(\mathcal{Z})$ as in Proposition 4.10. Define $I(\varphi) = m_1 - m_2$. Since I is linear on $(A_\sigma^*)^+$, it is easy to verify that I is an order-preserving linear transformation. The map I will be called the *standard representation* of A_σ^* in $M_\sigma^*(\mathcal{Z})$. We then have the following.

THEOREM 4.11. *The standard representation* I *of* A_σ^* *into* $M_\sigma(\mathcal{Z})$ *is a norm-preserving Riesz space isomorphism of* A_σ^* *onto* $M_\sigma(\mathcal{Z})$. *Furthermore,* I *maps* A_α^* *onto* $M_\alpha(\mathcal{Z})$ *for each infinite cardinal* α.

PROOF. If $I\varphi = 0$, then $I(\varphi^+) = I(\varphi^-)$. But then, by Proposition 4.10, $\varphi^+ = \varphi^-$. Thus I is one-one. Also if $0 \leqslant m \in M_\sigma(Z)$, then define $\varphi(f) = \int_X f\,dm$ for all $f \in A$. Since $0 \leqslant \varphi$, $\varphi \in A^*$, and the monotone convergence theorem implies that $\varphi \in A_\sigma^*$. The uniqueness statement in Proposition 4.10 now gives $I\varphi = m$. Thus I is onto $M_\sigma(Z)$.

In order to show that I is lattice preserving, it is sufficient to show that $(I\varphi)^+ = I(\varphi^+)$ for all $\varphi \in A_\sigma^*$. It is clear that $(I\varphi)^+ \leqslant I(\varphi^+)$. Now let $0 \leqslant m \in M_\sigma(Z)$ with $(I\varphi)^+ \leqslant m$. Since I is onto, there is $0 \leqslant \psi \in M_\sigma(Z)$ with $I\psi = m$. Since $I\varphi \leqslant (I\varphi)^+ \leqslant m = I\psi$, it follows that $0 \leqslant I(\psi - \varphi)$. Hence $\varphi \leqslant \psi$. Thus $\varphi^+ \leqslant \psi$ so that $I(\varphi^+) \leqslant I(\psi) = m$. Since $m \geqslant (I\varphi)^+$ was arbitrary, it follows that $(I\varphi)^+ = I(\varphi^+)$. The fact that I is norm preserving now follows from the fact that $|I\varphi| = I(|\varphi|)$ for all $\varphi \in A_\sigma^*$.

Now let α be an infinite cardinal, and let $\varphi \in A_\alpha^*$. Without loss of generality, assume that $0 \leqslant \varphi$. Let $m = I\varphi$ and let $I \subset Z$ be an α-system. For each $Z \in I$, let $f_Z \in A$ be such that $0 \leqslant f_Z \leqslant 1$, and $Z = \{x \in X\colon f_Z(x) = 1\}$. Let S denote the family of all finite subsets of I and for $\sigma \in S$, let $f_\sigma = \inf\{f_Z\colon Z \in \sigma\}$. Then $J = \{f_\sigma\colon \sigma \in S\}$ is an α-system in A. Hence $0 = \inf\{\varphi(f_\sigma)\colon \sigma \in S\}$. Hence, if $\epsilon > 0$, there is $\sigma \in S$ with $\varphi(f_\sigma) \leqslant \epsilon$. Take $Z \in I$ with $Z \subset \bigcap \sigma$. Then $m(Z) \leqslant \varphi(f_\sigma) \leqslant \epsilon$. Thus $\inf\{m(Z)\colon Z \in I\} = 0$. Hence $m \in M_\alpha(Z)$.

Now assume that $0 \leqslant m \in M_\alpha(Z)$ and take $\varphi \in A_\sigma^*$ with $I\varphi = m$. Let J be an α-system in A^+, and assume that $\inf\{\varphi(f)\colon f \in J\} = 2b(1 + \varphi(1))$ where $b > 0$. Without loss of generality, we may assume that $\|f\|_X \leqslant 1$ for all $f \in J$. For $f \in J$, define $Z_f = \{x \in X\colon f(x) \geqslant b\}$. Then $Z_f \in Z$ and $I = \{Z_f\colon f \in J\}$ is an α-system in Z. Since $m \in M_\alpha(Z)$, $\inf\{m(Z_f)\colon f \in J\} = 0$. Hence there is $f_0 \in J$ with $m(Z_{f_0}) \leqslant b$. It then follows that

$$\begin{aligned} 2b(1 + \varphi(1)) \leqslant \varphi(f_0) &= \int_{Z_{f_0}} f_0\,dm + \int_{X - Z_{f_0}} f_0\,dm \\ &\leqslant m(Z_{f_0}) + b\varphi(1) \\ &\leqslant b(1 + \varphi(1)). \end{aligned}$$

This is a contradiction. Hence $\varphi \in A_\alpha^*$. The proof is complete.

For the sake of completeness, we conclude this section with a brief discussion of strict topologies on the algebra A analogous to those considered by Sentilles in [15]. He showed that when A is the set of continuous bounded functions on a completely regular Hausdorff space, then A_σ^* $(= M_\sigma(Z(A)))$ and A_τ^* $(= M_\tau(Z(A)))$ are the dual spaces of A when A is given an appropriate strict topology. Since, in general, for an algebra A and paving W (with $M(W)$ representing A^*), $A_\sigma^*(A_\tau^*)$ may differ from $M_\sigma(M_\tau)$, the question arises as to which is the dual space of A for the corresponding strict topology.

Let $\mathcal{W}$ be a full paving on the underlying set X, and let $L_\alpha(\mathcal{W})$ (or L_α) denote the set of compact sets Q in $X_A - X$ for which there is a downward directed system $I \subset \overline{\mathcal{W}}$ with $\operatorname{card}(I) \leqslant \alpha$ with $Q = \bigcap I$. For each $Q \in L_\alpha$, let B_Q be the locally convex topology on A generated by the seminorms $\{p_g: g \in A, \bar{g} = 0 \text{ on } Q\}$ where $p_g(f) = \|fg\|_X$. The topology β_α is now defined to be the inductive limit of the topologies β_Q as Q ranges over L_α.

Let $M(\mathcal{W})$ represent A^*. The same arguments as in [13] and [15] give the following:

(1) $\beta_\tau \leqslant \beta_\alpha \leqslant \beta_\sigma \leqslant \|\cdot\|$,

(2) β_α is the finest locally convex topology on A agreeing with itself on norm bounded sets,

(3) the dual of A with the β_α topology is a Dedekind complete Riesz space. Furthermore, if $m \in M(\mathcal{W})$ represents $\varphi \in A^*$ and if $\mu \in M(\mathcal{G})$ represents $\bar{\varphi}$ where $\mathcal{G}$ is the family of closed sets in X_A and $\bar{\varphi}(\bar{f}) = \varphi(f)$ for all $f \in A$, then $m(W) = \mu(\overline{W})$ for all $W \in \mathcal{W}$. From this it follows that $m \in M_\alpha$ if and only if $\mu(Q) = 0$ for all $Q \in L_\alpha$. The following can then be shown as in [15].

THEOREM 4.12. *Let $M(\mathcal{W})$ represent A^*. Then the dual of A with the β_α topology is M_α.*

Many other results on the strict topologies in [15] and [20] can be shown to hold in the present setting. However, there are some exceptions. For example, it is shown in [20] that if A is the algebra of bounded, continuous functions on a completely regular Hausdorff space, then the following are equivalent for $B \subset (A_\sigma^*)^+$:

(1) B is β_σ-equicontinuous.

(2) B is $\sigma(A^*, A)$ relatively compact in A_σ^*.

(3) B is uniformly σ-smooth.

(4) If $H \subset M_\sigma$ is the set of representatives for the elements of B, then H is uniformly σ-additive.

In general, if $\mathcal{W}$ is a normal base or if $\mathcal{W} = Z(A)$, then it still follows that (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (4) and (2) $\Leftrightarrow$ (3). However, (4) $\nRightarrow$ (1) and (3) $\nRightarrow$ (4). (See (e) and (f) of appendix.) The authors do not know if (4) $\Rightarrow$ (3) holds

5. **Tightness.** In order to discuss tightness for M, it is necessary to develop a notion of compactness relative to $\mathcal{W}$.

DEFINITION 5.1. Let $\mathcal{W}$ be a paving on X. A set $\mathcal{U} \subset \mathcal{W}$ is a $\mathcal{W}$-filter if $\mathcal{U}$ satisfies the following.

1. $\mathcal{U} \neq \varnothing$ and $\varnothing \notin \mathcal{U}$.
2. If $W_1, W_2 \in \mathcal{U}$, then there is $W_3 \subset \mathcal{U}$ with $W_3 \subset W_1 \cap W_2$.

3. If $W_1 \in \mathcal{U}$, $W_2 \in \mathcal{W}$ and $W_1 \subset W_2$, then $W_2 \in \mathcal{U}$.

If $\mathcal{U}$ satisfies 1 and 2, it will be called a *$\mathcal{W}$-filter base.*

DEFINITION 5.2. A set $A \subset X$ is *$\mathcal{W}$-compact* if, for every $\mathcal{W}$-filter $\mathcal{U}$ with the property that $A \cap W \neq \emptyset$ for every $W \in \mathcal{U}$, it follows that $\bigcap\{A \cap W: W \in \mathcal{U}\} \neq \emptyset$.

In the following proposition, $\mathcal{W}_\tau$ denotes the family of all sets of the form $\bigcap I$ where I is a subset of $\mathcal{W}$. ($\mathcal{W}_\tau$ is again a paving.)

PROPOSITION 5.3. 1. *Let A, B be $\mathcal{W}$-compact. Then $A \cup B$ is also $\mathcal{W}$-compact.*

2. *Let A be $\mathcal{W}$-compact, $W_0 \in \mathcal{W}_\tau$ and $W_0 \subset A$. Then W_0 is $\mathcal{W}$-compact.*

PROOF. 1. Let $\mathcal{U}$ be a $\mathcal{W}$-filter and assume that $(A \cup B) \cap W \neq \emptyset$ for all $W \in \mathcal{U}$. Since $\mathcal{U}$ is closed under finite intersections, either $A \cap W \neq \emptyset$ for all $W \in \mathcal{U}$ or $B \cap W \neq \emptyset$ for all $W \in \mathcal{U}$. The result is immediate.

2. Let $\mathcal{U}$ be a $\mathcal{W}$-filter with $W_0 \cap W \neq \emptyset$ for all $W \in \mathcal{U}$. Let $I \subset \mathcal{W}$ be downward directed with $\bigcap I = W_0$. Without loss of generality, assume that $W_0 \neq \emptyset$. Let $\mathcal{H}$ denote the $\mathcal{W}$-filter on X with base $\{W \cap U: W \in I \text{ and } U \in \mathcal{U}\}$. Then $A \cap W \neq \emptyset$ for all $W \in \mathcal{H}$. Since A is $\mathcal{W}$-compact, $\emptyset \neq \bigcap\{A \cap W: W \in \mathcal{H}\} = \bigcap\{W_0 \cap W: W \in \mathcal{U}\}$. Hence W_0 is $\mathcal{W}$-compact.

DEFINITION 5.4. An element $m \in \mathcal{W}$ is *tight* (*$\mathcal{W}$-tight*) if, for every $\epsilon > 0$, there is a $\mathcal{W}$-compact set $W_0 \in \mathcal{W}_\tau$ such that $|m|(W) \leqslant \epsilon$ whenever $W \in \mathcal{W}$ and $W_0 \cap W = \emptyset$. Let M_t (or $M_t(\mathcal{W})$) denote the set of tight elements in $M(\mathcal{W})$.

It is clear that M_t is an ideal in M. (That M_t is closed under addition is a consequence of Proposition 5.3(1).)

PROPOSITION 5.5. *Let $\mathcal{W}$ be a full paving. Then M_t is a band in M.*

PROOF. Let (m_i) be a net in M_t^+ with $m_i \uparrow m \in M^+$. .Fix $\epsilon > 0$ and choose m_{i_0} with $m(X) < m_{i_0}(X) + \epsilon$. (Hence, $m(F) \leqslant m_{i_0}(F) + \epsilon$ for all $F \in \mathcal{F}(\mathcal{W})$.) Let $W_0 \in \mathcal{W}_\tau$ be $\mathcal{W}$-compact with $m_{i_0}(W) \leqslant \epsilon$ whenever $W \in \mathcal{W}$ and $W_0 \cap W = \emptyset$. Then $m(\mathcal{W}) \leqslant 2\epsilon$ whenever $W_0 \cap W = \emptyset$. Since $\epsilon > 0$ was arbitrary, the proof is complete.

PROPOSITION 5.6. *Let $m \in M$. If m is tight, then m is τ-additive.*

PROOF. Without loss of generality, let $m \in M^+$. Let $I \subset \mathcal{W}$ be downward directed with $\bigcap I = \emptyset$. Assume that $\inf\{m(W): W \in I\} = 2\epsilon > 0$. Since m is tight, there is a $\mathcal{W}$-compact set $W_0 \in \mathcal{W}_\tau$ with $m(W) \leqslant \epsilon$ whenever $W \in \mathcal{W}$ and $W_0 \cap W = \emptyset$. Since $m(W) \geqslant 2\epsilon$ for all $W \in I$, $W_0 \cap U \neq \emptyset$ for all $U \in \mathcal{U}$ where $\mathcal{U}$ is the $\mathcal{W}$-filter with base I. Since W_0 is $\mathcal{W}$-compact, it follows that $\emptyset \neq \bigcap\{W_0 \cap U: U \in \mathcal{U}\} \subset \bigcap I$. This is a contradiction.

DEFINITION 5.7. A set $S \subset M(\mathcal{W})$ is *tight* if for every $\epsilon > 0$, there is a $\mathcal{W}$-compact set $W_0 \in \mathcal{W}_\tau$ with $|m|(W) \leqslant \epsilon$ for all $m \in S$ and all $W \in \mathcal{W}$ with $W \cap W_0 = \emptyset$.

DEFINITION 5.8. Let $m \in M$. Then m has *compact support* if there is a $\mathcal{W}$-compact set $W_0 \in \mathcal{W}_\tau$ such that $|m|(W) = 0$ whenever $W \in \mathcal{W}$ and $W_0 \cap W = \emptyset$. Let M_c (or $M_c(\mathcal{W})$) denote the set of elements in M with compact support.

It is clear that M_c is an ideal in M_t. Furthermore, we have the following.

PROPOSITION 5.9. *Let $m \in M_t^+$. Then there is an increasing sequence (m_n) in M_c^+ with $m_n \uparrow m$.*

PROOF. Let $0 < m \in M_t^+(\mathcal{W})$. For each $n \in N$, choose $W_n \in \mathcal{W}_\tau$ (where τ is the $\mathcal{W}$-additivity index) with W_n $\mathcal{W}$-compact and with $m(W) \leqslant 1/n$ whenever $W \in \mathcal{W}$ and $W \cap W_n = \emptyset$.

For $W_0 \in \mathcal{W}_\tau$, define $\mu(W_0) = \inf\{m(W)\colon\ W \in \mathcal{W},\ W_0 \subset W\}$. For $F \in \mathcal{F}(\mathcal{W}_\tau)$, define $\mu(F) = \sup\{\mu(W_0)\colon\ W_0 \in \mathcal{W}_\tau,\ W_0 \subset F\}$. (Recall that $\mathcal{W}_\tau$ is a paving.) It can now be verified that $\mu \in M_t(\mathcal{W}_\tau)$. For each $n \in N$, define μ_n on $\mathcal{F}(\mathcal{W}_\tau)$ by $\mu_n(F) = \mu(F \cap W_n)$ for each $F \in \mathcal{F}(\mathcal{W}_\tau)$. It is not hard to verify that $\mu_n \in M(\mathcal{W}_\tau)$. Since $0 \leqslant \mu_n \leqslant \mu$, it follows that if m_n denotes the restriction of μ_n to $\mathcal{F}(\mathcal{W})$ then $m_n \in M^+(\mathcal{W})$. It is clear that $m_n \in M_c(\mathcal{W})$ since $m_n(W) = \mu_n(W) = 0$ whenever $W \in \mathcal{W}$ and $W_n \cap W = \emptyset$. By Proposition 5.3 we may assume that $W_n \subset W_{n+1}$ for all $n \in N$ so that (m_n) is an increasing sequence in $M_c^+(\mathcal{W})$. If it can be shown that $m_n \uparrow m$, the proof will be complete. Since (m_n) is increasing, this is equivalent to $m_n(F) \uparrow m(F)$ for all $F \in \mathcal{F}(\mathcal{W})$. Furthermore, using $\mathcal{W}$-regularity, this will follow if it can be shown that $m_n(W) \uparrow m(W)$ for all $W \in \mathcal{W}$.

In order to see that $m_n(W) \uparrow m(W)$ for all $W \in \mathcal{W}$, assume that this is false. Hence, there is $W_0 \in \mathcal{W}$ and $\epsilon > 0$ such that $m_n(W_0) + \epsilon \leqslant m(W_0)$ for all $n \in N$. Take $n_0 \in N$ with $1/n_0 < \epsilon/4$. Let $I \subset \mathcal{W}$ be downward directed with $W_{n_0} = \bigcap I$. Since $\mu \in M_\tau^+(\mathcal{W}_\tau)$, there is $W^* \in I$ with $m(W^*) = \mu(W^*) < \mu(W_{n_0}) + \epsilon/4$. Finally, since m is regular, there is a $W \in \mathcal{W}$ with $W \subset W_0 - W^*$ and $m(W_0 - W^*) < m(W) + \epsilon/4$. Since $W \cap W^* = \emptyset$, it follows that $W \cap W_{n_0} = \emptyset$ so that $m(W) \leqslant 1/n_0$. It now follows that

$$
\begin{aligned}
0 < \epsilon \leqslant m(W_0) - m_{n_0}(W_0) &= \mu(w_0) - \mu(W_0 \cap W_{n_0}) \\
&\leqslant \mu(W_0) - \mu(W_0 \cap W^*) + \epsilon/4 \\
&\leqslant m(W_0 - W^*) + \epsilon/4 \\
&\leqslant m(W) + \epsilon/2 \leqslant 1/n_0 + \epsilon/2 \leqslant 3\epsilon/4.
\end{aligned}
$$

This is a contradiction, and the result follows.

COROLLARY 5.10. *Let $\mathcal{W}$ be a full paving. Then M_t is the band in M generated by M_c.*

An element $m \in M(\mathcal{W})$ is called 0, 1-valued if $0 \neq m$ and $m(F) = 0$ or $m(F) = 1$ for each $F \in \mathcal{F}(\mathcal{W})$. Let X_0 denote the set of all 0, 1-valued elements in $M(\mathcal{W})$. For each infinite cardinal number α, let $X_\alpha = X_0 \cap M_\alpha$. The linear subspace of $M(\mathcal{W})$ generated by X_α (for $\alpha = 0$ or α an infinite cardinal) is denoted by L_α (or $L_\alpha(\mathcal{W})$).

Let $\hat{X}_0$ denote the set of all maximal $\mathcal{W}$-filters. For α an infinite cardinal number with $\alpha < \tau$, the filter $\xi \in \hat{X}_0$ satisfies the *α-condition* if $\bigcap I \neq \emptyset$ whenever $I \subset \xi$ and $\operatorname{card}(I) \leqslant \alpha$. Let $\hat{X}_\alpha = \{\xi \in \hat{X}_0\colon\ \xi$ satisfies the α-condition$\}$. Finally, let $\hat{X}_\tau = \{\xi \in \hat{X}_0\colon\ \bigcap \xi \neq \emptyset\}$.

REMARK. Let ξ be a $\mathcal{W}$-filter. Then ξ is a maximal $\mathcal{W}$-filter if and only if whenever $W \in \mathcal{W} - \xi$, there is $W' \in \xi$ with $W \cap W' = \emptyset$.

PROPOSITION 5.11. *For $m \in X_0$, define $\xi_m = \{W \in \mathcal{W}\colon\ m(W) = 1\}$. Then $\xi_m \in \hat{X}_0$. Furthermore, for $\alpha = 0$ or α any infinite cardinal number, the mapping $m \rightsquigarrow \xi_m$ is one-one from X_α onto $\hat{X}_\alpha$.*

PROOF. For $m \in X_0$ it is easy to verify that ξ_m is a filter, and it follows from the above remark that ξ_m is maximal. Hence $\xi_m \in \hat{X}_0$. It is also simple to verify that the map $m \rightsquigarrow \xi_m$ is one-one and that $\xi_m \in \hat{X}_\alpha$ if and only if $m \in X_\alpha$. All that need be verified is that the map is onto.

Let $\xi \in \hat{X}_0$, and define $m\colon\ \mathcal{F} \to R$ by

$$m(F) = \begin{cases} 1, & \text{if there is } W \in \xi \text{ with } W \subset F, \\ 0, & \text{otherwise,} \end{cases}$$

for all $F \in \mathcal{F}$. If it can be shown that m is finitely-additive, it will follow immediately that $m \in X_0$ and $\xi_m = \xi$. Let $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap F_2 = \emptyset$. It is clear that $m(F_1) + m(F_2) \leqslant m(F_1 \cup F_2)$. Hence, assume that $m(F_1 \cup F_2) = 1$. By Proposition 1.2 and the assumption that $F_1 \cap F_2 = \emptyset$, there are $W_1, \cdots, W_{n_1+n_2}, V_1, \cdots, V_{n_1+n_2} \in \mathcal{W}$ with $V_i \subset W_i$ for $i = 1, \cdots, n_1 + n_2$, $(W_i - V_i) \cap (W_j - V_j) = \emptyset$ for $i \neq j$ such that $F_1 = \bigcup\{W_i - V_i\colon\ i = 1, \cdots, n_1\}$ and $F_2 = \bigcup\{W_i - V_i\colon\ i = n_1 + 1, \cdots, n_1 + n_2\}$. Since $m(F_1 \cup F_2) = 1$, there is a $W_0 \in \xi$ with $W_0 \subset F_1 \cup F_2$. We will assume that $m(F_1) = m(F_2) = 0$ and derive a contradiction. We first deduce the following statement.

(1) For $i = 1, \cdots, n_1 + n_2$, there is $U_i \in \xi$ such that $U_i \cap (W_i - V_i) = \emptyset$.

In order to see (1), fix i. If $W_i \in \xi$, then either $V_i \in \xi$ or $V_i \notin \xi$. If $V_i \in \xi$, take $U_i = V_i$. If $V_i \notin \xi$, then there is $W \in \xi$ with $W \cap V_i = \emptyset$. Since $W_i \in \xi$, we have that $W \cap W_i \in \xi$ and $W \cap W_i \subset W_i - V_i$ so that either $m(F_1) = 1$ or $m(F_2) = 1$ contrary to the assumption that $m(F_1) = m(F_2) = 0$. Hence (1) follows if $W_i \in \xi$. If $W_i \notin \xi$, then there is $W \in \xi$ with $W \cap W_i = \emptyset$. Taking $U_i = W$, the claim follows.

Now by the claim, for each $i = 1, \cdots, n_1 + n_2$, there is $U_i \in \xi$ with $U_i \cap (W_i - V_i) = \emptyset$. Define $U = W_0 \cap \bigcap \{U_i: i = 1, \cdots, n_1 + n_2\}$. Then $U \in \xi$. However, $U \subset W_0 \subset F_1 \cup F_2$. But $U \cap (F_1 \cup F_2) = \emptyset$ so that $U = \emptyset$. This contradicts the fact that ξ is a filter. The proof of the proposition is complete.

Let A be a uniformly closed algebra of real-valued functions on the set X which contains the constants and separates points. A functional $\varphi \in A^*$ is called *tight* if for every uniformly bounded net in A which converges to 0 uniformly on τ_A-compact subsets of X, it follows that $|\varphi|(f_i) \rightarrow 0$. Let A_t^* denote the set of all tight functionals in A^*. It is not difficult to check that A_t^* is a band in A^* and that $A_t^* \subset A_\tau^*$. The proof of the following is due in essence to Varadarajan [21].

PROPOSITION 5.12. *Let $\mathcal{W}$ be a normal base for τ_A, and let $m \in M(\mathcal{W})$ represent $\varphi \in A^*$. Then $\varphi \in A_t^*$ if and only if $m \in M_t(\mathcal{W})$.*

PROOF. ($\Rightarrow$) Since $\mathcal{W}$ is a normal base for τ_A, $\mathcal{W}_\tau$ is the family of τ_A-closed sets; and a set $W \in \mathcal{W}_\tau$ is $\mathcal{W}$-compact if and only if it is τ_A-compact. Let $\varphi \in A_t^*$, and without loss of generality assume that $0 \leqslant \varphi$. Assume that $m \notin M_t$, and let $\mathcal{W}_c$ denote the set of all nonempty $\mathcal{W}$-compact subsets of $\mathcal{W}_\tau$. Since $m \notin M_t$, there is an $\epsilon_0 > 0$ such that, for every $W \in \mathcal{W}_c$, there is $U_W \in \mathcal{W}$ with $U_W \cap W = \emptyset$ and $m(U_W) \geqslant \epsilon_0$. Since W is τ_A-compact, there is $f_W \in A$ with $0 \leqslant f_W \leqslant 1$, $f_W = 1$ on U_W and $f_W = 0$ on W. Then $\{f_W: W \in \mathcal{W}_c\}$ is a net which converges to zero uniformly on $\mathcal{W}$-compact sets. But, for all $W \in \mathcal{W}_c$, $\varphi(f_W) = \int_X f_W \, dm \geqslant m(U_W) \geqslant \epsilon_0$. Thus $\varphi(f_W) \nrightarrow 0$ contrary to the assumption $\varphi \in A_t^*$.

($\Leftarrow$) Assume $m \in M_t$ and, without loss of generality, that $0 \leqslant m$. Let (f_i) be a uniformly bounded net in A with $f_i \rightarrow 0$ uniformly on the τ_A-compact sets in X. Let $\epsilon > 0$ be arbitrary. Take $W_0 \in \mathcal{W}_\tau$ with W_0 $\mathcal{W}$-compact and with $m(W) \leqslant \epsilon$ for all $W \in \mathcal{W}$ with $W_0 \cap W = \emptyset$. Let μ denote the extension of m to $M_\tau(\mathcal{W}_\tau)$ as in the proof of Proposition 5.9. Since W_0 is τ_A-compact, there is an index i_0 such that $|f_i(x)| \leqslant \epsilon$ for all $i \geqslant i_0$ and $x \in W_0$. Hence, by Theorem 2.7, for $i \geqslant i_0$,

$$|\varphi(f_i)| \leqslant \int_{X - W_0} |f_i| \, dm + \int_{W_0} |f_i| \, dm$$

$$\leqslant M\mu(X - W_0) + \epsilon\varphi(\underset{\sim}{1})$$

where M is a uniform bound for (f_i). Since $m(W) \leqslant \epsilon$ if $W \in \mathcal{W}$ and $W_0 \cap W = \emptyset$, it follows from the $\mathcal{W}_\tau$-regularity of μ that $\mu(X - W_0) \leqslant \epsilon$. Thus $|\varphi(f_i)| \leqslant \epsilon(M + \varphi(1))$ for $i \geqslant i_0$. Since $\epsilon > 0$ was arbitrary, $\varphi(f_i) \rightarrow 0$. The proof is complete.

Let $\mathcal{Z} = \mathcal{Z}(A)$ denote the family of A zero sets. Let I be the standard representation of A_σ^* in $M_\sigma(\mathcal{Z})$ as in Theorem 4.11. We then have the following.

THEOREM 5.13. *The standard representation I of A_σ^* in $M_\sigma(\mathcal{Z})$ maps A_t^* onto $M_t(\mathcal{Z})$.*

The proof of this theorem is left to the reader. The fact that $I_\varphi \in M_t(\mathcal{Z})$ if and only if $\varphi \in A_t^*$ follows as in Proposition 5.12. The only other thing that needs to be observed is that $\mathcal{Z}$ is a basis for the τ_A-closed sets and that a set is $\mathcal{Z}$-compact if and only if it is τ_A-compact.

As in [15] a topology β_0 on A may be defined for which the dual space is A_t^*. Indeed, let β_0 be defined as the finest locally convex topology on A which agrees on the norm bounded sets with the topology of uniform convergence on τ_A-compact sets. It can be shown as in [15] that $\beta_0 \leqslant \beta_T$.

If $M(\mathcal{W})$ represents A^*, it may happen that the $\mathcal{W}$-compact sets do not coincide with the τ_A-compact sets. In the case that $\mathcal{W}$ is a normal base, the two are the same and so the following.

PROPOSITION 5.14. *If $\mathcal{W}$ is a normal base with X_A the $\mathcal{W}$-compactification or if $\mathcal{W}$ is $\mathcal{Z}(A)$, then the dual space of A with the β_0 topology is $A_t^* = M_t^*$.*

Many of the other results in [15] and [20] concerned with the β_0 topology can be shown to hold when $\mathcal{W}$ is a normal base.

6. **Weak convergence.** Let A be an algebra and let $\mathcal{W}$ be a paving of τ_A-closed sets such that $M(\mathcal{W})$ represents A^*. The *weak topology* on $M(\mathcal{W})$ generated by A will be the unique topology on $M(\mathcal{W})$ which makes the standard representation of A^* in $M(\mathcal{W})$ a topological isomorphism when A^* has the $\sigma(A^*, A)$ topology. If (m_i) is a net in $M(\mathcal{W})$, then by Theorem 4.7 $m_i \to m$ in the weak topology if and only if $\int_X f\,dm_i \to \int_X f\,dm$ for all $f \in A$. Similarly, if $\mathcal{Z} = \mathcal{Z}(A)$, the weak topology on $M_\sigma(\mathcal{Z})$ will be the unique topology on $M_\sigma(\mathcal{Z})$ which makes the standard representation of A_σ^* onto $M_\sigma(\mathcal{Z})$ a topological isomorphism. The following version of the *Portmanteau theorem* holds. The proof is omitted as it is essentially that of [2, p. 12]. (Also see [21].)

THEOREM 6.1. *Let (m_i) be a net in $M_\sigma^+(\mathcal{Z})$. Then the following are equivalent.*

1. *$m_i \to m$ weakly.*
2. *$\limsup m_i(Z) \leqslant m(Z)$ for all $Z \in \mathcal{Z}$ and $m_i(X) \to m(X)$.*
3. *$\liminf m_i(Z^c) \geqslant m(Z^c)$ for all $Z \in \mathcal{Z}$, and $m_i(X) \to m(X)$.*
4. *$\lim m_i(F) = m(F)$ for all $F \in \mathcal{F}(\mathcal{Z})$ with $\sup\{m(Z^c)\colon Z \in \mathcal{Z}, Z^c \subset F\} = \inf\{m(Z)\colon Z \in \mathcal{Z}, F \subset Z\}$.*

LEMMA 6.2. *Let X be a compact Hausdorff space, let $\mathcal{W}$ be a base for the closed sets in X and let (m_i) be a net in $M^+(\mathcal{W})$ with $m_i(X) \to m(X)$ and* *lim sup $m_i(W) \leqslant m(W)$ for all $W \in \mathcal{W}$ where $m \in M^+(\mathcal{W})$. Then $m_i \to m$ in the weak topology.*

PROOF. Since $\mathcal{W}$ is a base for the closed sets, $M(\mathcal{W})$ represents $C(X)^*$ by Theorem 3.12. Let $\varphi_i(f) = \int_X f\,dm_i$ and $\varphi(f) = \int_X f\,dm$ for all $f \in C(X)$. Since $M(\mathcal{G})$ represents $C(X)^*$ (where $\mathcal{G}$ is the family of closed subsets of X), there are $\mu_i, \mu \in M^+(\mathcal{G})$ with $\varphi_i(f) = \int_X f\,d\mu_i$ and $\varphi(f) = \int_X f\,d\mu$ for all $f \in C(X)$. (Hence m_i and m are restrictions to $\mathcal{F}(\mathcal{W})$ of μ_i and μ respectively.) We will show that $\int_X f\,d\mu_i \to \int_X f\,d\mu$ for all $f \in A$ with $0 < f < 1$. (That is, $0 < f(x) < 1$ for all $x \in X$.) The result follows immediately from this.

Fix $f \in A$ with $0 < f < 1$. For each number $r \in (0, 1)$, define $Z_r = \{x \in X: f(x) = r\}$; and let $P = \{r \in (0, 1): \mu(Z_r) > 0\}$. Since μ is a finite measure, P is countable. Let n be a fixed natural number greater than 1. Let θ be a number with $1/n \leqslant \theta < 1/n - 1$ such that $k\theta \notin P$ for all $k = 1, \cdots, n$. (That such a number exists can be seen by considering an $H \subset [1/n, 1/n - 1)$ where H is a basis for the real numbers over the rational numbers. Since H is uncountable and linearly independent over the rationals, the fact that P is countable guarantees that there is $\theta \in H$ satisfying the required conditions.) Define $F_k = \{x \in X: k\theta \leqslant f(x)\}$ for each $k = 0, 1, \cdots, n$.

It now follows for $k = 1, \cdots, n$ that $\mu_i(F_k) \to \mu(F_k)$. Indeed, fix $\epsilon > 0$, and let $U_k = \{x \in X: k\theta < f(x)\}$. Since $\mathcal{W}$ is a base for the closed sets in X and since μ is τ-additive (as $\mu \in M(\mathcal{G}) = M_\tau(\mathcal{G})$), there are sets $W_1, W_2 \in \mathcal{W}$ with $W_1^c \subset U_k$, $F_k \subset W_2$, $\mu(U_k - W_1^c) \leqslant \epsilon$ and $\mu(W_2 - F_k) \leqslant \epsilon$. Since $\mu(F_k) = \mu(U_k) + \mu(Z_k) = \mu(U_k)$ (since $k\theta \notin P$), it follows that $m(W_1^c) \geqslant \mu(U_k) - \epsilon = \mu(F_k) - \epsilon$. Since $W_1^c \subset F \subset W_2$, it follows that

$$\limsup \mu_i(F_k) \leqslant \limsup m_i(W_2) \leqslant m(W_2) \leqslant \mu(F_k) + \epsilon,$$

and

$$\liminf \mu_i(F_k) \geqslant \liminf m_i(W_1^c) \geqslant m(W_1^c) \geqslant \mu(F_k) - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that $\mu_i(F_k) \to \mu(F_k)$ as claimed.

Let ν denote either μ_i or μ. Then we have that

$$\sum_{k=1}^{n} (k-1)\theta\nu(F_{k-1} - F_k) \leqslant \int_X f\,d\nu \leqslant \sum_{k=1}^{n} k\theta\nu(F_{k-1} - F_k).$$

(Note that $F_0 = X$ and $F_n = \varnothing$.) Rearranging the sums, we obtain that

$$\theta \sum_{k=1}^{n-1} \nu(F_k) \leqslant \int_X f\,d\nu \leqslant \theta\nu(X) + \theta \sum_{k=1}^{n-1} \nu(F_k). \tag{1}$$

Since $\mu_i(F_k) \to \mu(F_k)$ for all $k = 0, 1, \cdots, n$, it follows from (1) that

$$\limsup\left[\int_X f\,d\mu - \int_X f\,d\mu_i\right]$$
$$\leqslant \limsup\left[\theta\mu(X) + \theta\sum_{k=1}^{n-1}\mu(F_k) - \theta\sum_{k=1}^{n}\mu_i(F_k)\right]$$
$$\leqslant \theta\mu(X) \leqslant (1/n - 1)\mu(X).$$

Since $1 \neq n \in N$ was arbitrary, it then follows that $\limsup\,[\int_X f\,d\mu - \int_X f\,d\mu_i] \leqslant 0$. In a similar fashion, it follows that $\liminf\,[\int_X f\,d\mu - \int_X f\,d\mu_i] \geqslant 0$. Hence, $\lim \int_X f\,d\mu_i = \int_X f\,d\mu$. The proof is complete.

THEOREM 6.3. *Let $\mathcal{W}$ be a normal base for the τ_A topology on X whose Wallman compactification is X_A. Let (m_i) be a net $M^+(\mathcal{W})$ and let $m \in M^+(\mathcal{W})$. Then the following are equivalent.*

1. *$m_i \to m$ weakly.*
2. *$\limsup m_i(W) \leqslant m(W)$ for all $W \in \mathcal{W}$ and $m_i(X) \to m(X)$.*
3. *$\liminf m_i(W^c) \geqslant m(W^c)$ for all $W \in \mathcal{W}$ and $m_i(X) \to m(X)$.*

PROOF. ($1 \Rightarrow 2$) Let $W \in \mathcal{W}$, and fix $\epsilon > 0$. Since $M(\mathcal{W})$ represents A^* by Theorem 3.12, there is $f \in A$ with $\chi_W \leqslant f$ and $\int_X f\,dm \leqslant m(W) + \epsilon$. Hence it follows that

$$\limsup m_i(W) \leqslant \lim\int_X f\,dm_i = \int_X f\,dm \leqslant m(W) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the result follows immediately.

($2 \Leftrightarrow 3$) This is obvious.

($2 \Rightarrow 1$) By Proposition 3.11, $\overline{\mathcal{W}}$ is a basis for the closed sets in X_A. Let $\overline{\varphi}(\overline{f}) = \int_X f\,dm$ for all $f \in A$. Then $0 \leqslant \overline{\varphi} \in C(X_A)^*$. Since $M(\overline{\mathcal{W}})$ represents $C(X_A)^*$ by Theorem 3.12, there is $0 \leqslant \overline{m} \in M(\overline{\mathcal{W}})$ with $\overline{\varphi}(\overline{f}) = \int_{X_A} \overline{f}\,d\overline{m}$ for all $f \in A$. Hence for all $W \in \mathcal{W}$,

$$m(W) = \inf\{\varphi(f)\colon\ f \in A, \chi_W \leqslant f\}$$
$$= \inf\{\overline{\varphi}(\overline{f})\colon\ \overline{f} \in C(X_A), \chi_{\overline{W}} \leqslant \overline{f}\} = \overline{m}(\overline{W}).$$

Thus $\overline{m}_i(X_A) \to \overline{m}(X_A)$ and $\limsup \overline{m}_i(\overline{W}) \leqslant \overline{m}(\overline{W})$ for all $\overline{W} \in \overline{\mathcal{W}}$. Hence by Lemma 6.2, $\int_X f\,dm_i = \int_{X_A} \overline{f}\,d\overline{m}_i \to \int_{X_A} \overline{f}\,d\overline{m} = \int_X f\,dm$ for all $f \in A$. Hence $m_i \to m$ weakly as claimed. The proof is complete.

Theorem 6.3 need not hold if $\mathcal{W}$ is not a normal basis. (See (c) of appendix.)

For each $x \in X$, let m_x denote the measure assigning unit mass to the set $\{x\}$, and let $L(X)$ (or L) denote the linear hull of $\{m_x\colon\ x \in X\}$.

PROPOSITION 6.4. *Let $\mathcal{W}$ be a full paving on X with $M(\mathcal{W})$ representing A^*. Then $L(X)$ is weakly dense in $M(\mathcal{W})$. Furthermore, the map $x \rightsquigarrow m_x$ extends to* *a homeomorphism of* X_A *into* $M(\mathcal{W})$. *Finally,* $M(\mathcal{W})$ *is also dense in* $M_\sigma(\mathcal{Z}(A))$.

The fact that $L(X)$ is dense in $M(\mathcal{W})$ is immediate from the Hahn-Banach theorem and the fact that the dual of A^* with the topology $\sigma(A^*, A)$ is A. The proofs of the other statements are left to the reader.

If X is a metric space and (m_n) a sequence of Borel measures on X, it is well known that (m_n) converges weakly to a Borel measure m if $\int_X f\,dm_n \to \int_x f\,dm$ for every bounded, uniformly continuous function f on X. The following is a generalization of this fact.

PROPOSITION 6.5. *Let* A_1 *and* A_2 *be two algebras on* X *with* $A_2 \subset A_1$. *Assume that if* $Z_0 \in \mathcal{Z}_1 = \mathcal{Z}(A_1)$, *then* $Z_0 = \bigcap\{Z\colon Z \in \mathcal{Z}_2, Z_0 \subset Z\}$. *Let* (m_i) *be a net in* $M_\sigma^+(\mathcal{Z}_1)$, *let* $m \in M_\tau^+(\mathcal{Z}_1)$, *and assume that* $\int_X f\,dm_i \to \int_X f\,dm$ *for all* $f \in A_2$. *Then* $\int_X f\,dm_i \to \int_X f\,dm$ *for all* $f \in A_1$.

PROOF. Let $Z_0 \in \mathcal{Z}_1$ and fix $\epsilon > 0$. Since $m \in M_\tau^+(\mathcal{Z}_1)$, there is $Z_1 \in \mathcal{Z}_2$ with $Z_0 \subset Z_1$ and $m(Z_1 - Z_0) \leqslant \epsilon$. By Theorem 7.1, it follows that

$$\limsup m_i(Z_0) \leqslant \limsup m_i(Z_1) \leqslant m(Z_1) \leqslant m(Z_0) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the result follows by Theorem 6.1.

COROLLARY 6.6. *Let* X *be a completely regular Hausdorff space and let* A *be an algebra on* X *with* τ_A *the original topology of* X. *Let* $(\varphi_i) \in (C^b(X)^*)_\sigma^+$ *and* $\varphi \in (C^b(X)^*)_\tau^+$. *Then* $\varphi_i \to \varphi$ *weakly if and only if* $\varphi_i(f) \to \varphi(f)$ *for all* $f \in A$.

Appendix. The following space may be used to provide counterexamples for several natural conjectures.

EXAMPLE. Let Δ denote the closed unit disc in the plane. We give Δ a topology as follows. For $0 \neq z_0 = r_0 e^{i\theta_0}$, sets of the form $\{re^{i\theta_0}\colon r_0 - \epsilon < r < r_0 + \epsilon\}$ where $0 < \epsilon < r_0$ give a basis for the neighborhood system at z_0. A basis for the neighborhood system at 0 consists of sets $N(0; \theta_1, \cdots, \theta_n, r_1, \cdots, r_n) = \{z \in \Delta\colon z \neq re^{i\theta_k}$ where $k = 1, \cdots, n$ and $r_k \leqslant r\}$, where $\theta_1, \cdots, \theta_n$ are angles between 0 and 2π and $r_1, \cdots, r_n$ are positive numbers. It is not hard to check that Δ is a compact Hausdorff space. Let $X = \Delta - \{0\}$ and let A denote the algebra of restrictions to X of the continuous real-valued functions on Δ. It is clear that $X_A = \Delta$ and that τ_A is the restriction to X of the topology of Δ.

(a) *There is a paving* $\mathcal{W}$ *such that* $M(\mathcal{W})$ *represents* A^* *even though* $\mathcal{W}$ *is not a normal base.*

PROOF. Let $\mathcal{W}$ denote the full paving on X generated by the compact sets in X together with the sets $B_\epsilon = \{z \in X\colon |z| \leqslant \epsilon\}$ as ϵ runs through $(0, 1]$. It is clear that A strongly separates $\mathcal{W}$. It is also clear that $\overline{\mathcal{W}}$ is not a base for the closed sets in $X_A = \Delta$. Indeed, the set consisting of 0 together with annulus obtained by removing the open disc of radius ½ centered at 0 is closed in Δ but is not the intersection of elements of $\overline{\mathcal{W}}$. Hence by Proposition 3.11, $\mathcal{W}$ is not a normal base.

In order to show that $M(\mathcal{W})$ represents A^*, it is sufficient by Theorem 3.8 to show that condition (2′) of the remark following Theorem 3.8 holds. Let $0 \leq \mu \in M(\mathcal{G})$ where $\mathcal{G}$ is the family of closed sets in Δ. If μ is a multiple of the unit point measure at 0, it is clear that (2′) holds. Hence assume that $\inf(\mu, \mu_0) = 0$. Hence, given $\epsilon > 0$ and $G_0 \in \mathcal{G}$, there is $G_1 \in \mathcal{G}$ with $0 \notin G_1$, $G_1 \subset G_0^c$ and $\mu(G_0^c - G_1) \leq \epsilon$. But since $0 \notin G_1$, G_1 is a compact subset of X so that $G_1 \in \overline{\mathcal{W}}$. It is now clear that condition (2′) holds.

(b) *If $M(\mathcal{W})$ represents A^*, it is not necessary for $M_\alpha(\mathcal{W})$ to correspond with A_α^* even if $\mathcal{W}$ is a normal base.*

PROOF. The example will be given for $\alpha = \sigma$. (A similar example exists for every cardinal α.) Let $\mathcal{W}$ be the paving generated by the zero sets of A together with the sets $B_\epsilon = \{x \in X\colon |z| \leq \epsilon\}$. It is easy to check that any zero set of A is either compact in X or contains a set of the form $\{z \in X\colon z \neq re^{i\theta_n} \text{ for } n \in N\}$ where (θ_n) is a sequence from $[0, 2\pi)$. Hence in any pair of disjoint zero sets, one is compact. By Proposition 3.11, $\mathcal{W}$ is a normal base with $X_{\mathcal{W}} = X_A$. Hence $M(\mathcal{W})$ represents A^* by Theorem 3.12. Let $\varphi(f) = \bar{f}(0)$ for all $f \in A$. It is easy to see that if (f_n) is a sequence in A with $f_n \downarrow 0$ pointwise on X, then $\bar{f}_n(0) \downarrow 0$. Hence $\varphi \in A_\sigma^*$. If $m \in M(\mathcal{W})$ represents φ, then $m(B_\epsilon) = 1$ for all $\epsilon > 0$. Thus $B_{1/n} \downarrow \varnothing$ but $m(B_{1/n}) \not\to 0$. Hence $m \notin M_\sigma(\mathcal{W})$.

(c) *If $M(\mathcal{W})$ represents A^* and if $\mathcal{W}$ is not a normal base, it is possible to have a sequence (m_n) in $M^+(\mathcal{W})$ with $m_n(X) \to m(X)$ and* $\limsup m_n(W) \leq m(W)$ *for all $W \in \mathcal{W}$ even though (m_n) does not converge weakly to m.*

PROOF. Let $\mathcal{W}$ be the full paving generated by the compact sets in X together with the sets $D_\epsilon = \{z \in X\colon z = re^{i\theta} \text{ with } |r| \leq \epsilon \text{ and } \theta \neq 0\}$. Then $M(\mathcal{W})$ represents A^*. Let r_n be a sequence in $(0, 1)$ with $r_n \to 0$. For each $n \in N$, define m_{2n} to be the point measure at r_n and define m_{2n+1} to be the point measure at 1. It is clear that the sequence (m_k) does not converge weakly. However, as may easily be seen, $m_n(X) \to m_1(X)$ and $\limsup m_n(W) \leq m_1(W)$ for all $W \in \mathcal{W}$.

(d) *Let $M(\mathcal{W})$ represent A with $\mathcal{W}$ a normal base. It is not necessary that $M_\sigma(\mathcal{W})$ be weakly sequentially complete.*

PROOF. Let $\mathcal{W}$ be as in (c) above, and let $m_n \in M_\sigma^+(\mathcal{W})$ be the measure assigning mass 1 at the point $1/n$. Then $m_n \to m_0$ where m_0 is unit mass at 0.

However, $m_0 \notin M_\sigma^+(\mathcal{W})$. (The functional φ_0 representing m_0 on the other hand does belong to A_σ^+.)

(e) *Let $B \subset A_\sigma^*$ be represented by $H \subset M_\sigma(\mathcal{W})$. Then it can happen that H is uniformly σ-additive but B is not β_σ-equicontinuous even if $\mathcal{W}$ is a normal base or $\mathcal{Z}(A)$.*

PROOF. Take $\mathcal{W}$ to be the paving generated by $\mathcal{Z}(A)$ together with the set of $B_\epsilon = \{z \in X\colon |z| \leqslant \epsilon\}$ for $\epsilon > 0$. (Then $\mathcal{W}$ is a normal base for X_A.) Let $B = \{\varphi_z\colon |z| = ½\}$ where $\varphi_z(f) = f(z)$ for all $f \in A$. Then $H = \{m_z\colon |z| = ½\}$ where m_z is the measure assigning mass 1 to z. It is easy to see that if $I \subset \mathcal{W}$ is a σ-system then $\lim_{W \in I} m(W) = 0$ uniformly for $m \in H$. Hence H is uniformly σ-additive. However, B is not β_σ-equicontinuous. Indeed, if B were β_σ-equicontinuous, there is a $g \in A$ with $\overline{g}(0) = 0$ and $\{f\colon \|fg\|_X \leqslant 1\} \subset B$. But if $\overline{g}(0) = 0$, then $g = 0$ on all but at most a countable number of rays $\{I_{\theta_n}\colon n \in N\}$ where $I_{\theta_n} = \{z\colon z = re^{i\theta_n}$ for some $0 < r \leqslant 1\}$.

Now let $F_0 = \{x \in X_A\colon 0 \leqslant g(x) \leqslant ¼\}$ and $F_1 = \{x \in X_A\colon ½ \leqslant g(x)\}$. Let $f \in A$ be such that $0 \leqslant f \leqslant 2, f = 0$ on F_1 and $f = 2$ on F_0. Then $\|fg\|_X \leqslant 1$ so that $f \in B^0$. But if $\theta_0 \notin \{\theta_n\colon n \in N\}$ and if $z_0 = ½e^{i\theta_0}$, then $f(z_0) = \varphi_{z_0}(f) = 2$. Thus $f \notin B^0$ which is a contradiction.

(f) *For B and H as in* (e), *it may happen that B is uniformly σ-smooth but H is not uniformly σ-additive even if $\mathcal{W}$ is a normal base or $\mathcal{Z}(A)$.*

PROOF. Let $\mathcal{W}$ be as in (e) above, and take $B = \{\varphi_{1/k}\colon k \in N\}$. Then $H = \{m_{1/k}\colon k \in N\}$. It is clear that B is uniformly σ-smooth. However, $I = \{B_{1/n}\colon n \in N\}$ is a σ-system in $\mathcal{W}$ on which H does not converge uniformly to 0.

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