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I-RINGS

BY

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ABSTRACT. A ring R, possibly with no identity, is called an I_0-ring if each one-sided ideal not contained in the Jacobson radical $J(R)$ contains a nonzero idempotent. If, in addition, idempotents can be lifted modulo $J(R)$, R is called an I-ring. A survey of when these properties are inherited by related rings is given. Maximal idempotents are examined and conditions when I_0-rings have an identity are given. It is shown that, in an I_0-ring R, primitive idempotents are local and primitive idempotents in $R/J(R)$ can always be lifted. This yields some characterizations of I_0-rings R such that $R/J(R)$ is primitive with nonzero socle. A ring R (possibly with no identity) is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents can be lifted modulo $J(R)$. These rings are characterized in several new ways: among them as I_0-rings with no infinite orthogonal family of idempotents, and as I_0-rings R with $R/J(R)$ semisimple artinian. Several other properties are derived. The connection between I_0-rings and the notion of a regular module is explored. The rings R which have a regular module M such that $J(R) = \text{ann}(M)$ are studied. In particular they are I_0-rings. In addition, it is shown that, over an I_0-ring, the endomorphism ring of a regular module is an I_0-ring with zero radical.

1. Definitions and examples. Throughout this paper all rings are assumed to be associative but do not necessarily have an identity. When a ring has an identity, modules are assumed to be unital. Unless otherwise stated, all modules are left modules and homomorphisms are written on the right of their arguments. The *Jacobson radical* of a ring R will be denoted by $J(R)$.

LEMMA 1.1. *If R is a ring the following conditions are equivalent*:

(1) *Every left ideal $L \nsubseteq J(R)$ contains a nonzero idempotent.*

(2) *Every right ideal $T \nsubseteq J(R)$ contains a nonzero idempotent.*

(3) *If $a \notin J(R)$ then $xax = x$ for some $x \neq 0$.*

PROOF. Given (1), let $a \notin J(R)$. If $e = ra \in Ra$ is a nonzero idempotent,

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(3) follows with $x = rar$. The converse is clear and the proof that (1) $\Leftrightarrow$ (3) is analogous. $\square$

DEFINITION 1.2. *A ring R is called an I_0*-ring *if it satisfies the conditions of Lemma* 1.1. *An I_0-ring in which idempotents can be lifted modulo $J(R)$ is called an I*-ring.

An I-ring with a nil Jacobson radical will be called a *Zorn ring*. This terminology is used by Kaplansky [5, p. 19], and Bourbaki [3, p. 75]. However the reader is cautioned that Jacobson [4, p. 210] and Levitzky [8, p. 385] refer to our Zorn rings as I-rings. Many of the results below will be true for Zorn rings, but we shall not mention this fact in most instances. Our primary concern here is with I-rings. A deep study of Zorn rings can be found in Levitzky [8].

The class of I-rings is quite large. It obviously contains all division rings and, more generally, contains all local rings where, in this paper, a ring R will be called *local* if it has an identity and $R/J(R)$ is a division ring. In a different direction, each primitive ring with a minimal left ideal is an I-ring. In fact every semiprime ring with essential socle is an I-ring. On the other hand any I_0-ring with identity and no divisors of zero is a local ring, so the integers are an example of a noetherian semiprime ring which is not an I-ring.

An element a in a ring R is called (von Neumann) *regular* if $aba = a$ for some $b \in R$. The ring R is called *regular* if each of its elements is regular and R is called *π-regular* if some power of each element is regular. It is easily verified that every π-regular ring is an I-ring (in fact a Zorn ring). In particular, every algebraic algebra is an I-ring [4, p. 210].

LEMMA 1.3. *Let R be a ring, let L be a left (right) ideal of R and let $x \in L$. If there is an idempotent f such that $f - x \in J(R)$, then there exists an idempotent $e \in L$ such that $e - x \in J(R)$.*

PROOF. If $f^2 = f$ and $f - x \in J(R)$, choose $a \in J(R)$ such that $a + (f - x) = a(f - x)$. Then $faf + f - fxf = faf - faxf$ so $f = f(x - ax)f$. Take $e = f(x - ax)$. $\square$

PROPOSITION 1.4. *Let R be a ring in which idempotents can be lifted modulo $J(R)$. Then R is an I-ring if and only if $R/J(R)$ is an I-ring.*

A ring R (possibly with no identity) will be called *semiperfect* if $R/J(R)$ is artinian and idempotents can be lifted modulo $J(R)$.

COROLLARY 1.5. *Every semiperfect ring is an I-ring.*

PROPOSITION 1.6. *If R is an I-ring (I_0-ring) so is each one-sided ideal of R.*

PROOF. Let L be a left ideal of R. Then $J(L) = \{a \in L | La \subseteq J(R)\}$ [1,

p. 113]. If M is a left ideal of L and $M \not\subseteq J(L)$ then $LM \not\subseteq J(R)$ so there exists $0 \neq e^2 = e \in LM \subseteq M$. Suppose now that $x \in L$, $x^2 - x \in J(L)$. Then $x^4 - x^2 \in J(R)$ so, by Lemma 1.3, choose $e^2 = e \in L$ such that $e - x^2 \in J(R)$. Then $e - x \in J(L)$. □

COROLLARY 1.7. *If R is an I-ring (I_0-ring) so is each subring aRb where $a, b \in R$.*

PROOF. aRb is a left ideal of aR. □

Note that Proposition 1.6 and its corollary are true for Zorn rings as well. We remark here that the center of an I-ring need not be an I-ring (see Example 1.9).

The $n \times n$ matrix ring over a ring R will be denoted by $M_n(R)$. If $r \in R$ let $E_{ij}(r)$ denote the matrix with r in the (i, j)-position and zeros elsewhere.

PROPOSITION 1.8. *If R is an I_0-ring so is the ring $M_n(R)$ of all $n \times n$ matrices over R.*

PROOF. Let L be a left ideal of $M_n(R)$ with $L \not\subseteq J[M_n(R)] = M_n[J(R)]$. There exists a matrix $A = (a_{ij}) \in L$ with $a_{pq} \notin J(R)$ for some p, q. Then $L_0 = \{x \in R | x = x_{pq}$ for some $(x_{ij}) \in L\}$ is a left ideal of R and $L_0 \not\subseteq J(R)$. If $0 \neq e^2 = e \in L_0$, let $X = (x_{ij}) \in L$ with $x_{pq} = e$. One verifies that $E_{qp}(e)X = \Sigma_j E_{qj}(ex_{pj})$ is a nonzero idempotent in L. □

A natural question is the following: Is $M_n(R)$ an I-ring whenever R is an I-ring? It is sufficient to answer this in the case $n = 2$. The answer is affirmative if R is semiperfect or if $J(R)$ is locally nilpotent. More generally, one can ask: If a ring R is such that idempotents can be lifted modulo $J(R)$, does $M_2(R)$ have the same property? We remark in this connection that it is an open question whether $M_2(R)$ is a nil ring whenever R is a nil ring (see [7]).

It is obvious that a direct sum of rings is an I-ring (an I_0-ring) if and only if the same is true of each summand. It is also clear that R/A is an I-ring (I_0-ring) if R is an I-ring (I_0-ring) and $A \subseteq J(R)$ is an ideal. However the class of I-rings is not closed under homomorphic images.

EXAMPLE 1.9. Let Δ be a division ring and let $S \subseteq \Delta$ be any subring. Let R be the ring of all countably infinite square matrices of the form

$$\begin{pmatrix} A & & & 0 \\ & \delta & & \\ & & \delta & \\ 0 & & & \ddots \end{pmatrix}$$

where $A \in M_n(\Delta)$ for some $n \geq 1$ and $\delta \in S$. Denote this matrix by (A, δ). If

$(A, \delta) \neq 0$ we can assume $A \neq 0$ so, by Proposition 1.8, choose a matrix B such that BA is a nonzero idempotent. Then $(B, 0)(A, \delta) = (BA, 0)$ is an idempotent in R, and it follows that R is an I_0-ring with $J(R) = 0$. The map $R \longrightarrow S$ given by $(A, \delta) \longmapsto \delta$ is a ring epimorphism so it is clear that a homomorphic image of an I-ring need not be an I-ring. If S is in the center of Δ, the center of R is isomorphic to S and so need not be an I_0-ring. □

2. **Primitive idempotents.** In any ring R there is a natural partial ordering of the idempotents defined by $f \leqslant e$ if $f \in eRe$. A nonzero idempotent which is minimal in this partial ordering is called *primitive*. It is easily verified that e is primitive if and only if $0 \neq f^2 = f \in Re$ implies $Rf = Re$.

We say that $e = e^2$ is a *local idempotent* if eRe is a local ring. It is well known that every primitive idempotent in a semiperfect or regular ring is local. We generalize this as follows:

PROPOSITION 2.1. *The following conditions are equivalent for an idempotent e in an I_0-ring R:*

(1) *e is primitive.*

(2) *If $L \subseteq Re$ is a left ideal and $L \not\subseteq J(R)$ then $L = Re$.*

(3) *e is local.*

PROOF. (1) ⇒ (2) by the above remark (since R is an I_0-ring) and (3) ⇒ (1) is obvious. Assume (2). If $a \in eRe$ and $a \notin J(eRe) = eRe \cap J(R)$, then $Ra = Re$ by (2). Hence a has a left inverse in eRe, proving (3). □

An immediate consequence of this is that an I_0-ring has a unique nonzero idempotent if and only if it has the form $L \oplus A$ where L is local and $J(A) = A$.

Two idempotents e and f in a ring R are said to be *equivalent* if there exist $x \in eRf$ and $y \in fRe$ such that $e = xy$ and $f = yx$.

COROLLARY 2.2. *Two primitive idempotents e and f in an I_0-ring R are equivalent if and only if $eRf \not\subseteq J(R)$.*

PROOF. If $x \in eRf$, $x \notin J(R)$, we have $Rx = Rf$ by Proposition 2.1. Write $f = yx$ with $y \in fRe$. Then $0 \neq (xy)^2 = xy \in eRe$ so $xy = e$. The converse is trivial. □

We have the following result on the existence of primitive idempotents:

PROPOSITION 2.3. *An I_0-ring with $J(R) = 0$ has a primitive idempotent if and only if it has a maximal left (right) annihilator.*

PROOF. Let $L = \{a \in R | aS = 0\}$ be a maximal left annihilator. If $0 \neq x \in S$ and $0 \neq e^2 = e \in xR$, we have $L = \{a | ae = 0\}$ by maximality. We claim

e is primitive. If $0 \neq f^2 = f \in eRe$ we have $L = \{a | af = 0\}$ so $e - f \in L$. But then $0 = (e - f)e = e - f$. The converse follows from Proposition 2.1. □

We have immediately the following which retrieves a result of Koh [6] when R is regular.

COROLLARY 2.4. *An I_0-ring R with $J(R) = 0$ is primitive with nonzero socle if and only if it is a prime ring with a maximal left (right) annihilator.*

The next result shows that, in an I_0-ring, primitive idempotents in $R/J(R)$ can always be lifted to R. It will be referred to several times below.

LEMMA 2.5. *Let R be an I_0-ring and suppose $x \in R$ is such that $x + J(R)$ is a primitive idempotent in $R/J(R)$. There exists an idempotent $e \in R$ such that $e - x \in J(R)$.*

PROOF. Choose $0 \neq f^2 = f \in Rx$. Then, in $\bar{R} = R/J(R)$, $\bar{f} \in \bar{R}\bar{x}$ so $\bar{R}\bar{f} = \bar{R}\bar{x}$ by Proposition 2.1. If we set $e = f + xf - fxf$, then $e^2 = e \neq 0$ and $\bar{e} = \bar{x}$. □

PROPOSITION 2.6. *The following are equivalent for a ring R:*

(1) *R is an I_0-ring and every nonzero idempotent contains a primitive idempotent.*

(2) *Each left (right) ideal $L \not\subseteq J(R)$ contains a primitive idempotent.*

PROOF. If (2) holds and $e^2 = e \neq 0$ let $f^2 = f \in Re$ be primitive. Then $(ef)^2 = ef \in eRe$ and ef is primitive by Proposition 2.1 since $Ref = Rf$. Hence (2) ⇒ (1); the converse is clear. □

We say a ring R *has primitive idempotents* if it satisfies these conditions. Lemmas 2.5 and 1.3 immediately yield

COROLLARY 2.7. *An I_0-ring R has primitive idempotents if and only if the same is true of $R/J(R)$.*

Hence every semiperfect ring has primitive idempotents. It is well known that a primitive ring with nonzero socle has primitive idempotents (the socle is large as a left ideal). The ring in Example 1.9 is easily shown to be primitive with nonzero socle and so a homomorphic image of a ring with primitive idempotents need not have this property. On the other hand, the methods of §1 show that, if R has primitive idempotents, the same is true of any one-sided ideal of R, any subring of the form aRb, and any matrix ring $M_n(R)$.

If R is a ring with identity, a module $M \neq 0$ is called *local* if it is projective and $Rx = M$ for each $x \in M - \text{rad } M$. If $e^2 = e \in R$ then $\text{rad}(Re) = J(R)e$ and it follows easily from Proposition 2.1 that Re is local if and only if e is a local idempotent.

THEOREM 2.8. *The following are equivalent for an I_0-ring R:*

(1) *R has primitive idempotents and any two primitive idempotents are equivalent.*

(2) *R contains a primitive idempotent and $J(R)$ is a prime ideal.*

(3) *$R/J(R)$ is a primitive ring with nonzero socle.*

If R has an identity these are equivalent to

(4) *R has a local module and $J(R)$ is a prime ideal.*

PROOF. If R has an identity, (2) $\Leftrightarrow$ (4) by the remark above; (3) $\Rightarrow$ (1) by Corollary 2.7 and [4, p. 51]; and (2) $\Rightarrow$ (3) is clear. If (1) holds let $aRb \subseteq J(R)$ where $a, b \in R - J(R)$. Choose primitive idempotents $e \in Ra$ and $f \in bR$. By hypothesis, $e = xy$ where $x \in eRf$ and $y \in fRe$ so $e \in RaRbR \subseteq J(R)$, a contradiction. This means $J(R)$ is a prime ideal so (1) $\Rightarrow$ (2). □

In fact, the equivalence of (2) and (4) can be proved under the weaker assumption that $J(R)$ is small in R as a left ideal. In this form it generalizes the fact that a prime ring is primitive with nonzero socle if and only if it has an irreducible projective module.

3. **Maximal idempotents.** An idempotent e in a ring R is said to be the *greatest* idempotent if $e \geqslant f$ for every idempotent f in R. Such an idempotent is central since $e \geqslant e + er - ere$ and $e \geqslant e + re - ere$ for each $r \in R$. Hence:

LEMMA 3.1. *An I_0-ring with a greatest idempotent has the form $S \oplus A$ where S has an identity and $J(A) = A$.*

An idempotent e is called *maximal* if it is not the greatest idempotent and $e \leqslant f$, $f^2 = f$ implies $e = f$ or f is the greatest idempotent.

LEMMA 3.2. *Let R be a ring with no greatest idempotent. The following are equivalent for an idempotent $e \in R$:*

(1) *e is maximal.*

(2) *If $Re \subseteq Rf$ where $f^2 = f$ then $Re = Rf$.*

(3) *If $ef = 0$ where $f^2 = f$ then $f = 0$.*

Moreover the left-right analogs of (2) *and* (3) *are equivalent to* (1).

PROOF. (1) $\Rightarrow$ (2). If $Re \subseteq Rf$ let $g = e + f - fe$. Then $g^2 = g \geqslant e$ and (2) follows.

(2) $\Rightarrow$ (3). If $ef = 0$ let $g = e + f - fe$. Then $g^2 = g$ and $Re \subseteq Rg$ so $g = ge = e$ by (2). Hence $f = f^2 = (fe)f = 0$.

(3) $\Rightarrow$ (1). If $g^2 = g \geqslant e$ then $e(g - e) = 0$ so $g = e$. □

THEOREM 3.3. *If an I_0-ring R has a maximal idempotent then $R/J(R)$ has* *an identity and the following are equivalent*:

(1) $x \in Rx$ *for each* $x \in R$.

(2) $J(R)$ *is small as a right ideal.*

(3) R *has a left identity.*

PROOF. If R has a greatest idempotent the result follows from Lemma 3.1. Otherwise, let $e^2 = e$ be maximal. The right ideal $T = \{a \in R | ea = 0\}$ is contained in $J(R)$ by Lemma 3.2 so $r - er \in J(R)$ for every $r \in R$. Similarly $r - re \in J(R)$ so $R/J(R)$ has an identity.

Now (3) ⇒ (2) is well known and (2) ⇒ (3) follows since $R = eR + J(R)$. If (1) holds let $r \in R$ and write $x = r - er$ where $e^2 = e$ is maximal. If $x = tx$, $t \in R$ then $x = (t - te)x$ so $x = 0$ (since $t - te \in J(R)$). Hence (1) ⇒ (3); the converse is obvious. □

In particular a regular ring with a maximal idempotent has an identity. The next result generalizes the well-known fact that a semisimple artinian ring has an identity.

COROLLARY 3.4. *Let* R *be an* I_0*-ring with* $J(R) = 0$. *If* R *has a minimal right annihilator of the form* $r(e) = \{a \in R | ea = 0\}$, $e^2 = e$ *then* R *has an identity.*

PROOF. It suffices to show R has a greatest idempotent by Lemma 3.1. If not, let $r(e)$ be minimal. If $e \leqslant f$, $f^2 = f$ then $r(f) \subseteq r(e)$ so either $r(f) = r(e)$ (so $e = f$) or $r(f) = 0$ (so $fR = R$). It follows that R has a maximal idempotent and so has an identity, a contradiction. □

We remark finally that an I-ring R has a maximal idempotent if and only if $R/J(R)$ has an identity and, in this case, the maximal idempotents are just the pre-images of the identity in $R/J(R)$.

4. Finiteness conditions. In this section we study I_0-rings which satisfy chain conditions on the set of idempotents and obtain some new characterizations of semiperfect rings.

LEMMA 4.1. *Let* R *be any ring. The following are equivalent*:

(1) R *has maximum condition on idempotents.*

(2) R *has maximum condition on left ideals* Re, $e^2 = e$ (*on right ideals* eR, $e^2 = e$).

(3) R *has no infinite orthogonal family of idempotents.*

PROOF. The proofs of (2) ⇒ (3) ⇒ (1) are omitted. Given (1) let $Re_1 \subseteq Re_2 \subseteq \cdots$, $e_i^2 = e_i$. Define $f_1, f_2, \ldots$ as follows: $f_1 = e_1$, $f_{k+1} = f_k + e_{k+1} - e_{k+1}f_k$ for $k \geqslant 1$. Then $f_k \in Re_k$ for each k and consequently $f_k^2 = f_k$

and $f_1 \leqslant f_2 \leqslant \cdots$. If $f_n = f_{n+1} = \cdots$ for some n then $Re_{n+1} = Re_{n+2} = \cdots$. Hence (1) ⇒ (2). □

We remark that a ring R with minimum condition in left annihilators of idempotents has maximum condition on idempotents. Moreover, the converse is true if R has no total right annihilators; that is if $Ra = 0$ implies $a = 0$.

It is surprising that the minimum condition on idempotents is, in general, weaker than the maximum condition. (Consider $R = \Delta \oplus \Delta \oplus \cdots$ where Δ is a division ring.) The two are equivalent for rings with identity as the next result shows.

LEMMA 4.2. *The following are equivalent in any ring R:*

(1) *R has minimum condition on idempotents.*

(2) *R has minimum condition on left ideals Re, $e^2 = e$ (on right ideals eR, $e^2 = e$).*

(3) *Any bounded ascending chain of idempotents terminates.*

PROOF. Given (1) let $Re_1 \supseteq Re_2 \supseteq \cdots$, $e_k^2 = e_k$. Then $e_1 e_2 \cdots e_k$ is an idempotent for each k, $Re_1 e_2 \cdots e_k = Re_k$ and $e_1 \geqslant e_1 e_2 \geqslant \cdots$. Hence (1) ⇒ (2). We leave (2) ⇒ (3) ⇒ (1) to the reader. □

The next result contains some new characterizations of semiperfect rings (defined following Proposition 1.4).

THEOREM 4.3. *The following are equivalent for any ring R:*

(1) *R is semiperfect.*

(2) *R is an I_0-ring with maximum condition on idempotents.*

(3) *If $L \subseteq R$ is a left (right) ideal there exist an idempotent $e \in L$ and a left (right) ideal $M \subseteq J(R)$ such that $L = Re + M$.*

(4) *R is an I_0-ring and $R/J(R)$ is semisimple artinian.*

PROOF. (1) ⇒ (2). If R is semiperfect it is an I_0-ring by Corollary 1.5. If $e_1 \leqslant e_2 \leqslant \cdots$ are idempotents in R then $\bar{e}_n = \bar{e}_{n+1} = \cdots$ in $R/J(R)$ for some n. Hence $(e_{k+1} - e_k)^2 = (e_{k+1} - e_k) \in J(R)$ for each $k \geqslant n$ and so $e_n = e_{n+1} = \cdots$.

(2) ⇒ (3). If $L \subseteq J(R)$ take $e = 0$ and $M = L$. Otherwise let $0 \neq e^2 = e \in L$ be maximal in L. Then $L = Re + M$ where $M = \{x \in L | xe = 0\}$. If $M \not\subseteq J(R)$ choose $0 \neq f^2 = f \in M$. Then $g = e + f - ef$ is an idempotent in L and $e \leqslant g$. Hence $e = g$ by the choice of e and so $f = f^2 = f(ef) = 0$, a contradiction. Hence $M \subseteq J(R)$.

(3) ⇒ (4). This is obvious.

(4) ⇒ (1). Since R is an I_0-ring, primitive idempotents in $R/J(R)$ can be

lifted by Lemma 2.5. But every idempotent in $R/J(R)$ is a sum of orthogonal primitive idempotents and so can be lifted by standard techniques. □

If R has an identity it is not hard to show (using Lemma 2.3 of [2]) that condition (3) is equivalent to the condition that every cyclic R-module has a projective cover. The following consequences of Theorem 4.3 give, in the case when R has an identity, some new proofs of well-known facts.

COROLLARY 4.4. *Let R be a semiperfect ring. A subring of R is semiperfect if and only if it is an I_0-ring. In particular, one-sided ideals of R and subrings of the form aRb, $a, b \in R$, are semiperfect.*

PROOF. Simply observe that the maximum condition on idempotents is inherited by subrings and use Proposition 1.6. □

COROLLARY 4.5. *A ring R is semiperfect if and only if the ring $M_n(R)$ is semiperfect for all $n \geqslant 1$ (some $n \geqslant 1$).*

PROOF. If R is semiperfect then $M_n(R)$ is an I_0-ring by Proposition 1.8 and $M_n(R)/J[M_n(R)] \cong M_n[R/J(R)]$ is semisimple artinian. Conversely, R can be embedded in $M_n(R)$ as the subring of all matrices with zeros in all positions except (1, 1). Since this is the intersection of a left and a right ideal, Corollary 4.4 completes the proof. □

COROLLARY 4.6. *Every homomorphic image of a semiperfect ring is semiperfect.*

PROOF. If R is semiperfect and $A \subseteq R$ is an ideal, let $L \supseteq A$ be a left ideal. Choose an idempotent $e \in L$ and a left ideal $M \subseteq J(R)$ such that $L = Re + M$. Then, in $\bar{R} = R/A$, $\bar{L} = \bar{R}\bar{e} + \bar{M}$ and $\bar{M} \subseteq J(\bar{R})$. Hence R/A is semiperfect. □

We remark that if R is a semiperfect ring and A is an ideal of R then $J(R/A) = (A + J)/A$ where J denotes $J(R)$. Indeed, there is a ring epimorphism $R/A \longrightarrow R/(A + J)$ with kernel $(A + J)/A$. But $R/(A + J)$ is a homomorphic image of R/J so $J(R/A) \subseteq (A + J)/A$. The reverse inclusion always holds.

5. **Regular idempotents.** The notion of regularity has been extended to projective modules by Ware [9] and to a wider class of modules by Zelmanowitz [10]. In this section we shall study the connection between these modules and the I_0-rings. In particular we show that, over an I_0-ring, the endomorphism ring of a regular module is an I_0-ring with zero radical.

Let M be a left R-module and let $M^* = \mathrm{Hom}_R(M, R)$. Zelmanowitz calls *$M$ regular* if, for any $x \in M$, there exists $\alpha \in M^*$ such that $(x\alpha)x = x$. If M is an R-module, $x \in M$ and $\alpha \in M^*$, define a map $[\alpha, x] \in \mathrm{end}(M)$ by $y[\alpha, x] = (y\alpha)x$

for every $y \in M$. We say M is *projective* if it has a *dual basis*, that is if there exist subsets $\{x_\nu | \nu \in I\} \subseteq M$ and $\{\alpha_\nu | \nu \in I\} \subseteq M^*$ (indexed by the same set I) such that, for each $x \in M$, $x\alpha_\nu = 0$ for all but a finite number of $\nu \in I$ and $x = \Sigma_\nu x[\alpha_\nu, x_\nu]$. The following easily verified facts from [10] will be needed.

LEMMA 5.1. *Let M be a regular module, let $x \in M$, and suppose $\alpha \in M^*$ satisfies $(x\alpha)x = x$. Then*:

(1) *$e = x\alpha$ is an idempotent and $ex = x$.*

(2) *$\alpha_{|Rx}: Rx \longrightarrow Re$ is an isomorphism so Rx is projective.*

(3) *$M = Rx \oplus \ker[\alpha, x]$.*

This shows that, if M is regular and $x \in M$, then Rx is a projective summand of M. Zelmanowitz proves the converse [10, Theorem 2.2], and also shows that $Rx_1 + \cdots + Rx_n$ is a projective summand for any $x_i \in M$.

LEMMA 5.2. *If $e^2 = e \in R$, the following are equivalent*:

(1) *Re is a regular R-module.*

(2) *For each $x \in Re$ there exists $y \in R$ such that $xyx = x$.*

(3) *For each $x \in Re$ there exists $f^2 = f \in Re$ with $Rx = Rf$.*

The easy proof is left to the reader. An idempotent e in a ring R will be called *left regular* if Re is a regular R-module. Clearly, every idempotent in a regular ring is left regular. Also, eRe is a regular ring if e is a left regular idempotent (the converse is false as an example below will show).

The following is a strengthening of a remark in [10, p. 346].

LEMMA 5.3. *Let R be a ring and let M be a regular R-module. If $a \notin$ ann M then aR contains a nonzero left regular idempotent. In particular $J(R) \subseteq$ ann M.*

PROOF. If $ax \neq 0$, $x \in M$, choose $\alpha \in M^*$ such that $[(ax)\alpha]ax = ax$. Then $(ax)\alpha \in aR$ is a nonzero left regular idempotent by Lemma 5.1 and the fact that submodules of a regular module are regular. □

Prompted by this, we say that a ring R *has left regular idempotents* if each *right* ideal $T \nsubseteq J(R)$ contains a nonzero left regular idempotent.

THEOREM 5.4. *The following are equivalent for a ring R*:

(1) *$J(R)$ is an intersection of annihilators of regular left modules.*

(2) *$J(R) =$ ann M for some regular left module M.*

(3) *R has left regular idempotents.*

Moreover, when this is the case, $J(R)$ is the intersection of the annihilators of the cyclic regular left ideals Re, $e^2 = e \in R$.

PROOF. (1) ⇒ (2) since a direct sum of regular modules is again regular [10, Theorem 2.8]. (2) ⇒ (3) by Lemma 5.3. Hence assume (3). If $a \notin J(R)$ choose $0 \neq e^2 = e \in aR$ such that Re is regular. Then $a \notin \text{ann}(Re)$ so $J(R) \supseteq \bigcap\{\text{ann}(Re) | e^2 = e \text{ left regular}\}$. This must be equality by Lemma 5.3 so the proof is complete. □

COROLLARY 5.5. *If M is a regular left R-module and $A = \text{ann}\, M$, then R/A has left regular idempotents and $J(R/A) = 0$.*

PROPOSITION 5.6. *Let R be a ring, let $A \subseteq J(R)$ be an ideal, let $T \subseteq R$ be a right ideal and let $e^2 = e \in R$. If R has left regular idempotents so do the rings R/A, T and eRe.*

PROOF. Let M be a regular R module with $\text{ann}\, M = J(R)$. Then M is a regular R/A-module and $\text{ann}_{R/A}(M) = J(R/A)$.

Turning to eRe, eM is an eRe-module and $\text{ann}_{eRe}(eM) = J(R) \cap eRe = J(eRe)$. We show that eM is regular. If $x \in eM$ let $\alpha: M \longrightarrow R$ satisfy $(x\alpha)x = x$. Define $\bar{\alpha}: eM \longrightarrow eRe$ by $y\bar{\alpha} = (y\alpha)e$. Then $\bar{\alpha} \in (eM)^*$ and $(x\bar{\alpha})x = (x\alpha)ex = x$.

Finally, TM is a T-module and $\text{ann}_T(TM) = \{t \in T | tTM = 0\} = \{t \in T | tT \subseteq J(R)\} = J(T)$. Now let $x \in TM$ and suppose $\alpha: M \longrightarrow R$ satisfies $(x\alpha)x = x$. Since $(TM)\alpha \subseteq T(M\alpha) \subseteq T$, the map $\alpha_{|TM}: TM \longrightarrow T$ is a T-homomorphism and $(x\alpha_{|TM})x = x$. This shows TM is a regular T-module. □

PROPOSITION 5.7. *If a ring R has left regular idempotents the same is true of the matrix ring $M_n(R)$.*

PROOF. Let $T \not\subseteq J[M_n(R)]$ be a right ideal. As in the proof of Proposition 1.8, there exists an idempotent X in T of the form $X = \Sigma_{i=1}^n E_{ip}(r_i e)$. It is easy to show that each element in $M_n(R)X$ has the form $Y = \Sigma_{i=1}^n E_{ip}(t_i)$ where each $t_i \in Re$. By Lemma 5.2, we must show each such Y is regular in $M_n(R)$. By Lemma 5.2, let $t_1 s_1 t_1 = t_1$. Then $YE_{p1}(s_1)Y - Y = \Sigma_{i=2}^n E_{ip}(t_i s_1 t_1 - t_i)$ and it suffices to show this is regular in $M_n(R)$. In other words, we may assume Y has the form $Y = \Sigma_{i=2}^n E_{ip}(t_i)$, $t_i \in Re$. This procedure can be continued to complete the proof. □

EXAMPLES AND REMARKS. 1. Every regular ring has left (and right) regular idempotents.

2. Every ring R with $J(R) = 0$ and which has primitive idempotents has left (and right) regular idempotents.

3. If a prime ring R has a nonzero regular left module then it has left regular idempotents and $J(R) = 0$.

4. Any local ring R with $J(R) \neq 0$ is an example of an I-ring which does not have regular left (or right) idempotents.

5. The ring R in Example 1.9 is primitive with nonzero socle and so has left (and right) regular idempotents. Hence, if R has left regular idempotents, the same need not be true of the center of R or of a homomorphic image of R.

6. Let Δ be a division ring and let $R = \left[\begin{smallmatrix}\Delta & \Delta\\ 0 & 0\end{smallmatrix}\right] \subseteq M_2(\Delta)$. If $e = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ then Re is a regular module and $\operatorname{ann}(Re) = J(R)$. On the other hand $fR = R$ for any $0 \neq f^2 = f \in R$ so R has *no* regular right modules. In particular, Re can be regular while eR is not even when eRe is a division ring.

We conclude this section with some remarks about the endomorphism ring $E(M)$ of a regular module M. We shall write $A(M) = \{\alpha \in E(M) | \ker \alpha \text{ is large in } M\}$. It is well known that $A(M)$ is an ideal of $E(M)$.

THEOREM 5.8. *If M is a regular R-module, each one-sided ideal of $E(M)$ which is not contained in $A(M)$ contains a nonzero idempotent.*

PROOF. If $\alpha \in E(M) - A(M)$, choose $0 \neq x \in M$ such that $Rx \cap \ker \alpha = 0$. Then $\alpha_{|Rx}: Rx \longrightarrow Rx\alpha$ is an isomorphism. We have $M = Rx\alpha \oplus K$ by Lemma 5.1 so define $\beta: M \longrightarrow M$ by setting $K\beta = 0$ and $\beta_{|Rx\alpha} = (\alpha_{|Rx})^{-1}$. Then $\beta\alpha\beta = \beta \neq 0$ and the result follows. $\square$

COROLLARY 5.9. *Let M be a regular module such that every monomorphism in $E(M)$ has a right inverse. Then $E(M)$ is an I_0-ring and $J[E(M)] = A(M)$.*

PROOF. Since $\ker \alpha \cap \ker(1 - \alpha) = 0$ for every $\alpha \in E(M)$, our hypothesis implies $A(M) \subseteq J[E(M)]$. This must be equality since $J[E(M)]$ contains no nonzero idempotent. $\square$

COROLLARY 5.10. *Let M be a regular R-module with the property that $M\gamma \subseteq J(R)$ for every $\gamma \in M^*$ with large kernel. Then $E(M)$ is an I_0-ring with $J[E(M)] = 0$.*

PROOF. Suppose $\alpha \in A(M)$ and let $y \in M\alpha$. There exists $\gamma \in M^*$ such that $(y\gamma)y = y$. But $\ker \alpha \subseteq \ker(\alpha\gamma)$ and consequently $M\alpha\gamma \subseteq J(R)$. Thus $y\gamma$ is an idempotent in $J(R)$ and so $y\gamma = 0$. It follows that $M\alpha = 0$ and hence that $A(M) = 0$. $\square$

One situation where the condition in Corollary 5.10 is met is when R is an I_0-ring.

COROLLARY 5.11. *If R is an I_0-ring and M is a regular module then $E(M)$ is an I_0-ring with $J[E(M)] = 0$.*

PROOF. Suppose $\gamma \in M^*$ has large kernel and $M\gamma \not\subseteq J(R)$. Let $e = y\gamma$ be a nonzero idempotent and put $x = ey$. Then $x\gamma = e \neq 0$ and $x = ex$ and it follows that $Rx \cap \ker \gamma = 0$. This is a contradiction since $Rx \neq 0$ and so the result follows from Corollary 5.10. $\square$

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