

SIMILARITY OF QUADRATIC FORMS AND ISOMORPHISM OF THEIR FUNCTION FIELDS

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ABSTRACT. This paper considers the question: Given anisotropic quadratic forms Q and Q' over a field K ($\text{char } K \neq 2$), if their function fields are isomorphic must Q and Q' be similar? It is proved that the answer is yes if Q is a Pfister form or the pure part of a Pfister form, or a 4-dimensional form. The argument for Pfister forms and their pure parts does not generalize because these are the only anisotropic forms which attain maximal Witt index over their function fields. To handle the 4-dimensional case the following theorem is proved: *If Q and Q' are two 4-dimensional forms over K with the same determinant d , then Q and Q' are similar over K iff they are similar over $K[\sqrt{d}]$.* The example of Pfister neighbors suggests that quadratic forms arguments are unlikely to settle the original question for other kinds of forms.

Let Q be a nonsingular quadratic form defined on an n -dimensional vector space V ($n \geq 3$) over a field K ($\text{char } K \neq 2$), with diagonal representation $(a_0, a_1, \dots, a_{n-1})$. The function field of Q , K_Q , is the quotient field of $K[x_1, \dots, x_{n-1}] / (a_0 + a_1x_1^2 + \dots + a_{n-1}x_{n-1}^2)$, i.e., $K(x_1, \dots, x_{n-2})[\sqrt{-\alpha}]$, where $\alpha = a_{n-1}^{-1}(a_0 + a_1x_1^2 + \dots + a_{n-2}x_{n-2}^2)$. The field K_Q is uniquely determined (up to K -isomorphism) by Q , independent of the choice of diagonal representation. Further, if Q' is another quadratic form which is isometric to Q , or even similar to Q (i.e., with a diagonal representation $(aa_0, aa_1, \dots, aa_{n-1})$ for some $a \in K^*$), then $K_{Q'} \cong K_Q$.

We consider here the converse of the last sentence:

(†) If Q and Q' are two anisotropic forms over K , such that $K_{Q'} \cong K_Q$, must Q and Q' be similar?

We must confine attention exclusively to anisotropic forms when $\dim Q \geq 4$, since K_Q is purely transcendental over K iff Q is isotropic.

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It was proved long ago by Witt [7], using the theory of algebraic function fields in one variable, that the answer to (†) is yes if the forms are three-dimensional. In §1 we will give an affirmative answer if Q is a Pfister form or the pure part of a Pfister form (in particular, reproving Witt's result). In §2 we will show that the answer is again yes if $\dim Q = 4$. We will indicate why the arguments used here do not generalize to other cases. Indeed, the example of Pfister neighbors suggests that quadratic-form-theoretic arguments will not settle (†) except in the cases analyzed here.(2)

We begin with a few remarks on notation and terminology, and a standard lemma. Throughout the discussion the field K will be fixed, with $\text{char } K \neq 2$. Field isomorphisms will be K -isomorphisms. Each quadratic form Q will be nonsingular and (unless indicated otherwise) will be a K -form, i.e., a quadratic form defined on some finite-dimensional K -vector space V . As usual, $\dim Q$ is taken to be the dimension of V . The *discriminant* of Q , $\text{disc } Q = \bar{d}$, is the image in $K^*/(K^*)^2$ of the determinant d of some matrix representing Q . (No sign is attached.) For Q' another K -form, $Q' \cong Q$ means Q and Q' are isometric. Q' is a *subform* of Q if there is another K -form Q'' such that $Q \cong Q' \perp Q''$. $Q \sim a$ means Q represents a . For $a_1, a_2, \dots \in K^*$, (a_1, \dots, a_n) is an n -dimensional K -form which is represented by a diagonal matrix with entries a_1, \dots, a_n . If $Q \cong (a_1, \dots, a_n)$, then $aQ \cong (aa_1, \dots, aa_n)$. Q is *nearly hyperbolic* if it is the orthogonal sum of a hyperbolic form and a 1-dimensional form. If L is a field containing K , Q_L denotes the L -form defined on $V \otimes_K L$ induced by Q on V . For standard quadratic form results quoted without reference, the reader is referred to [4] or [2].

A *Pfister form* P is a K -form representable as a product $P \cong \bigotimes_{j=1}^r (1, b_j)$, for some $b_1, \dots, b_r \in K^*$. Recall that for any $a \in K^*$, if $P \sim a$, then $P \cong aP$ and if P is isotropic, then P is hyperbolic. Of course, P_L is again a Pfister form for any field $L \supseteq K$. Since $P \sim 1$ we have the decomposition $P \cong (1) \perp Q$. Q is called the *pure part* of the Pfister form P . Observe that Q_L is the pure part of P_L , and if Q_L is isotropic, then it is nearly hyperbolic. A *Pfister neighbor* (following Knebusch's definition) is a quadratic form R which is similar to a subform of a Pfister form P , with $\dim R > \frac{1}{2} \dim P$. For details on Pfister forms, see Pfister's original paper [6], or the excellent accounts in [2] and [3].

We state for the reader's convenience a well-known lemma which pro-

(2) All of the results given here for the function field of a form apply equally well to its homogeneous function field, which is a simple purely transcendental extension of the function field.

vides the opening wedge for our consideration of function fields. See, for example, [2, p. 200] for a proof.

Lemma 1. *Take $d \in K^*$ and let $L = K[\sqrt{d}]$. Any anisotropic K -form Q can be decomposed $Q \cong (1, -d) \otimes Q_0 \perp Q_1$, where Q_{1L} is anisotropic. (In particular, if Q_L is hyperbolic, the Q_1 term disappears.)*

1. Function fields of Pfister forms. The following notation will be used throughout this section. Q will be a K -form of dimension $n \geq 3$, $m = n - 2$. $F = K(x_1, \dots, x_m)$ and $M = F(x_{m+1})$, where the x_i 's are independent indeterminates. Take a diagonal representation (a_0, \dots, a_m, a) of Q , and let $\alpha = a^{-1}(a_0 + a_1x_1^2 + \dots + a_mx_m^2)$ and $\beta = \alpha + x_{m+1}^2$. So $K_Q \cong F[\sqrt{-\alpha}]$.

Our first theorem collects the information needed to consider isomorphism of function fields of Pfister forms and their pure parts.

Theorem 2. *Suppose Q' is an anisotropic quadratic form over K . Assume further (with Q, α, β , as given above):*

(a) Q' becomes hyperbolic in K_Q .

Then,

(b) there is an F -form Q_0 , such that $Q'_F \cong (1, \alpha) \otimes Q_0$;

(c) $Q'_M \cong \beta Q'_M$;

(d) Q is similar to a subform of Q' . (Hence $\dim Q' \geq \dim Q$.)

Proof. We show (a) \implies (b) \implies (c) \implies (d). Recall that the Witt index is preserved under purely transcendental field extensions. Thus, Q'_F is anisotropic. To obtain (b) from (a), apply Lemma 1 over F with $d = -\alpha$. Since $(1, \alpha)_M \sim \beta$, we have $(1, \alpha)_M \cong \beta(1, \alpha)_M$. (c) now follows at once. In particular, taking any $c \in K^*$ represented by Q' , $Q'_M \sim c\beta$. Applying the Cassels-Pfister subform theorem [6] (inhomogeneous form, but the same proof holds), it follows that $(ca^{-1}a_0, \dots, ca^{-1}a_m, c)$ is a subform of Q' . But this subform is just $ca^{-1}Q$, proving (d). Q.E.D.

Remark. (c) \implies (a) holds as well. ((c) \implies (b) can be deduced by the Cassels-Pfister subform theorem.) Indeed, the equivalence of (a) and (c) is a special case of a beautiful theorem of Knebusch [1].

Theorem 3. *If Q' is an anisotropic Pfister form ($\dim Q' \geq 4$) and $K_Q \cong K_{Q'}$, then Q and Q' are similar.*

Proof. Being a Pfister form Q' becomes not merely isotropic, but actually hyperbolic in $K_{Q'}$. Since $K_Q \cong K_{Q'}$, Theorem 2 implies Q is similar to a subform of Q' . But $\dim Q = \dim Q'$ because K_Q and $K_{Q'}$ have the same transcendence degree over K . So the subform must be all of Q' , completing the proof.

Theorem 4. *If Q' is the pure part of an anisotropic Pfister form ($\dim Q' \geq 3$) and $K_Q \cong K_{Q'}$, then Q and Q' are similar.*

Proof. Say $P \cong (1) \perp Q'$, a Pfister form. Over K_Q , Q' becomes isotropic, so P becomes hyperbolic. By Theorem 2, Q is similar to a subform of P . But, by the transcendence degree argument just used, $\dim Q = \dim Q'$. Thus, $P \cong cQ \perp (d)$ for some $c, d \in K^*$. Since $P \sim d$, $P \cong dP \cong cdQ \perp (1)$. By the Witt cancellation theorem, $Q' \cong cdQ$. Q.E.D.

Note that the same kind of proof shows that if Q' is a Pfister neighbor of the anisotropic Pfister form P and $K_Q \cong K_{Q'}$, then Q is also a Pfister neighbor of P and $\dim Q = \dim Q'$. (But see the comments at the end of §2.)

Since there is only one similarity class of isotropic 3-dimensional forms, the condition that Q' be anisotropic can be deleted when $\dim Q' = 3$. Thus, we have reproved Witt's result which was the starting point of this investigation.

One might hope to generalize Theorems 3 and 4 to anisotropic forms which become hyperbolic or nearly hyperbolic over their function fields. The next two theorems show that there is no further generalization.⁽³⁾

Theorem 5. *If Q becomes hyperbolic over its function field ($\dim Q \geq 4$), then Q is already hyperbolic over K , or Q is similar to an anisotropic Pfister form.*

Proof. If Q is isotropic, then K_Q is purely transcendental over K . So Q hyperbolic over K_Q forces Q hyperbolic over K .

Now assume Q is anisotropic. Applying a suitable similarity factor, we may assume $Q \sim 1$. In the notation established above (with $a = 1$), we take $Q' = Q$, and Theorem 2 implies $Q_M \cong \beta Q_M$. That is, Q is strongly multiplicative in the sense of Pfister [6] (inhomogeneous form). Being anisotropic, Q must be a Pfister form. Q.E.D.

Theorem 6. *If Q becomes nearly hyperbolic over its function field ($\dim Q \geq 3$), then Q is already nearly hyperbolic over K , or Q is similar to the pure part of an anisotropic Pfister form.*

Proof. The isotropic case is handled just as in the preceding proof. So assume Q is anisotropic. We simplify the notation by applying a similarity to obtain $Q \sim 1$ (which permits us to take $a = 1$ in the expressions for α and β). Let $\bar{d} = \text{disc } Q$.

Now, Q_F is anisotropic, but Q becomes nearly hyperbolic over $F[\sqrt{-\bar{d}}] = K_Q$. By Lemma 1, $Q_F \cong (1, \alpha) \otimes Q_0 \perp (\delta)$, for some F -form Q_0 .

⁽³⁾Theorems 5 and 6 have been obtained independently by Knebusch.

(δ) can be determined by a comparison of discriminants. There are two possibilities, the first of which will be ruled out.

Case 1. $\dim Q \equiv 1 \pmod{4}$. Then $\dim Q_0$ is even, so that $\text{disc}((1, \alpha) \otimes Q_0) = \bar{1}$. Hence, we can take $\delta = d$, which shows that $Q_F \sim d$. However, as F is purely transcendental over K , Cassel's theorem [3, p. 18] implies $Q \sim d$. This gives a decomposition of Q over K : $Q \cong Q_1 \perp (d)$. Cancelling $(d)_F$ from the two decompositions of Q_F yields $Q_{1F} \cong (1, \alpha) \otimes Q_0$. By Theorem 2 ((b) \Rightarrow (d)), Q must be similar to a subform of Q_1 , which is absurd, as $\dim Q_1 = \dim Q - 1$. Thus, Case 1 can never occur.

Case 2. $\dim Q \equiv 3 \pmod{4}$. This time $\dim Q_0$ is odd, so that $\text{disc}((1, \alpha) \otimes Q_0) = \bar{\alpha}$, and $(\delta) \cong (\alpha d)$. Let P be the K -form $Q \perp (d)$, i.e., $P \cong (a_0, \dots, a_m, 1, d)$. To complete the proof it suffices to show that P is an anisotropic Pfister form. (Then $P \cong dP \cong (1) \perp dQ$. So dQ is the pure part of P .)

The decomposition of Q_F yields $P_F \cong (1, \alpha) \otimes (Q_0 \perp (d))$. So, by Theorem 2 ((b) \Rightarrow (c)), $P_M \cong \beta P_M'$. Hence, $P_M' \cong \beta P_M'$, where P' is the anisotropic part of P . By Theorem 2 ((c) \Rightarrow (d)), Q is similar to a subform of P' . Since $\dim Q = \dim P - 1$, P must be anisotropic.

Now take new independent indeterminates y_0, \dots, y_n and set $K' = K(y_0, \dots, y_n)$ and $M' = M(y_0, \dots, y_n) = K'(x_1, \dots, x_{m+1})$. Of course, $P_{M'} \cong \beta P_{M'}$. Take any $c \in K'$, such that $P_{K'} \sim c$. Then $P_{M'} \sim c\beta$, so that, by the Cassels-Pfister subform theorem (treating $c\beta$ as a polynomial in the x_i 's over K'), $(ca_0, ca_1, \dots, ca_m, c)$ is a subform of $P_{K'}$. A comparison of discriminants shows that $P_{K'} \cong (ca_0, ca_1, \dots, c, cd) \cong cP_{K'}$. In particular, we may take $c = a_0y_0^2 + a_1y_1^2 + \dots + a_my_m^2 + y_{m+1}^2 + dy_n^2$, showing that P is strongly multiplicative, hence a Pfister form. Q.E.D.

2. 4-dimensional forms. We now proceed to give an affirmative answer to question (\dagger) for 4-dimensional forms. This is done by using the Pfister form results over a quadratic extension L of K , then working back to K . The key step is the transition from L to K , which is provided by the following theorem.

Theorem 7. *Let Q and Q' be two 4-dimensional K -forms, each representing 1 and each with discriminant \bar{d} . Let $L = K[\sqrt{d}]$. Then Q and Q' are similar iff $Q_L \cong Q'_L$.*

Proof. Necessity. Q_L and Q'_L are Pfister forms, and hence isometric whenever they are similar.

Sufficiency. Suppose $Q_L \cong Q'_L$. Assume $d \notin (K^*)^2$ —otherwise $L = K$ and there is nothing to prove. If Q is isotropic, then so are Q_L and Q'_L ,

hence so is Q' (a well-known fact which can be deduced from Lemma 1 by a discriminant argument). But then $\text{disc } Q = \text{disc } Q'$ implies that Q and Q' are similar. Thus, we may assume Q and Q' are anisotropic.

Take decompositions $Q \cong (1) \perp R$ and $Q' \cong (1) \perp R'$, and let $S = R \perp (-R')$. Since $Q_L \cong Q'_L$, S_L must be hyperbolic. If S were anisotropic, Lemma 1 would force $\text{disc } S = -\bar{d}$. But $\text{disc } S = \text{disc } R \cdot (-\text{disc } R') = -\bar{1} \neq -\bar{d}$. Therefore, S must be isotropic. Since R and R' are anisotropic it follows that there is an $a \in K^*$, such that $R \sim a$ and $R' \sim a$. Hence, there exist diagonal representations $Q \cong (1, a, b, bad)$ and $Q' \cong (1, a, b', b'ad)$, for some $b, b' \in K^*$.

We will demonstrate the similarity of Q and Q' "piecewise", by finding a $t \in K^*$ with $(1, a) \cong t(1, a)$ and $(b', b'ad) \cong t(b, bad)$. To this end, consider the form $T = (1, a, -bb', -bb'ad)$, whose discriminant is \bar{d} . Working over L , where d is a square, we have

$$bT_L \cong (b, ba, -b', -b'ad)_L \cong (b, bad, -b', -bad)_L.$$

Thus, bT_L is a subform of the hyperbolic form $Q_L \perp -Q'_L$, with complement $(1, a, -1, -a)_L$ which is also hyperbolic. Therefore, T_L is hyperbolic. Using again the well-known fact quoted above, T itself is isotropic. From a nontrivial representation of 0 by T , we obtain $t \in K^*$, such that $(1, a) \sim t$ and $(bb', bb'ad) \sim t$. The two piecewise similarities now follow at once. Thus, $Q' \sim tQ$. Q.E.D.

This theorem may be of some interest in its own right. One immediate consequence is that a Hasse principle for similarity of 4-dimensional forms over a global field may be deduced from the Hasse principle for isometry.⁽⁴⁾ (The global square theorem [4, 65:15] shows that equality of discriminants can be verified locally.)

For the proof of our final theorem, it is convenient to recast Theorem 7 as follows: *If Q and Q' are 4-dimensional forms each with discriminant \bar{d} and $L = K[\sqrt{d}]$, then Q and Q' are similar iff Q_L and Q'_L are similar.*

Theorem 8. *Let Q and Q' be anisotropic 4-dimensional forms with $K_Q \cong K_{Q'}$. Then Q and Q' are similar.*

Proof. Let $\bar{d} = \text{disc } Q$ and $\bar{d}' = \text{disc } Q'$. If $\bar{d} = \bar{1}$, then Q is similar to an anisotropic Pfister form (with the same function field), and Theorem 3 applies. Therefore, we may assume that $d, d' \notin (K^*)^2$.

Let $L = K[\sqrt{d}]$. Note that because K_Q is a regular extension of K , the K -isomorphism of K_Q and $K_{Q'}$, induces an L -isomorphism of L_{Q_L} of $L_{Q'_L}$.

⁽⁴⁾ Ono [5] has shown that the Hasse principle for similarity holds for forms of any dimension over a global field.

By Theorem 3 (Q_L being similar to a Pfister form), Q_L and Q'_L are similar. In particular, $\text{disc } Q_L = \text{disc } Q'_L$, i.e., $d' \in (L^*)^2$. Since $d' \in K^* - (K^*)^2$, an easy computation (or an application of Lemma 1 to $(1, -d')$) shows that $\overline{d'} = \overline{d}$. Thus, we may apply Theorem 7 (as recast) to conclude that Q and Q' are similar. Q.E.D.

The techniques used for Pfister forms and 4-dimensional forms do not generalize readily to other kinds of forms. To illustrate the problems that can be encountered, it suffices to consider the example of Pfister neighbors. Let Q and Q' be two *dissimilar* forms of the same dimension, which are each a Pfister neighbor of the same anisotropic Pfister form P . (So $\dim P = 2^r$, for $r \geq 3$, and $2^{r-1} < \dim Q < 2^r - 1$.) Then in any extension field F of K , Q_F is isotropic iff P_F is hyperbolic iff Q'_F is isotropic. In particular, Q is isotropic over $K_{Q'}$, and vice versa. (In fact, it can be shown that for any K -form R , R is isotropic (resp. hyperbolic) over K_Q iff R is isotropic (resp. hyperbolic) over $K_{Q'}$.) Further, for every algebraic extension L of K , L_{Q_L} is purely transcendental over L iff $L_{Q'_L}$ is purely transcendental over L . Whether K_Q and $K_{Q'}$ can be isomorphic is unclear. It appears that new techniques will be necessary to settle this question.

Added in proof. M. Knebusch has shown that dissimilar Pfister neighbors often have isomorphic function fields. Specifically, if Q and Q' are neighbors of the Pfister form P , with $\dim P = 2^r$, and if Q and Q' each have subforms similar to the Pfister form R , with $\dim R = 2^{r-1}$, and $\dim Q = \dim Q'$, then $K_Q \cong K_{Q'}$. This is one of many interesting results in his excellent paper *Generic splitting of quadratic forms. I*, to appear in Proc. London Math. Soc.

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