TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 208, 1975

CONICAL VECTORS IN INDUCED MODULES

BY

J. LEPOWSKY[1]

ABSTRACT. Let $\mathfrak{g}$ be a real semisimple Lie algebra with Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. A conical vector in a $\mathfrak{g}$-module is defined to be a nonzero $\mathfrak{m} \oplus \mathfrak{n}$-invariant vector. The $\mathfrak{g}$-modules which are algebraically induced from one-dimensional $(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$-modules on which the action of $\mathfrak{m}$ is trivial have "canonical generators" which are conical vectors. In this paper, all the conical vectors in these $\mathfrak{g}$-modules are found, in the special case $\dim \mathfrak{a} = 1$. The conical vectors have interesting expressions as polynomials in two variables which factor into linear or quadratic factors. Because it is too difficult to determine the conical vectors by direct computation, metamathematical "transfer principles" are proved, to transfer theorems about conical vectors from one Lie algebra to another; this reduces the problem to a special case which can be solved. The whole study is carried out for semisimple symmetric Lie algebras with splitting Cartan subspaces, over arbitrary fields of characteristic zero. An exposition of the Kostant-Mostow double transitivity theorem is included.

1. Introduction. The theory of Verma modules, as developed by D.-N. Verma [10(a), (b)] and by I. N. Bernšteĭn, I. M. Gel′fand and S. I. Gel′fand [1(a), (b)], is becoming increasingly important. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$. The associated *Verma modules* are the $\mathfrak{g}$-modules induced, in the algebraic sense, by the one-dimensional $\mathfrak{b}$-modules (see [2, Chapter 7]). As we shall see in this introduction, a corresponding theory of $\mathfrak{g}$-modules induced from more general parabolic subalgebras of $\mathfrak{g}$ should also be developed, and the purpose of this paper is to begin such a study.

Here is our main reason for interest in this problem: Let $G = KAN$ be an Iwasawa decomposition of a real semisimple Lie group with finite center, and

Received by the editors March 14, 1974 and, in revised form, July 15, 1974.
AMS (MOS) subject classifications (1970). Primary 17B10; Secondary 16A64, 17B20, 17B35, 22E45.
Key words and phrases. Conical vectors, highest weight vectors, induced modules, Verma modules, real semisimple Lie algebras, real rank one, semisimple symmetric Lie algebras, splitting Cartan subspaces, restricted roots, restricted weight vectors, restricted Weyl group, universal enveloping algebra, double transitivity theorem, polynomial invariants.
(1) Partially supported by NSF GP-33893.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ the corresponding decomposition of the complexified Lie algebra of G. Let M be the centralizer of A in K, and $\mathfrak{m}$ its complexified Lie algebra. The infinitesimal nonunitary principal series of G is the family of $\mathfrak{g}$-modules obtained by taking the K-finite subspaces of the nonunitary principal series representations—those Hilbert space representations of G induced from the finite-dimensional irreducible representations of MAN (see for example [7(a)]). This family of $\mathfrak{g}$-modules is of great importance because every irreducible $\mathfrak{g}$-module which splits into a direct sum of finite-dimensional irreducible $\mathfrak{k}$-modules exponentiating to K-modules is a subquotient of an infinitesimal nonunitary principal series module (see [4], [7(a)], [9] and [2, Chapter 9]). But roughly speaking, the infinitesimal nonunitary principal series modules may be identified with certain "large" subspaces of the contragredient $\mathfrak{g}$-modules to $\mathfrak{g}$-modules *algebraically* induced by finite-dimensional irreducible modules of the parabolic subalgebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$ (cf. [2, §§9.3.1, 9.7.10]). Other important families of induced representations of G are similarly related to $\mathfrak{g}$-modules algebraically induced from parabolic subalgebras of $\mathfrak{g}$.

In a sense, the algebraically induced modules may be thought of as modules of distributions supported at the identity element of G, and their duals—algebraically "produced" modules—as modules of formal power series at the identity element of G. The K-finite elements of the produced modules (the K-finite formal power series) then correspond to analytic functions on G which are also the K-finite elements of the Hilbert space induced representations.

The Verma modules that can be embedded in a given Verma module are completely known ([10] and [1(a)]; see also [2, Théorème 7.6.23]). Suppose one could correspondingly determine the $\mathfrak{g}$-module maps between pairs of $\mathfrak{g}$-modules algebraically induced from $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Looking at the dual maps between the K-finite subspaces of the contragredient modules, one would have intertwining operators between nonunitary principal series G-modules, and these intertwining operators, which might be Kunze-Stein integral operators, would now be given by *differential* formulas. Furthermore, since an algebraically induced module is generated by a "highest weight vector" ($\mathfrak{n}$-invariant vector), the $\mathfrak{g}$-maps from one of the algebraically induced modules to another are closely related to the highest weight vectors in the target module. These give rise to highest weight vectors in the dual of the K-finite subspace of the Hilbert space induced G-module, and therefore are intimately connected with S. Helgason's conical distributions [5(a), (b)].(2) The submodule structure of

(2) See also M. Hu's thesis [12], whose results on conical distributions are related to our results on conical vectors.

the algebraically induced $\mathfrak{g}$-modules must also shed light on the subquotient structure of the nonunitary principal series modules (see M. Duflo [3] and [2, §9.6] for the case of complex G, using Verma modules), but examples show that the relation will be subtle. For instance, irreducibility of the algebraically induced module is not equivalent to irreducibility of the related contragredient nonunitary principal series module. On the other hand, the subquotient structure of the nonunitary principal series is notoriously complicated, but the structure of the algebraically induced modules already appears to be more regular and perhaps more fundamental. For example, the inclusion relations among the Verma submodules of certain Verma modules recover the inclusion relations among the closures of the Bruhat cells for complex semisimple Lie groups (see [10]), and it is likely that this situation will generalize to real semisimple Lie groups, using the modules algebraically induced from $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Now that we want to find the highest weight vectors in a given $\mathfrak{g}$-module X algebraically induced from a finite-dimensional irreducible $(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$-module, how do we do it? The following seemed at first like a good starting point: Let $\mathfrak{l}$ be a Cartan subalgebra of $\mathfrak{m}$, so that $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ and $\mathfrak{n}$. Then it is easy to see that X is a $\mathfrak{g}$-module quotient of a certain Verma module V induced from $\mathfrak{b}$ (cf. [2, Lemma 9.3.2]). Hence one can try to use the well-developed theory of highest weight vectors in Verma modules to study highest weight vectors in X. Unfortunately, however, highest weight vectors in V can vanish when one passes to the quotient X, even in simple examples. Moreover, it turns out that there are, in general, highest weight vectors in X which do not come from highest weight vectors in V. This subtlety, which made the problem much more difficult than we expected it to be, forced us to work in a relatively special case and to develop new tools to handle even this case.

Now we shall describe our main results, and then we shall say what is interesting about our methods.

By analogy with Helgason's conical distributions, we call a nonzero vector in a $\mathfrak{g}$-module (or more generally, in an $\mathfrak{m} \oplus \mathfrak{n}$-module) *conical* if it is $\mathfrak{m} \oplus \mathfrak{n}$-invariant. The space of conical vectors, together with 0, is called the *conical space* of the module. Let $\mathcal{G}$ be the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{P} \subset \mathcal{G}$ the universal enveloping algebra of $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Define $\rho \in \mathfrak{a}^*$ (* denotes dual) by the condition $\rho(a) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} a|\mathfrak{n})$ for all $a \in \mathfrak{a}$, so that ρ is half the sum of the positive restricted roots with multiplicities counted. For all $\nu \in \mathfrak{a}^*$, the linear functional on $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ which is zero on $\mathfrak{m} \oplus \mathfrak{n}$ and

$\nu - \rho$ on $\mathfrak{a}$ defines a one-dimensional representation of $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Regarding $\mathbf{C}$ as the associated one-dimensional $\mathcal{P}$-module, and $\mathcal{G}$ as a right $\mathcal{P}$-module by right multiplication, we can form the $\mathcal{G}$-module $X^\nu = \mathcal{G} \otimes_{\mathcal{P}} \mathbf{C}$. This is a "twisted induced module" in the sense of [2, §5.2]. The vector $x_0 = 1 \otimes 1 \in X^\nu$ is a conical vector which generates X^ν, and is called the *canonical generator* of X^ν. Let $\mathfrak{n}^- \subset \mathfrak{g}$ be the sum of the negative restricted root spaces of $\mathfrak{g}$ with respect to $\mathfrak{a}$, and $\mathfrak{N}^- \subset \mathcal{G}$ its universal enveloping algebra. Then $X^\nu = \mathfrak{N}^- \cdot x_0$.

We are aiming for a description of the conical vectors in X^ν in case G has real rank 1, i.e., $\dim \mathfrak{a} = 1$. Assume this, and let $\alpha \in \mathfrak{a}^*$ be the unique simple restricted root. Then $\mathfrak{n}^-$ is the direct sum of the restricted root spaces $\mathfrak{g}^{-\alpha}$ and $\mathfrak{g}^{-2\alpha}$; here $\mathfrak{g}^{-2\alpha}$ may be zero. There are natural M-invariant nonsingular symmetric bilinear forms on $\mathfrak{g}^{-\alpha}$ and $\mathfrak{g}^{-2\alpha}$. Let $q_{-\alpha} \in \mathfrak{N}^-$ and $q_{-2\alpha} \in \mathfrak{N}^-$ be the sums of the squares of orthonormal bases of $\mathfrak{g}^{-\alpha}$ and $\mathfrak{g}^{-2\alpha}$, respectively, so that $q_{-\alpha}$ and $q_{-2\alpha}$ are quadratic M-invariant elements of $\mathfrak{N}^-$, and $q_{-2\alpha} = 0$ if $\mathfrak{g}^{-2\alpha} = 0$. Let $(\mathfrak{N}^-)^M$ be the algebra of all M-invariants in $\mathfrak{N}^-$. Then $(\mathfrak{N}^-)^M$ is a polynomial algebra on either one or two generators, depending on whether $\mathfrak{g}^{-2\alpha} = 0$ or $\mathfrak{g}^{-2\alpha} \neq 0$ (see §5), and in the difficult case when $\dim \mathfrak{g}^{-2\alpha} > 1$, the two generators are $q_{-\alpha}$ and $q_{-2\alpha}$; this follows from the Kostant-Mostow double transitivity theorem (see §4) on M-orbits in $\mathfrak{n}^-$ (or more precisely, M-orbits in the intersection of $\mathfrak{n}^-$ with the real Lie algebra of G). With this as background, we now state our main results (see §10):

Theorem 1.1. *Assume* $\dim \mathfrak{a} = 1$ *and let* $\nu \in \mathfrak{a}^*$. *Then the conical space of* X^ν *is either one- or two-dimensional, according to whether* ν *is a positive integral multiple of* α *(of* $\frac{1}{2}\alpha$ *if* $\dim \mathfrak{g}^\alpha = 1$*) or not. If* ν *is not of this form, then the conical space of* X^ν *is spanned by the canonical generator* x_0 *of* X^ν. *Suppose* $\nu = l\alpha$, l *a positive integer. (If* $\dim \mathfrak{g}^\alpha = 1$, *take instead* $\nu = \frac{1}{2} l\alpha$.*) Then* $q_{-\alpha}$ *and* $q_{-2\alpha}$ *can be suitably renormalized (independently of* l*) so that the following is true: Suppose* $\dim \mathfrak{g}^\alpha > 1$. *Define* $\zeta_l \in \mathfrak{N}^-$ *by the formula*

$$\zeta_l = \begin{cases} \displaystyle\prod_{j=1;\, j\ odd}^{l-1} (q_{-\alpha}^2 + j^2 q_{-2\alpha}), & l \text{ even}, \\ \displaystyle q_{-\alpha} \prod_{j=2;\, j\ even}^{l-1} (q_{-\alpha}^2 + j^2 q_{-2\alpha}), & l \text{ odd}. \end{cases}$$

If $\dim \mathfrak{g}^\alpha = 1$, *define* $\zeta_l = f^l \in \mathfrak{N}^-$, *where* f *is a nonzero element of* $\mathfrak{g}^{-\alpha}$. *Then the conical space of* X^ν *has basis* $\{x_0, \zeta_l \cdot x_0\}$. *Moreover, the* $\mathfrak{g}$*-submodule of* X^ν *generated by* $\zeta_l \cdot x_0$ *is isomorphic to* $X^{-\nu}$.

Theorem 1.2. *Let* $\mu, \nu \in \mathfrak{a}^*$. *Then* $\dim \operatorname{Hom}_{\mathfrak{g}}(X^\mu, X^\nu) \leq 1$. *Moreover,* $\dim \operatorname{Hom}_{\mathfrak{g}}(X^\mu, X^\nu) = 1$ *if and only if either* $\mu = \nu$, *or else* $\mu = -\nu$ *and* ν *is a nonnegative integral multiple of* α (*of* $\frac{1}{2}\alpha$ *if* $\dim \mathfrak{g}^\alpha = 1$). *This is exactly the case in which* X^μ *is isomorphic to a* $\mathfrak{g}$*-submodule of* X^ν.

(The annoying exceptional case $\dim \mathfrak{g}^\alpha = 1$ in these two theorems is essentially the case $G = SL(2, \mathbf{R})$, and is trivial.)

Considering how rare it is for a polynomial in two variables to factor into linear or quadratic factors, the factored form of the ζ_l in Theorem 1.1 seems remarkable. We shall say more about this below.

It turns out that Theorem 1.2 follows easily from Theorem 1.1, so we shall explain what is involved in proving Theorem 1.1. First, it is easy to see that the space of $\mathfrak{m}$-invariants in X^ν is the space $(\mathfrak{N}^-)^{\mathfrak{m}} \cdot x_0$ (here $(\mathfrak{N}^-)^{\mathfrak{m}}$ is the space of $\mathfrak{m}$-invariants in $\mathfrak{N}^-$ and equals $(\mathfrak{N}^-)^M$). From the above, $(\mathfrak{N}^-)^{\mathfrak{m}}$ is a polynomial algebra in one or two generators. If $\mathfrak{g}^{2\alpha} = 0$, we have one generator, and Theorem 1.1 is not terribly hard in this case (see §6). Suppose now that $\dim \mathfrak{g}^{2\alpha} > 1$, so that $(\mathfrak{N}^-)^{\mathfrak{m}}$ is the polynomial algebra $\mathbf{C}[q_{-\alpha}, q_{-2\alpha}]$. The whole problem is to determine those polynomials p in two variables such that $p(q_{-\alpha}, q_{-2\alpha}) \cdot x_0$ is $\mathfrak{n}$-invariant. Clearly, this involves computing commutators of elements of $\mathfrak{n}$ with $q_{-\alpha}$ and $q_{-2\alpha}$, and also commutators of *these* commutators with $q_{-\alpha}$ and $q_{-2\alpha}$. We were able to compute the necessary commutators (see §§6, 7), but the resulting condition on the polynomial p is immensely complicated, and it is not feasible to analyze it directly (see the last remark in §8).

However, when attempting to unravel this condition on p for some special G's, we noticed that the computations, even though we could not do them for any one G, did not seem to depend on G. The key was then to prove *a priori* that the conical vectors would look the same for any one G (for which $\dim \mathfrak{g}^{2\alpha} > 1$) as for any other such G, and then to use possibly special methods to solve the problem for one "small" G. Specifically, we first proved what we call the "fundamental commutation relation in $\mathfrak{N}^-$": There is a nonzero constant $c \in \mathbf{C}$ such that $[[f, q_{-\alpha}], q_{-\alpha}] = cfq_{-2\alpha}$ for all $f \in \mathfrak{g}^{-\alpha}$ (see Theorem 7.4). This is called "fundamental" because of the next result: If f is chosen more carefully, then this relation and a trivial one ($[f, q_{-2\alpha}] = 0$) generate *all* relations which are linear in f in the associative subalgebra of $\mathfrak{N}^-$ generated by f, $q_{-\alpha}$ and $q_{-2\alpha}$ (see Theorem 8.1). This in turn implies the following metamathematical "transfer principle for $\mathfrak{N}^-$": If $a_1, \dots, a_r$, $b_1, \dots, b_r$ are complex polynomials in two variables, then the truth of any assertion of the form "$\sum_{i=1}^r a_i(q_{-\alpha}, q_{-2\alpha}) f b_i(q_{-\alpha}, q_{-2\alpha}) = 0$ in $\mathfrak{N}^-$" is independent of G (see Theorem 8.4). But the condition that $p(q_{-\alpha}, q_{-2\alpha}) \cdot x_0$ be conical in X^{ν} can be expressed in this form (see Lemma 8.5), where the a_i and b_i depend only on p and the complex number c such that $\nu = c\alpha$. Thus we could prove the "transfer principle for conical vectors", another metatheorem which says that if $p(q_{-\alpha}, q_{-2\alpha}) \cdot x_0$ is conical in $X^{c\alpha}$ for some G with $\dim \mathfrak{g}^{2\alpha} > 1$, then the same is true for any such G (see Theorem 8.6). Furthermore, the above metatheorems have analogues for the case $\dim \mathfrak{g}^{2\alpha} = 1$, enabling us even to transfer theorems about conical vectors from any one G with $\dim \mathfrak{g}^{2\alpha} = 1$ to any G with *either* $\dim \mathfrak{g}^{2\alpha} = 1$ *or* $\dim \mathfrak{g}^{2\alpha} > 1$ (see Theorems 8.4 and 8.6).

The conical vectors still had to be computed for some special G with $\dim \mathfrak{g}^{2\alpha} \geq 1$. The only cases which we were able to do directly, aided by a crucial observation of L. Corwin, were the cases $G = SU(n, 1)$—essentially all the G's such that $\dim \mathfrak{g}^{2\alpha} = 1$. In these cases, $(\mathfrak{N}^-)^{\mathfrak{m}}$ is the polynomial algebra in $q_{-\alpha}$ and $r_{-2\alpha}$, where $r_{-2\alpha}$ is a nonzero element of the one-dimensional space $\mathfrak{g}^{-2\alpha}$. We reformulated the condition that $p(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$ be conical in X^{ν} (where p is a complex polynomial in two variables) in terms of a complicated system of linear equations whose unknowns were essentially the coefficients of p. These equations implied uniqueness of the conical vectors, but it was not clear that the equations had a consistent solution (and hence it was not clear that the conical vectors in Theorem 1.1 existed) until Corwin noticed that a solution vector could be constructed from the coefficients of a certain polynomial which factored into certain linear factors. This meant that if p were this polynomial, then $p(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$ would be conical. This was enough to prove Theorem 1.1 for these G's. To place the case $\dim \mathfrak{g}^{2\alpha} = 1$ in perspective, we further note the following: In this case, $r_{-2\alpha}^2 = q_{-2\alpha}$ in $\mathfrak{N}^-$, and therefore the factors $q_{-\alpha}^2 + j^2 q_{-2\alpha}$ in Theorem 1.1 themselves factor into linear factors: $(q_{-\alpha} + (-1)^{1/2} j r_{-2\alpha})(q_{-\alpha} - (-1)^{1/2} j r_{-2\alpha})$. It was this which made it feasible to carry out the necessary computations (see the Remark following Lemma 9.1).

Actually, in writing up the special case in §9, we dealt only with $G = SU(2, 1)$, and following a suggestion of N. Wallach, we used the theory of Verma modules to prove the uniqueness of the conical vectors. (For $G = SU(2, 1)$, the $\mathfrak{g}$-module induced from $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is actually a Verma module, not just a quotient.) Thus the original approach, using the complicated system of linear equations, is not carried out in this paper.

The above results are stated for G of real rank 1, but they imply a result for arbitrary real rank, included in Theorems 10.1 and 10.2.

There is another direction in which Theorems 1.1 and 1.2 are extended in this paper—to arbitrary fields of characteristic zero. In fact, throughout this paper, we work with semisimple symmetric Lie algebras with splitting Cartan subspaces, over fields of characteristic zero (see [2] and [7(b)] for background on these). This accounts for most of the length of §§2–4, in which we wanted to give a self-contained elementary treatment of the Kostant-Mostow double transitivity theorem and its consequences for algebras of polynomial invariants, valid over general fields of characteristic zero, without using any theory of Lie or algebraic groups. Instead of group orbits, we use "infinitesimal transitivity and double transitivity" conditions. We essentially give Wallach's modified version of Kostant's proof of the double transitivity theorem. See §§3 and 4 for a more detailed discussion of this theorem and its consequences.

Incidentally, it is not surprising that theorems about real semisimple Lie algebras, Cartan decompositions and Iwasawa decompositions should also hold for more general semisimple symmetric Lie algebras, since joint work with G. McCollum has shown that assertions about such structures whose truth is preserved under field extension and restriction are true for any one field of characteristic zero if and only if they are true for any other; see [8(e)]. This gives a generalization of H. Weyl's "unitary trick", which enables one to transfer theorems from compact semisimple Lie algebras to semisimple Lie algebras over arbitrary fields of characteristic zero.

After the work for this paper was completed, we found a simpler proof of the uniqueness of the conical vectors, avoiding the use of the double transitivity theorem; see [8(d)]. (But the existence and explicit form of the conical vectors still require the fundamental commutation relation and transfer principles.) This proof uses an observation of Kostant on the limitations imposed on conical vectors by the action of the center of $\mathfrak{G}$. The proof also uses an a priori argument that the first assertion of Theorem 1.2 holds—that $\dim \operatorname{Hom}_{\mathfrak{g}}(X^{\mu}, X^{\nu}) \leq 1$. In fact, we have generalized this last inequality to all parabolic subalgebras (see [8(c)]) by extending the method that Verma originally used (see [2, Théorème 7.6.6]) to prove the corresponding fact about Verma modules.

We remarked above that a $\mathfrak{g}$-module X induced from a finite-dimensional irreducible $(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$-module is a quotient of a certain Verma module V, but that one cannot very well use V to determine the highest weight vectors in X. On the other hand, since Theorems 1.1 and 1.2 *are* true, we can use *them* as a tool in investigating the composition series of the Verma module V. Interesting things happen: First, recall that in [1(a)], Bernšteĭn, Gel′fand and

Gel′fand found an example of a Verma module for $\mathfrak{sl}(4, \mathbf{C})$ having two strange properties: It contains a proper submodule not generated by Verma submodules, and its composition series contains a certain irreducible subquotient with multiplicity two. But it now turns out that if one regards $\mathfrak{sl}(4, \mathbf{C})$ as the complexification of $\mathfrak{su}(3, 1)$, then one can explain all of this pathology by means of the existence of a certain conical vector in X which does not come from a highest weight vector in V. In effect, Bernšteĭn, Gel′fand and Gel′fand were actually dealing with the case $l = 1$, $\zeta_l = q_{-\alpha}$ in Theorem 1.1. Moreover, using Theorem 1.1, we can generate whole families of examples of the same two "strange" phenomena for many Lie algebras. Thus a "bad" phenomenon for Verma modules becomes "good" when one interprets the situation using a larger parabolic subalgebra than a Borel subalgebra. This further emphasizes the importance of studying modules induced from general parabolic subalgebras.

Along the same lines, we comment that the results of [1] and [10] do not, in general, give explicit expressions for the highest weight vectors in a Verma module, or equivalently, explicit formulas for the embedding of one Verma module into another; they usually give only the *existence* of the vectors or the embeddings. But we can use the polynomials ζ_l in Theorem 1.1 to give explicit expressions for certain of these highest weight vectors or embeddings which have not yet been described explicitly.

We would like to thank G. D. Mostow for informing us about his approach to the double transitivity theorem.

Notations. We shall write $\mathbf{Z}_+$ for the set of nonnegative integers and $\mathbf{Q}$ for the field of rational numbers. Throughout this paper, k is a field of characteristic zero. The dual of a vector space V over k is denoted V^*. The symmetric algebra of V is written $S(V)$, and for all $r \in \mathbf{Z}_+$, the rth symmetric power is denoted $S^r(V)$, so that

$$S(V) = \coprod_{r \in \mathbf{Z}_+} S^r(V).$$

$S(V^*)$ is naturally isomorphic to the algebra of polynomial functions on V (i.e., the algebra of sums of products of linear functions on V), and we shall often identify these two algebras. Let $\mathfrak{g}$ be a Lie algebra over k, and let V be a $\mathfrak{g}$-module. Then $\mathfrak{g}$ may be canonically embedded in the universal enveloping algebra $\mathcal{G}$ of $\mathfrak{g}$, and V may be regarded naturally as a $\mathcal{G}$-module. The action of $\mathcal{G}$ on V will be denoted $x \cdot v$ $(x \in \mathcal{G}, v \in V)$. If $\mathfrak{s}$ and T are subsets of $\mathfrak{g}$ and V, respectively, let $T^{\mathfrak{s}}$ be the set of $\mathfrak{s}$-invariants in T, i.e., $\{t \in T \mid s \cdot t = 0 \text{ for all } s \in \mathfrak{s}\}$. Regard $\mathcal{G}$ and $S(\mathfrak{g})$ as $\mathfrak{g}$-modules by the natural extensions by derivations of the adjoint action of $\mathfrak{g}$ on itself. Then for

$x \in \mathfrak{g}$ and $y \in \mathfrak{G}$, $x \cdot y = [x, y]$, where we use $[\cdot, \cdot]$ to denote the commutator in associative algebras, as well as the bracket in Lie algebras. In particular, if $\mathfrak{s} \subset \mathfrak{g}$ and $T \subset \mathfrak{G}$, then $T^{\mathfrak{s}}$ is the ordinary centralizer of $\mathfrak{s}$ in T. Note that for all $x \in \mathfrak{g}$, $y \in \mathfrak{G}$ and $v \in V$, we have $x \cdot (y \cdot v) = [x, y] \cdot v + y \cdot (x \cdot v)$. Regard V^* as the $\mathfrak{g}$-module contragredient to the $\mathfrak{g}$-module V.

2. **The setting.** Here we shall summarize the necessary preliminaries and fix notation to be used throughout most of this paper.

Let $(\mathfrak{g}, \theta)$ be a semisimple symmetric Lie algebra over k, i.e., $\mathfrak{g}$ is a semisimple Lie algebra over k and θ is an automorphism of $\mathfrak{g}$ such that $\theta^2 = 1$. (See [2] and [7(b)] for background information on semisimple symmetric Lie algebras.) Denote by $\mathfrak{k}$ and $\mathfrak{p}$ the $+1$ and -1 eigenspaces for θ, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the symmetric decomposition of $(\mathfrak{g}, \theta)$, orthogonal with respect to the Killing form of $\mathfrak{g}$. Assume that there is a splitting Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$. That is, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ whose adjoint action on $\mathfrak{g}$ can be simultaneously diagonalized.

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and for all k-linear functionals $\phi: \mathfrak{a} \to k$, define

$$\mathfrak{g}^{\phi} = \{x \in \mathfrak{g} \mid [a, x] = \phi(a)x \text{ for all } a \in \mathfrak{a}\}.$$

Then $\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$. Let

$$\Sigma = \{\phi \in \mathfrak{a}^* \mid \phi \neq 0 \text{ and } \mathfrak{g}^{\phi} \neq 0\},$$

the set of *restricted roots* of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \coprod_{\phi \in \Sigma} \mathfrak{g}^{\phi} = \mathfrak{m} \oplus \mathfrak{a} \oplus \coprod_{\phi \in \Sigma} \mathfrak{g}^{\phi}.$$

Moreover, $[\mathfrak{g}^{\phi}, \mathfrak{g}^{\psi}] \subset \mathfrak{g}^{\phi \oplus \psi}$ and $\theta \mathfrak{g}^{\phi} = \mathfrak{g}^{-\phi}$ for all $\phi, \psi \in \mathfrak{a}^*$.

Let B be the Killing form of $\mathfrak{g}$. Then B is nonsingular on $\mathfrak{a}$ (see [7(b)]), so that B induces naturally a nonsingular symmetric k-bilinear form $(\cdot, \cdot)$ on $\mathfrak{a}^*$, as well as a natural isometry between $\mathfrak{a}$ and $\mathfrak{a}^*$. Let $\mathfrak{a}^*_{\mathbf{Q}}$ denote the rational span of Σ in $\mathfrak{a}^*$. Then $\mathfrak{a}^*$ is naturally isomorphic to $\mathfrak{a}^*_{\mathbf{Q}} \otimes_{\mathbf{Q}} k$, and the form $(\cdot, \cdot)$ is rational-valued and positive definite on the rational space $\mathfrak{a}^*_{\mathbf{Q}}$ (see [7(b)]). In particular, $(\phi, \phi) \neq 0$ for all $\phi \in \Sigma$.

For all $\phi \in \Sigma$, let s_{ϕ} denote the orthogonal reflection of $\mathfrak{a}^*$ through the hyperplane perpendicular to ϕ, and let W be the group of isometries of $\mathfrak{a}^*$ generated by the s_{ϕ} $(\phi \in \Sigma)$. W is called the *restricted Weyl group* of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Σ spans $\mathfrak{a}^*$ and forms a (not necessarily reduced) system of roots in $\mathfrak{a}^*$ with Weyl group W (see [7(b), §2]).

Let Σ_+ be a positive system in Σ, and define

$$\mathfrak{n} = \coprod_{\phi \in \Sigma_+} \mathfrak{g}^{\phi} \quad \text{and} \quad \mathfrak{n}^- = \coprod_{\phi \in \Sigma_+} \mathfrak{g}^{-\phi}.$$

Then $\mathfrak{n}$ and $\mathfrak{n}^-$ are nilpotent subalgebras of $\mathfrak{g}$, and we have the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Define the bilinear form B_θ on $\mathfrak{g}$ by the condition $B_\theta(x, y) = -B(x, \theta y)$ $(x, y \in \mathfrak{g})$. Then B_θ is a nonsingular symmetric form, and the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \coprod_{\phi \in \Sigma} \mathfrak{g}^{\phi}$ is a B_θ-orthogonal decomposition (see [7(b), Lemma 3.2]). Hence B_θ is nonsingular on each $\mathfrak{g}^{\phi}$ $(\phi \in \Sigma)$ on $\mathfrak{m}$ and on $\mathfrak{a}$. Moreover, B_θ is clearly a $\mathfrak{k}$-invariant and θ-invariant form on $\mathfrak{g}$.

For all $\phi \in \Sigma$, let $x_\phi \in \mathfrak{a}$ denote the image of ϕ under the canonical isometry from $\mathfrak{a}^*$ to $\mathfrak{a}$, so that $B(x_\phi, a) = \phi(a)$ for all $a \in \mathfrak{a}$, and $B(x_\phi, x_\psi) = (\phi, \psi)$ for all $\phi, \psi \in \Sigma$. Then for all $e \in \mathfrak{g}^{\phi}$, $[e, \theta e] \in \mathfrak{a}$, and in fact

$$[e, \theta e] = B(e, \theta e) x_\phi = -B_\theta(e, e) x_\phi$$

[7(b), Lemma 3.3]. Since $(\phi, \phi) \neq 0$, we can define $h_\phi = 2x_\phi/(\phi, \phi) \in \mathfrak{a}$. Then $\phi(h_\phi) = 2$.

Suppose now that k is algebraically closed, so that every element in k has a square root. Since B_θ is a symmetric nonsingular form on $\mathfrak{g}^{\phi}$, $\mathfrak{g}^{\phi}$ contains a nonisotropic vector e_0 with respect to the form B_θ (i.e., $B_\theta(e_0, e_0) \neq 0$). Set

$$e_\phi = (2/(\phi, \phi) B_\theta(e_0, e_0))^{1/2} e_0$$

and $f_\phi = -\theta e_\phi$. Then $B_\theta(e_\phi, e_\phi) = 2/(\phi, \phi)$, and so $[h_\phi, e_\phi] = 2e_\phi$, $[h_\phi, f_\phi] = -2f_\phi$ and $[e_\phi, f_\phi] = h_\phi$. Hence $\{h_\phi, e_\phi, f_\phi\}$ spans a three-dimensional simple subalgebra $\mathfrak{u}_\phi$ of $\mathfrak{g}$.

Now drop the algebraic closure assumption on k. Let $\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{g}$, and let $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{N}$ and $\mathfrak{N}^-$ denote the universal enveloping algebras of $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{n}^-$, respectively, regarded as canonically embedded in $\mathfrak{G}$. Then the multiplication map in $\mathfrak{G}$ induces a linear isomorphism

$$\mathfrak{G} \simeq \mathfrak{N}^- \otimes \mathfrak{M} \otimes \mathfrak{A} \otimes \mathfrak{N}.$$

Let $\nu \in \mathfrak{a}^*$. Then the linear form on the subalgebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$ which is ν on $\mathfrak{a}$ and zero on $\mathfrak{m} \oplus \mathfrak{n}$ vanishes on the commutator subalgebra of $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and thus corresponds to a one-dimensional representation π of $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and hence of its universal enveloping algebra $\mathfrak{M}\mathfrak{A}\mathfrak{N}$. Let V^ν be the $\mathfrak{g}$-module induced by the $(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$-module defined by π (see [2, §5.1]). That is,

$$V^\nu = \mathcal{G} \otimes_{\mathfrak{M}\mathfrak{A}\mathfrak{N}} k,$$

where $\mathcal{G}$ is regarded as a right $\mathfrak{M}\mathfrak{A}\mathfrak{N}$-module by right multiplication, and k is regarded as the $\mathfrak{M}\mathfrak{A}\mathfrak{N}$-module defined by π. The vector $v_0 = 1 \otimes 1 \in V^\nu$ generates V^ν as a $\mathcal{G}$-module, and is called the *canonical generator* of V^ν. It is clear that the map $\omega : \mathfrak{N}^- \to V^\nu$ given by $x \mapsto x \cdot v_0$ is a linear isomorphism.

Let V be a $\mathfrak{g}$-module, $v \in V$ a nonzero vector and $\lambda \in \mathfrak{a}^*$. Then v is called a *restricted weight vector* and λ a *restricted weight* for V if $x \cdot v = \lambda(x)v$ for all $x \in \mathfrak{a}$. For all $\lambda \in \mathfrak{a}^*$, the subspace of V consisting of 0 and the restricted weight vectors for λ is called the *restricted weight space* for λ; it is nonzero if and only if λ is a restricted weight for V.

The following definitions are central to this paper: Let V be a $\mathfrak{g}$-module, and let $v \in V$ be nonzero. Then v is a *conical vector* for V if $v \in V^{\mathfrak{m}\oplus\mathfrak{n}}$, i.e., if $(\mathfrak{m} \oplus \mathfrak{n}) \cdot v = 0$. The subspace $V^{\mathfrak{m}\oplus\mathfrak{n}}$ consisting of 0 and the conical vectors is called the *conical space* of V.

Now let $\nu \in \mathfrak{a}^*$ and let v_0 be the canonical generator of the induced module V^ν. Then v_0 is clearly a conical restricted weight vector in V^ν with restricted weight ν. It is also clear that the conical space of V^ν is $\mathfrak{a}$-invariant and hence is the direct sum of its intersections with the restricted weight spaces of V^ν.

The standard universal property of the induced module V^ν (see [2, §5.1]) say that if U is a $\mathfrak{g}$-module and $u \in U$ is a conical restricted weight vector with restricted weight ν, then there is a unique $\mathfrak{g}$-module homomorphism $f: V^\nu \to U$ such that $f(v_0) = u$. If u generates U, then f is surjective. If $U = V^\mu$ for some $\mu \in \mathfrak{a}^*$, then f is injective; this follows from the fact that $\mathfrak{N}^-$ has no zero divisors. Let $Z \subset V^\mu$ be the intersection of the conical space and the restricted weight space for ν. Then we have a natural linear isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(V^\nu, V^\mu) \to Z, \qquad f \mapsto f(v_0).$$

Let ν and v_0 be as above. Since $v_0 \in (V^\nu)^{\mathfrak{m}}$, the linear isomorphism $\omega : \mathfrak{N}^- \to V^\nu$ (see above) is also an $\mathfrak{m}$-module isomorphism, where $\mathfrak{N}^-$ is regarded as an $\mathfrak{m}$-submodule of $\mathcal{G}$ under the adjoint action. In particular, $(V^\nu)^{\mathfrak{m}} = (\mathfrak{N}^-)^{\mathfrak{m}} \cdot v_0$, and in fact ω restricts to a linear isomorphism

$$\omega : (\mathfrak{N}^-)^{\mathfrak{m}} \to (V^\nu)^{\mathfrak{m}}, \qquad x \mapsto x \cdot v_0.$$

Define $\rho \in \mathfrak{a}^*$ by the formula

$$\rho(a) = \tfrac{1}{2}\mathrm{tr}(\mathrm{ad}\ a \mid \mathfrak{n})$$

for all $a \in \mathfrak{a}$, i.e.,

$$\rho = \frac{1}{2} \sum_{\phi \in \Sigma_+} (\dim \mathfrak{g}^{\phi})\phi.$$

For all $\nu \in \mathfrak{a}^*$, define the $\mathfrak{g}$-module X^{ν} to be the induced module $V^{\nu-\rho}$. As above, let π be the one-dimensional representation of $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ defined by ν. Then X^{ν} can be interpreted as the twisted induced module induced by the one-dimensional $(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$-module corresponding to π, in the sense of [2, §5.2]. That is, for all $m \in \mathfrak{m}$, $a \in \mathfrak{a}$ and $n \in \mathfrak{n}$, the trace of the action of $m + a + n$ on $\mathfrak{g}/(\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n})$ is $-\mathrm{tr}(\mathrm{ad}\, a|\mathfrak{n}) = -2\rho(a)$. But we shall not need this fact.

The *canonical linear isomorphism* $\lambda: S(\mathfrak{g}) \to \mathfrak{G}$ is defined by the formula

$$\lambda(g_1 \cdots g_n) = \frac{1}{n!} \sum_{\sigma} g_{\sigma(1)} \cdots g_{\sigma(n)}$$

for all $n \in \mathbf{Z}_+$ and $g_i \in \mathfrak{g}$; here the product on the left is taken in $S(\mathfrak{g})$, the products on the right are taken in $\mathfrak{G}$, and σ ranges through the group of permutations of $\{1, \ldots, n\}$ (see [2, §2.4]). For all $g \in \mathfrak{g}$ and $n \in \mathbf{Z}_+$, $\lambda(g^n) = g^n$. Also, λ is a $\mathfrak{g}$-module isomorphism (see [2, §2.4.10]).

Let $\bar{k}$ be a field extension of k, $\bar{\mathfrak{g}} = \mathfrak{g} \otimes_k \bar{k}$, $\bar{\mathfrak{k}} = \mathfrak{k} \otimes_k \bar{k}$, etc., and let $\bar{\theta}$ be the $\bar{k}$-linear extension of θ to $\bar{\mathfrak{g}}$. Then $(\bar{\mathfrak{g}}, \bar{\theta})$ is a semisimple symmetric Lie algebra over $\bar{k}$ with symmetric decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}}$, $\bar{\mathfrak{a}}$ is a splitting Cartan subspace of $\bar{\mathfrak{p}}$, etc. We shall often use the technique of extension to a "sufficiently large" field $\bar{k}$, which can always be taken to be an algebraic closure of k. For example, the construction of the subalgebra $\mathfrak{u}_{\phi}$ above might have to be carried out over an extension field $\bar{k}$ of k, but results about $(\bar{\mathfrak{g}}, \bar{\theta})$ proved using $\mathfrak{u}_{\phi}$ can often be transferred to $(\mathfrak{g}, \theta)$.

3. **General results on polynomial invariants.** Let U be a finite-dimensional real Euclidean space and $SO(U)$ the rotation group of U. There is a natural $SO(U)$-invariant quadratic element t of the second symmetric power $S^2(U^*)$ given by the sum of the squares of the members of the dual basis of any orthonormal basis of U (t is the "square of the radius"). Let I be the algebra of $SO(U)$-invariant polynomial functions on U, or equivalently, the algebra of $SO(U)$-invariants in the symmetric algebra $S(U^*)$. A standard result of classical invariant theory states that I is exactly the set of polynomials in t if $\dim U > 1$. (If $\dim U = 1$, then $SO(U)$ acts trivially on U, and so $I = S(U^*)$.)

Clearly, I is exactly the set of polynomial functions on U constant on the $SO(U)$-orbits in U, i.e., the spheres centered at the origin if $\dim U > 1$, and the points if $\dim U = 1$. If M is any Lie group which acts as isometries on U in such a way that M acts transitively on the $SO(U)$-orbits in U (i.e., the M-orbits in U are the same as the $SO(U)$-orbits), then the set of M-invariant polynomial functions on U must coincide with the set I of $SO(U)$-invariants.

Now suppose that M also acts as isometries on a second finite-dimensional Euclidean space V so that M acts transitively on the $SO(V)$-orbits in V. Then the set of M-invariant polynomial functions on V is the set $J \subset S(V^*)$ of $SO(V)$-invariants, and J is a polynomial algebra as above.

Now M and $SO(U) \times SO(V)$ both act naturally on $U \oplus V$. Let L be the set of M-invariants in $S((U \oplus V)^*) = S(U^*) \otimes S(V^*)$. It is easy to see that the set of $SO(U) \times SO(V)$-invariants in $S((U \oplus V)^*)$ is exactly $I \otimes J$, and that $I \otimes J \subset L$. It is important to know that $I \otimes J = L$ in certain situations. In this case, for example, L will be a polynomial algebra on two generators. In order to insure this, it is natural to assume that the M-orbits in $U \oplus V$ are the same as the $SO(U) \times SO(V)$-orbits, i.e., the products of the $SO(U)$-orbits in U with the $SO(V)$-orbits in V. This assumption is equivalent to the "double-transitivity" hypothesis—that if A is an $SO(U)$-orbit in U and B is an $SO(V)$-orbit in V, then the isotropy group of M at any point of A acts transitively on B. If $\dim U > 1$ and $\dim V > 1$, this is equivalent to saying that M acts transitively on the product of the unit sphere in U with the unit sphere in V. Under the double transitivity hypothesis, $L = I \otimes J$.

The present section is devoted to algebraic analogues of these facts, valid over the field k of characteristic zero, assumed for convenience to be algebraically closed throughout this section. Here we are concerned with a Lie algebra $\mathfrak{m}_0$ (over k) which acts on modules U and V with nonsingular symmetric $\mathfrak{m}_0$-invariant bilinear forms. Replacing the orbit hypotheses for M by corresponding "infinitesimal transitivity and double-transitivity" assumptions, we show that the $\mathfrak{m}_0$-invariant polynomial functions on U, V and $U \oplus V$ are exact analogues of the spaces of M-invariants above. We also transfer these results to the symmetric algebras $S(U)$, $S(V)$ and $S(U \oplus V) = S(U) \otimes S(V)$; the invariants here are essentially the same as for the spaces of polynomial functions. We do not need any theory of algebraic groups. The setup in this section is entirely independent of §2; the results here will be applied to the setting of §2 in the next section.

Let $\mathfrak{m}_0$ be a Lie algebra over k, U a nonzero finite-dimensional $\mathfrak{m}_0$-module, and B_0 a nonsingular symmetric $\mathfrak{m}_0$-invariant bilinear form on U. The homogeneous quadratic polynomial function $x \mapsto B_0(x, x)$ on U defines a canonical nonzero element $t_0 \in S^2(U^*)^{\mathfrak{m}_0}$ under the natural identification between the algebra of polynomial functions on U and $S(U^*)$. B_0 also induces a canonical $\mathfrak{m}_0$-module isomorphism $\xi_0: U^* \to U$ which extends to an $\mathfrak{m}_0$-module and algebra isomorphism $\xi_0: S(U^*) \to S(U)$. Let $p_0 = \xi_0(t_0)$, so that $p_0 \in S^2(U)^{\mathfrak{m}_0}$.

For every element $e \in U$, denote by $e^\perp$ the B_0-orthogonal complement of e in U. Recall that e is called *isotropic* (resp., *nonisotropic*) with respect to B_0 if $B_0(e, e) = 0$ (resp., $B_0(e, e) \neq 0$). Note that e is B_0-nonisotropic if and only if $U = ke \oplus e^\perp$.

Lemma 3.1. *For all* $e \in U$, $\mathfrak{m}_0 \cdot e \subset e^\perp$.

Proof. Let $x \in \mathfrak{m}_0$. Then $B_0(x \cdot e, e) = -B_0(e, x \cdot e) = -B_0(x \cdot e, e)$ since B_0 is $\mathfrak{m}_0$-invariant and symmetric, and so $B_0(x \cdot e, e) = 0$. Q.E.D.

We now make the key assumption that for every B_0-nonisotropic vector $e \in U$, we have $\mathfrak{m}_0 \cdot e = e^\perp$. This can be thought of as an "infinitesimal transitivity" hypothesis. Our goal now is to compute $S(U)^{\mathfrak{m}_0}$, and in fact to prove:

Theorem 3.2. *If* $\dim U = 1$, *then* $S(U)^{\mathfrak{m}_0} = S(U)$. *If* $\dim U \geq 2$, *then* $S(U)^{\mathfrak{m}_0}$ *is the polynomial algebra generated by* p_0. *In particular,* $S(U)^{\mathfrak{m}_0}$ *is a polynomial algebra on one generator.*

The proof will be carried out in a series of lemmas. First we settle the easy one-dimensional case:

Lemma 3.3. *Suppose* $\dim U = 1$. *Then* $\mathfrak{m}_0$ *acts trivially on* U. *In particular,* $S(U)^{\mathfrak{m}_0} = S(U)$.

Proof. Any nonzero element e of U is B_0-nonisotropic, and so $e^\perp = 0$. Thus $\mathfrak{m}_0 \cdot e = 0$ (Lemma 3.1). Q.E.D.

It is also convenient to handle the two-dimensional case separately:

Lemma 3.4. *Suppose* $\dim U = 2$. *Then* $S(U)^{\mathfrak{m}_0}$ *is the polynomial algebra generated by* p_0.

Proof. Since k is algebraically closed, we may choose a B_0-orthonormal basis $\{e_1, e_2\}$ of U. Then $p_0 = e_1^2 + e_2^2 \in S^2(U)$. By hypothesis, there exists $x \in \mathfrak{m}_0$ such that $x \cdot e_1 = e_2$. Since B_0 is $\mathfrak{m}_0$-invariant, we have $B_0(e_1, x \cdot e_2) = -B_0(x \cdot e_1, e_2) = -B_0(e_2, e_2) = -1$. But $x \cdot e_2$ is a multiple of e_1, and so $x \cdot e_2 = -e_1$.

Again since k is algebraically closed, k contains a square root i of -1. Let $v_1 = e_1 + ie_2$ and $v_2 = e_1 - ie_2$, so that $\{v_1, v_2\}$ is a basis of U. Then $x \cdot v_1 = e_2 - ie_1 = -iv_1$ and $x \cdot v_2 = e_2 + ie_1 = iv_2$.

Let $f \in S(U)^{\mathfrak{m}_0}$. Then f is a polynomial of the form

$$f = \sum_{\alpha,\beta \in \mathbf{Z}_+} c_{\alpha\beta} v_1^{\alpha} v_2^{\beta}$$

in v_1 and v_2 $(c_{\alpha\beta} \in k)$. Since $x \cdot f = 0$, we have

$$\sum_{\alpha,\beta \in \mathbf{Z}_+} i c_{\alpha\beta}(\beta - \alpha) v_1^{\alpha} v_2^{\beta} = 0,$$

so that $c_{\alpha\beta} = 0$ unless $\alpha = \beta$. Thus $f = \sum_{\alpha \in \mathbf{Z}_+} c_{\alpha\alpha}(v_1 v_2)^{\alpha}$. But $v_1 v_2 = e_1^2 + e_2^2 = p_0$, and so f is a polynomial in p_0. Conversely, it is clear that any polynomial in p_0 is in $S(U)^{\mathfrak{m}_0}$. The lemma now follows from the fact that the subalgebra of $S(U)$ generated by p_0 is isomorphic to the polynomial algebra generated by p_0. Q.E.D.

In order to compute $S(U)^{\mathfrak{m}_0}$ in general, we shall use the following result:

Lemma 3.5. *Let $e \in U$ be B_0-nonisotropic, and let $r \in \mathbf{Z}_+$. Then e^r generates $S^r(U)$ as an $\mathfrak{m}_0$-module. In particular,*

$$S^r(U) = ke^r + \mathfrak{m}_0 \cdot S^r(U).$$

Proof. The second statement clearly follows from the first, and so it is sufficient to prove by induction on $j = 0, \ldots, r$ that the smallest $\mathfrak{m}_0$-invariant subspace T of $S^r(U)$ containing e^r also contains $e^{r-j}S^j(e^{\perp})$. This is clearly true for $j = 0$, so assume it is true for $0, \ldots, j$ $(j < r)$. Let $x \in \mathfrak{m}_0$ and $s \in S^j(e^{\perp})$. Then

$$x \cdot e^{r-j}s = (r-j)e^{r-(j+1)}(x \cdot e)s + e^{r-j}(x \cdot s).$$

The left-hand side and the second term on the right are in T by the induction hypothesis, and so $e^{r-(j+1)}(x \cdot e)s \in T$ since $r - j > 0$. The lemma now follows from the assumption that $\mathfrak{m}_0 \cdot e = e^{\perp}$. Q.E.D.

The point is the following:

Lemma 3.6. *Let $e \in U$ be B_0-nonisotropic, $r \in \mathbf{Z}_+$ and $f \in S^r(U^*)^{\mathfrak{m}_0}$. Regard f as a polynomial function on U. Then f is determined by its value at e. Equivalently, if $f(e) = 0$, then $f = 0$.*

Proof. There is a natural pairing $\{\cdot,\cdot\}$ between $S^r(U^*)$ and $S^r(U)$ given as follows:

$$\{f_1 \cdots f_r, u_1 \cdots u_r\} = \sum_{\sigma} \prod_{i=1}^{r} \langle f_i, u_{\sigma(i)} \rangle,$$

where $u_1, \ldots, u_r \in U$, $f_1, \ldots, f_r \in U^*$, $\langle \cdot,\cdot \rangle$ is the natural pairing between U^* and U and σ ranges through the group of permutations of $\{1, \ldots, r\}$. Then $\{f, u^r\} = r!f(u)$ for all $f \in S^r(U^*)$ and $u \in U$, where f is regarded as a polynomial function on U on the right-hand side. It follows that $\{\cdot,\cdot\}$ is nonsingular. Also, the natural actions of $\mathfrak{m}_0$ on $S^r(U^*)$ and $S^r(U)$ are contragredient with respect to $\{\cdot,\cdot\}$ (see for example the proof of [7(b), Lemma 3.6]).

Now let f and e be as in the statement of the lemma. If $f(e)=0$, then $\{f, e^r\}=0$. Since f is $\mathfrak{m}_0$-invariant, $\{f, x\cdot s\} = -\{x\cdot f, s\}=0$ for all $x \in \mathfrak{m}_0$ and $s \in S^r(U)$. Thus $\{f, S^r(U)\}=0$ by Lemma 3.5, and so $f=0$ by the nonsingularity of $\{\cdot,\cdot\}$. Q.E.D.

Theorem 3.2 now follows by applying the canonical isomorphism $\xi_0 : S(U^*) \to S(U)$ to the following result:

Lemma 3.7. *Let* $\dim U \geq 3$. *Then* $S(U^*)^{\mathfrak{m}_0}$ *is the polynomial algebra generated by* t_0. *Equivalently, if* $r \in \mathbf{Z}_+$ *is odd, then* $S^r(U^*)^{\mathfrak{m}_0}=0$, *and if* $r=2m$, $m \in \mathbf{Z}_+$, *then* $S^r(U^*)^{\mathfrak{m}_0}$ *is spanned by* t_0^m.

Proof. Since $S(U^*)^{\mathfrak{m}_0}$ is the direct sum of its homogeneous components, it is sufficient to compute $S^r(U^*)^{\mathfrak{m}_0}$ for $r \in \mathbf{Z}_+$. Let $V \subset U$ be the algebraic set defined by the equation $t_0(v)=0$ $(v \in U)$. Then V is exactly the set of B_0-isotropic vectors in U. Let $f \in S^r(U^*)^{\mathfrak{m}_0}$. If f has a zero outside V, then $f=0$ by Lemma 3.6. Hence we may assume that all the zeros of f lie in V. But then by the Hilbert Nullstellensatz, f divides some power of t_0. Choose a B_0-orthonormal basis of U, and let $X_1, \ldots, X_n \in U^*$ be the corresponding dual basis. Then $S(U^*)$ can be identified with the polynomial algebra $k[X_1, \ldots, X_n]$, and $t_0 = X_1^2 + \cdots + X_n^2$. Since $\dim U \geq 3$, t_0 is an irreducible polynomial. The fact that f divides a power of t_0 thus implies that f is itself a power of t_0 up to a scalar multiple. Q.E.D.

Theorem 3.2 is now proved.

Remark. The last assertion of Lemma 3.7 (the case $r=2m$) can also be proved more directly (even when $\dim U \leq 2$) as follows: Let $f \in S^r(U^*)^{\mathfrak{m}_0}$, let $e \in U$ be a B_0-nonisotropic vector, and set $c=(t_0^m)(e)=t_0(e)^m \in k$. Since $t_0(e)=B_0(e,e)\neq 0$, we have $c \neq 0$. But $f(e)t_0^m$ and cf are two elements of $S^r(U^*)^{\mathfrak{m}_0}$ which take the same value $cf(e)$ at e. Hence $f=c^{-1}f(e)t_0^m$, by Lemma 3.6, proving the assertion.

The following consequence is interesting, but it will not be needed:

Corollary 3.8 (to Theorem 3.2). *Every* $\mathfrak{m}_0$*-invariant symmetric bilinear form on* U *is a scalar multiple of* B_0.

Proof. From Theorem 3.2, $S^2(U)^{\mathfrak{m}_0}=kp_0$, and so $S^2(U^*)^{\mathfrak{m}_0}=kt_0$. The corollary now follows by polarization. Q.E.D.

Remark. Corollary 3.8 has a direct proof which does not use either Lemma 3.4 or Lemma 3.6: Let C be an $\mathfrak{m}_0$-invariant symmetric bilinear form on U. Then the unique linear operator $A: U \longrightarrow U$ defined by $C(u, v) = B_0(Au, v)$ for all $u, v \in U$ is an $\mathfrak{m}_0$-module map which is symmetric with respect to B_0. Let $e \in U$ be a B_0-nonisotropic vector, and let $e' \in e^{\perp}$. By hypothesis, there exists $x \in \mathfrak{m}_0$ such that $x \cdot e = e'$. Then

$$B_0(Ae, e') = B_0(Ae, x \cdot e) = -B_0(x \cdot Ae, e) = -B_0(A(x \cdot e), e)$$

$$= -B_0(Ae', e) = -B_0(e, Ae') = -B_0(Ae, e'),$$

and so $B_0(Ae, e') = 0$. Thus every B_0-nonisotropic vector of U is an eigenvector for A. Since every two B_0-orthogonal B_0-nonisotropic vectors have a B_0-nonisotropic linear combination not proportional to either of them, we see that they must have the same eigenvalue for A. Applying this to a B_0-orthogonal basis of U consisting of B_0-nonisotropic vectors shows that A is a scalar, and this completes the proof.

Another general result is required for the next section. Let V be a nonzero finite-dimensional $\mathfrak{m}_0$-module with a nonsingular symmetric $\mathfrak{m}_0$-invariant bilinear form B_1. Let $p_1 \in S^2(V)^{\mathfrak{m}_0}$ be the corresponding canonical invariant. The symmetric algebra of the direct sum $\mathfrak{m}_0$-module $U \oplus V$ is naturally isomorphic to $S(U) \otimes S(V)$, and $\mathfrak{m}_0$ acts on $S(U \oplus V)$ according to the tensor product of its actions on $S(U)$ and $S(V)$. In particular, $S(U)^{\mathfrak{m}_0} \otimes S(V)^{\mathfrak{m}_0} \subset S(U \oplus V)^{\mathfrak{m}_0}$. The next theorem gives an important case in which this inclusion becomes an equality.

Theorem 3.9. *In the context of Theorem 3.2, suppose in addition that for every B_0-nonisotropic vector $e_0 \in U$ and every B_1-nonisotropic vector $e_1 \in V$, we have $\mathfrak{m}_0' \cdot e_1 = e_1^{\perp}$ in V, where $\mathfrak{m}_0'$ is the centralizer of e_0 in $\mathfrak{m}_0$. Then*

$$S(U \oplus V)^{\mathfrak{m}_0} = S(U)^{\mathfrak{m}_0} \otimes S(V)^{\mathfrak{m}_0},$$

$S(U)^{\mathfrak{m}_0}$ is given by Theorem 3.2, and $S(V)^{\mathfrak{m}_0}$ is either $S(V)$ or the polynomial algebra generated by p_1, depending on whether $\dim V = 1$ *or* $\dim V \geq 2$. *In particular, $S(U \oplus V)^{\mathfrak{m}_0}$ is a polynomial algebra on two generators.*

Proof. Let $e_0 \in U$ be B_0-nonisotropic, and let $\mathfrak{m}_0'$ be the centralizer of e_0 in $\mathfrak{m}_0$. For every B_1-nonisotropic vector $e_1 \in V$, we have $e_1^{\perp} = \mathfrak{m}_0' \cdot e_1 \subset \mathfrak{m}_0 \cdot e_1 \subset e_1^{\perp}$ by Lemma 3.1, so that $\mathfrak{m}_0 \cdot e_1 = e_1^{\perp}$. Thus Theorem 3.2 applies to $\mathfrak{m}_0$, V, B_1 and p_1, and so to prove the theorem all we must show is that $S(U \oplus V)^{\mathfrak{m}_0} \subset S(U)^{\mathfrak{m}_0} \otimes S(V)^{\mathfrak{m}_0}$.

We shall now apply a technique used in [7(b), §5]. It is clear that

$S(U \oplus V)^{\mathfrak{m}_0}$ is the direct sum of its homogeneous components of the form $(S^r(U) \otimes S(V))^{\mathfrak{m}_0}$, where $r \in \mathbf{Z}_+$, and so it is sufficient to show that $(S^r(U) \otimes S(V))^{\mathfrak{m}_0} \subset S^r(U)^{\mathfrak{m}_0} \otimes S(V)^{\mathfrak{m}_0}$.

Recall the nonsingular $\mathfrak{m}_0$-invariant pairing $\{\cdot, \cdot\}$ between $S^r(U^*)$ and $S^r(U)$ (see the proof of Lemma 3.6). Also recall the canonical $\mathfrak{m}_0$-module and algebra isomorphism $\xi_0: S(U^*) \to S(U)$. Then ξ_0 restricts to an $\mathfrak{m}_0$-module isomorphism $\xi_0: S^r(U^*) \to S^r(U)$. Define a bilinear map

$$\omega: S^r(U) \otimes S(V) \times S^r(U) \to S(V)$$

by the condition $s \otimes w, t \mapsto \{\xi_0^{-1}(s), t\}w$ for all $s, t \in S^r(U)$ and $w \in S(V)$. Then for all $x \in \mathfrak{m}_0$, $y \in S^r(U) \otimes S(V)$ and $t \in S^r(U)$, we have

$$\omega(x \cdot y, t) + \omega(y, x \cdot t) = x \cdot \omega(y, t).$$

Moreover, let X be any subspace of $S(V)$. We claim that for all $y \in S^r(U) \otimes S(V)$, $\omega(y, S^r(U)) \subset X$ implies $y \in S^r(U) \otimes X$. In fact, choose a basis $\{w_i\}$ for a complement of X in $S^r(V)$ and write $y = \sum_i s_i \otimes w_i + z$ ($s_i \in S^r(U)$, $z \in S^r(U) \otimes X$). Then for all $t \in S^r(U)$, we have

$$\sum_i \omega(s_i \otimes w_i, t) + \omega(z, t) \in X,$$

and so $\sum_i \{\xi_0^{-1}(s_i), t\}w_i \in X$. Hence $\{\xi_0^{-1}(s_i), S^r(U)\} = 0$ for all i, so that each $s_i = 0$, proving the claim.

Let $y \in (S^r(U) \otimes S(V))^{\mathfrak{m}_0}$, and let e_0 and $\mathfrak{m}_0'$ be as in the statement of the theorem. Then for all $x \in \mathfrak{m}_0'$,

$$x \cdot \omega(y, e_0^r) = \omega(x \cdot y, e_0^r) + \omega(y, r(x \cdot e_0)e_0^{r-1}) = 0$$

since $x \cdot y = 0$ and $x \cdot e_0 = 0$. Hence $\omega(y, e_0^r) \in S(V)^{\mathfrak{m}_0'}$. But by hypothesis, $\mathfrak{m}_0' \cdot e_1 = e_1^{\perp}$ in V, for every B_1-nonisotropic vector $e_1 \in V$. Thus Theorem 3.2 applies to $\mathfrak{m}_0'$, V, B_1 and p_1, as well as to $\mathfrak{m}_0$, V, B_1 and p_1. In particular, $S(V)^{\mathfrak{m}_0'} = S(V)^{\mathfrak{m}_0}$, and so $\omega(y, e_0^r) \in S(V)^{\mathfrak{m}_0}$. But the set Z of B_0-nonisotropic vectors in U is Zariski dense since it is the set on which the polynomial function $t_0 \in S^2(U^*)$ does not vanish. Hence the powers e_0^r ($e_0 \in Z$) span $S^r(U)$ (see for example [7(b), Lemma 3.5(ii)]). It follows that $\omega(y, S^r(U)) \subset S(V)^{\mathfrak{m}_0}$. But now the above claim applied to $X = S(V)^{\mathfrak{m}_0}$ implies that $y \in S^r(U) \otimes S(V)^{\mathfrak{m}_0}$.

The rest is easy: Let $\{a_i\}$ be a basis of $S(V)^{\mathfrak{m}_0}$, and write $y = \sum_i b_i \otimes a_i$ ($b_i \in S^r(U)$). Since $\mathfrak{m}_0 \cdot y = 0$, we must have $\sum_i x \cdot b_i \otimes a_i = 0$ for all $x \in \mathfrak{m}_0$, so that $\mathfrak{m}_0 \cdot b_i = 0$ for each i. Hence $y \in S^r(U)^{\mathfrak{m}_0} \otimes S(V)^{\mathfrak{m}_0}$, and the theorem is proved. Q.E.D.

4. The Kostant-Mostow double transitivity theorem. In this section, we return to the setting of §2. For every $\phi \in \Sigma$, $\mathfrak{m}$ acts naturally on the subalgebra $\mathfrak{n}_\phi = \mathfrak{g}^\phi \oplus \mathfrak{g}^{2\phi}$ of $\mathfrak{g}$. (Here $\mathfrak{g}^{2\phi}$ might be zero.) Our main goal at this point is to determine the algebra $S(\mathfrak{n}_\phi)^\mathfrak{m}$ of $\mathfrak{m}$-invariants in the symmetric algebra $S(\mathfrak{n}_\phi)$. It will turn out to be a polynomial algebra on one or two generators (Theorem 4.6). The method will be to verify the hypotheses of §3 and then to apply the results of §3.

Suppose that $\dim \mathfrak{a} = 1$, ϕ is the unique simple restricted root, $\dim \mathfrak{g}^{2\phi} > 1$, $k = \mathbf{R}$, θ is a Cartan involution of $\mathfrak{g}$ in the sense that the Killing form of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, G is a connected Lie group corresponding to $\mathfrak{g}$, K is the connected Lie subgroup of G corresponding to $\mathfrak{k}$, and M is the centralizer of $\mathfrak{a}$ in K. Then $S(\mathfrak{n}_\phi)^\mathfrak{m}$ is the space $S(\mathfrak{n}_\phi)^M$ of M-invariants in $S(\mathfrak{n}_\phi)$, and determining $S(\mathfrak{n}_\phi)^M$ amounts to proving a double transitivity theorem for the action of M on $\mathfrak{g}^\phi \oplus \mathfrak{g}^{2\phi}$. Specifically, let S_1 be the unit sphere in $\mathfrak{g}^\phi$, and S_2 the unit sphere in $\mathfrak{g}^{2\phi}$, with respect to the bilinear form B_θ, which is positive definite on $\mathfrak{g}$. The issue is to prove that M acts transitively on $S_1 \times S_2$. This theorem was proved by B. Kostant [6, §2.1] (in a somewhat different formulation) and independently by G. D. Mostow (oral communication; related ideas are discussed in [8, §19]). Kostant's proof, as modified slightly by N. Wallach [11, Theorem 8.11.3], is purely algebraic. In order to show that this proof applies in our general setting, and for our later reference, we shall give an exposition of Kostant's proof below. (Mostow's proof is based on explicit case-by-case checking; only the case of the exceptional group F_4 is difficult.) We have been discussing the rather subtle situation in which $\dim \mathfrak{g}^{2\phi} > 1$; if $\dim \mathfrak{g}^{2\phi} \leq 1$, the appriate results are very easy.

Return now to the general setting of §2.

Fix $\phi \in \Sigma$. We shall describe a canonical element $p_\phi \in S^2(\mathfrak{g}^\phi)^\mathfrak{m}$. The symmetric bilinear form B_θ is nonsingular on $\mathfrak{g}^\phi$ (see §2). Since B_θ is $\mathfrak{k}$-invariant and hence $\mathfrak{m}$-invariant, and since $\mathfrak{g}^\phi$ is $\mathfrak{m}$-stable, the restriction of B_θ to $\mathfrak{g}^\phi$ is $\mathfrak{m}$-invariant. As in §3, we get a nonzero homogeneous quadratic polynomial function $x \mapsto B_\theta(x, x)$ on $\mathfrak{g}^\phi$, and this defines a nonzero element $t_\phi \in S^2((\mathfrak{g}^\phi)^*)^\mathfrak{m}$. B_θ induces a canonical $\mathfrak{m}$-module isomorphism $\xi_\phi : (\mathfrak{g}^\phi)^* \to \mathfrak{g}^\phi$ which extends to an $\mathfrak{m}$-module and algebra isomorphism $\xi_\phi : S((\mathfrak{g}^\phi)^*) \to S(\mathfrak{g}^\phi)$. Let $p_\phi = \xi_\phi(t_\phi)$, so that $p_\phi \in S^2(\mathfrak{g}^\phi)^\mathfrak{m}$.

Now we shall verify that the key assumption of the beginning of §3 holds in the present context, with $\mathfrak{m}_0 = \mathfrak{m}$ acting on $U = \mathfrak{g}^\phi$ by the adjoint action, and $B_0 = B_\theta | \mathfrak{g}^\phi \times \mathfrak{g}^\phi$. The word "nonisotropic" and the symbol $e^\perp$ have the meanings of §3.

Lemma 4.1 (cf. [6, Theorem 2.1.7]). *Let* $e_0 \in \mathfrak{g}^{\phi}$ *be a* B_θ*-nonisotropic vector. Then* $[\mathfrak{m}, e_0] = e_0^{\perp}$. *In particular,* $\mathfrak{g}^{\phi} = ke_0 \oplus [\mathfrak{m}, e_0]$.

Proof. It is clearly sufficient to assume that k is algebraically closed. As in §2, we may choose a multiple e_ϕ of e_0 such that $B_\theta(e_\phi, e_\phi) = 2/(\phi, \phi)$. Setting $h_\phi = 2x_\phi/(\phi, \phi) \in \mathfrak{a}$ and $f_\phi = -\theta e_\phi$, we have the bracket relations $[h_\phi, e_\phi] = 2e_\phi$, $[h_\phi, f_\phi] = -2f_\phi$ and $[e_\phi, f_\phi] = h_\phi$ (see §2), so that $\{h_\phi, e_\phi, f_\phi\}$ spans a three-dimensional simple Lie subalgebra $\mathfrak{u}_\phi$ of $\mathfrak{g}$. Let $\mathfrak{g}_\phi$ be the $\mathfrak{u}_\phi$-submodule $\coprod_{j=-2}^{2} \mathfrak{g}^{j\phi}$ of $\mathfrak{g}$. Since the eigenspaces of ad h_ϕ in $\mathfrak{g}_\phi$ with eigenvalues 0 and 2 are $\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$ and $\mathfrak{g}^{\phi}$, respectively, the representation theory of a three-dimensional simple Lie algebra implies that $[e_\phi, \mathfrak{m} \oplus \mathfrak{a}] = \mathfrak{g}^{\phi}$. But $[e_\phi, \mathfrak{m}] \subset e_\phi^{\perp}$ by Lemma 3.1, and since $[e_\phi, \mathfrak{a}] = ke_\phi$, we must have $[e_\phi, \mathfrak{m}] = e_\phi^{\perp}$. The lemma is now clear. Q.E.D.

Before applying Theorem 3.2, we shall derive two more results:

Lemma 4.2. *We have* $[\mathfrak{g}^{\phi}, \mathfrak{g}^{\phi}] = \mathfrak{g}^{2\phi}$.

Proof. We may assume that k is algebraically closed. As in §2 (or the last proof), we have the three-dimensional simple Lie subalgebra $\mathfrak{u}_\phi$ of $\mathfrak{g}$ spanned by h_ϕ, e_ϕ and f_ϕ. Let $\mathfrak{g}_\phi$ be the $\mathfrak{u}_\phi$-submodule $\coprod_{j=-2}^{2} \mathfrak{g}^{j\phi}$ of $\mathfrak{g}$. The eigenspaces of h_ϕ in $\mathfrak{g}_\phi$ with eigenvalues 2 and 4 are $\mathfrak{g}^{\phi}$ and $\mathfrak{g}^{2\phi}$, respectively, and so the representation theory of $\mathfrak{u}_\phi$ implies that $[e_\phi, \mathfrak{g}^{\phi}] = \mathfrak{g}^{2\phi}$. Q.E.D.

The following consequence will be useful later:

Corollary 4.3. *Let* X *be a* $\mathfrak{g}$*-module and* $x \in X$ *an* $\mathfrak{m}$*-invariant vector annihilated by some* B_θ*-nonisotropic vector* $e_0 \in \mathfrak{g}^{\phi}$. *Then* $(\mathfrak{g}^{\phi} \oplus \mathfrak{g}^{2\phi}) \cdot x = 0$. *In particular, if* $\dim \mathfrak{a} = 1$ *and* ϕ *is the unique simple restricted root, then* x *is a conical vector in* X.

Proof. For all $y \in \mathfrak{m}$, $[y, e_0] \cdot x = y \cdot (e_0 \cdot x) - e_0 \cdot (y \cdot x) = 0$, and so $\mathfrak{g}^{\phi} \cdot x = 0$ by Lemma 4.1. Lemma 4.2 now implies that $\mathfrak{g}^{2\phi} \cdot x = 0$. The last assertion is clear. Q.E.D.

Theorem 3.2, Lemma 4.1 and the field extension technique imply:

Theorem 4.4. *If* $\dim \mathfrak{g}^{\phi} = 1$, *then* $S(\mathfrak{g}^{\phi})^{\mathfrak{m}} = S(\mathfrak{g}^{\phi})$. *If* $\dim \mathfrak{g}^{\phi} \geq 2$, *then* $S(\mathfrak{g}^{\phi})^{\mathfrak{m}}$ *is the polynomial algebra generated by* p_ϕ. *In particular,* $S(\mathfrak{g}^{\phi})^{\mathfrak{m}}$ *is a polynomial algebra on one generator.*

Corollary 4.5. *Every* $\mathfrak{m}$*-invariant symmetric bilinear form on* $\mathfrak{g}^{\phi}$ *is a scalar multiple of* B_θ.

The corollary follows from either Theorem 4.4 or Corollary 3.8; see the remark following Corollary 3.8 for a simple proof. We shall not have to use Corollary 4.5.

Our next goal is to verify the hypothesis of Theorem 3.9 for $U = \mathfrak{g}^{\phi}$ and $V = \mathfrak{g}^{2\phi}$ in case $2\phi \in \Sigma$ (see Lemma 4.7). This amounts to proving the Kostant-Mostow double transitivity theorem. For reasons mentioned above, we shall essentially repeat Kostant's proof [6, §2.1], with a couple of modifications (the proofs of Lemmas 4.18 and 4.20) taken from Wallach's exposition [11, Theorem 8.11.3]. The result is:

Theorem 4.6. *Suppose $\phi \in \Sigma$, and let $\mathfrak{n}_{\phi}$ be the subalgebra $\mathfrak{g}^{\phi} \oplus \mathfrak{g}^{2\phi}$ of $\mathfrak{g}$. Then $S(\mathfrak{n}_{\phi})^{\mathfrak{m}} = S(\mathfrak{g}^{\phi})^{\mathfrak{m}} \otimes S(\mathfrak{g}^{2\phi})^{\mathfrak{m}}$, and this is a polynomial algebra. Moreover, let $p_{\phi} \in S^2(\mathfrak{g}^{\phi})^{\mathfrak{m}}$ be the canonical quadratic $\mathfrak{m}$-invariant defined by B_{θ}, and if $2\phi \in \Sigma$, let $p_{2\phi} \in S^2(\mathfrak{g}^{2\phi})^{\mathfrak{m}}$ be the same for 2ϕ. Then there are four possibilities*:

Case 1. $\dim \mathfrak{g}^{\phi} = 1$ *and* $\mathfrak{g}^{2\phi} = 0$. *Let $x \in \mathfrak{g}^{\phi}$, $x \neq 0$. Then $S(\mathfrak{n}_{\phi})^{\mathfrak{m}} = S(\mathfrak{g}^{\phi}) = k[x]$, and $k[x]$ is the polynomial algebra generated by x.*

Case 2. $\dim \mathfrak{g}^{\phi} > 1$ *and* $\mathfrak{g}^{2\phi} = 0$. *Then $S(\mathfrak{n}_{\phi})^{\mathfrak{m}} = k[p_{\phi}]$, and $k[p_{\phi}]$ is the polynomial algebra generated by p_{ϕ}.*

Case 3. $\dim \mathfrak{g}^{\phi} > \dim \mathfrak{g}^{2\phi} = 1$. *Then $S(\mathfrak{g}^{\phi})^{\mathfrak{m}} = k[p_{\phi}]$ and $S(\mathfrak{g}^{2\phi})^{\mathfrak{m}} = S(\mathfrak{g}^{2\phi}) = k[y]$, where $y \in \mathfrak{g}^{2\phi}$, $y \neq 0$. Both algebras are polynomial algebras in the indicated generators, so that $S(\mathfrak{n}_{\phi})^{\mathfrak{m}}$ is the polynomial algebra $k[p_{\phi}, y]$ in the two generators p_{ϕ} and y.*

Case 4. $\dim \mathfrak{g}^{\phi} > \dim \mathfrak{g}^{2\phi} > 1$. *Then $S(\mathfrak{g}^{\phi})^{\mathfrak{m}}$ and $S(\mathfrak{g}^{2\phi})^{\mathfrak{m}}$ are the polynomial algebras $k[p_{\phi}]$ and $k[p_{2\phi}]$, respectively, so that $S(\mathfrak{n}_{\phi})^{\mathfrak{m}}$ is the polynomial algebra $k[p_{\phi}, p_{2\phi}]$.*

Proof. We may, and do, assume that k is algebraically closed. The fact that $\dim \mathfrak{g}^{\phi} > \dim \mathfrak{g}^{2\phi}$ will be proved in Lemma 4.8. Also, Cases 1 and 2 are covered in Theorem 4.4. The rest of Theorem 4.6 follows immediately from Theorem 3.9, Lemma 4.1 and:

Lemma 4.7. *Suppose $\phi, 2\phi \in \Sigma$. Let $e_0 \in \mathfrak{g}^{\phi}$ and $e_1 \in \mathfrak{g}^{2\phi}$ be B_{θ}-nonisotropic, and let $\mathfrak{m}_0$ be the centralizer of e_0 in $\mathfrak{m}$. Then $[\mathfrak{m}_0, e_1] = e_1^{\perp}$ in $\mathfrak{g}^{2\phi}$.*

This result will follow from the next series of lemmas. Note that only Case 4 of Theorem 4.6 remains to be proved, since Lemma 4.7 is trivial if $\dim \mathfrak{g}^{2\phi} = 1$. But it will not be necessary in the following proof to impose any restriction on $\dim \mathfrak{g}^{2\phi}$, and in fact the proof holds even if $\mathfrak{g}^{2\phi} = 0$.

We shall use the notation of the proof of Lemma 4.1, so that e_{ϕ} is a certain multiple of e_0, and $\{h_{\phi}, e_{\phi}, f_{\phi}\}$ spans a three-dimensional simple subalgebra $\mathfrak{u}_\phi$ of $\mathfrak{g}$. Also as in the proof of Lemma 4.1, let $\mathfrak{g}_\phi$ be the $\mathfrak{u}_\phi$-submodule $\amalg_{j=-2}^{2} \mathfrak{g}^{j\phi}$ of $\mathfrak{g}$. The natural representation of $\mathfrak{u}_\phi$ on $\mathfrak{g}_\phi$ decomposes $\mathfrak{g}_\phi$ into a direct sum of irreducible $\mathfrak{u}_\phi$-submodules. Since the eigenvalues of ad h_ϕ on $\mathfrak{g}_\phi$ are among 0, ±2 and ±4 (with corresponding eigenspaces $\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$, $\mathfrak{g}^{\pm\phi}$ and $\mathfrak{g}^{\pm 2\phi}$), the dimensions of the irreducible components can only be 1, 3 and 5. A five-dimensional irreducible module occurs if and only if $\mathfrak{g}^{2\phi} \neq 0$, and a three-dimensional irreducible module always occurs—$\mathfrak{u}_\phi$ itself. Let $\mathfrak{g}_i \subset \mathfrak{g}_\phi$ be the sum of all the $(2i+1)$-dimensional irreducible $\mathfrak{u}_\phi$-submodules of $\mathfrak{g}_\phi$ $(i = 0, 1, 2)$, so that $\mathfrak{g}_\phi = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Also, let $\mathfrak{g}_i^j = \mathfrak{g}_i \cap \mathfrak{g}^{j\phi}$ $(0 \leq i \leq 2, -2 \leq j \leq 2)$; then $\mathfrak{g}_i = \amalg_{j=-i}^{i} \mathfrak{g}_i^j$ for each $i = 0, 1, 2$. Also, $\mathfrak{g}^{\pm 2\phi} = \mathfrak{g}_2^{\pm 2}$, $\mathfrak{g}^{\pm\phi} = \mathfrak{g}_1^{\pm 1} \oplus \mathfrak{g}_2^{\pm 1}$ and $\mathfrak{g}^0 = \mathfrak{g}_0^0 \oplus \mathfrak{g}_1^0 \oplus \mathfrak{g}_2^0$.

Lemma 4.8. *We have* $\dim \mathfrak{g}^\phi > \dim \mathfrak{g}^{2\phi}$.

Proof. This is clear since $\mathfrak{g}^\phi = \mathfrak{g}_1^1 \oplus \mathfrak{g}_2^1$, $\mathfrak{g}^{2\phi} = \mathfrak{g}_2^2$, $\dim \mathfrak{g}_2^1 = \dim \mathfrak{g}_2^2$ and $\dim \mathfrak{g}_1^1 \geq 1$ (since $e_\phi \in \mathfrak{g}_1^1$). Q.E.D.

Lemma 4.9. *The decomposition* $\mathfrak{g}_\phi = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *is both* B_θ*-orthogonal and* B*-orthogonal.*

Proof. First we shall show that $B_\theta(\mathfrak{g}_1^1, \mathfrak{g}_2^1) = 0$. Let $x \in \mathfrak{g}_1^1$, $y \in \mathfrak{g}_2^1$. Then $y = [f_\phi, z]$ for some $z \in \mathfrak{g}^{2\phi} = \mathfrak{g}_2^2$, and so

$$B_\theta(x, y) = -B(x, \theta y) = -B(x, [-e_\phi, \theta z]) = -B([e_\phi, x], \theta z) = 0$$

since $[e_\phi, x] = 0$. Hence $B_\theta(\mathfrak{g}_1^1, \mathfrak{g}_2^1) = 0$, and similar arguments show that

$$B_\theta(\mathfrak{g}_1^{-1}, \mathfrak{g}_2^{-1}) = B_\theta(\mathfrak{g}_0^0, \mathfrak{g}_1^0) = B_\theta(\mathfrak{g}_0^0, \mathfrak{g}_2^0) = 0$$

and

$$B(\mathfrak{g}_1^1, \mathfrak{g}_2^{-1}) = B(\mathfrak{g}_1^{-1}, \mathfrak{g}_2^1) = B(\mathfrak{g}_0^0, \mathfrak{g}_0^1) = B(\mathfrak{g}_0^0, \mathfrak{g}_0^2) = 0.$$

Since $B_\theta(\mathfrak{g}^{j\phi}, \mathfrak{g}^{k\phi}) = 0$ unless $j = k$, and $B(\mathfrak{g}^{j\phi}, \mathfrak{g}^{k\phi}) = 0$ unless $j = -k$, all that remains is to show that $B_\theta(\mathfrak{g}_1^0, \mathfrak{g}_2^0) = B(\mathfrak{g}_1^0, \mathfrak{g}_2^0) = 0$. Let $u \in \mathfrak{g}_1^0, v \in \mathfrak{g}_2^0$. Then $v = [f_\phi, w]$ for some $w \in \mathfrak{g}_2^1$, so that

$$B_\theta(u, v) = -B(u, \theta v) = -B(u, [-e_\phi, \theta w])$$

$$= -B([e_\phi, u], \theta w) = B_\theta([e_\phi, u], w) = 0$$

by the above, since $[e_\phi, u] \in \mathfrak{g}_1^1$ and $w \in \mathfrak{g}_2^1$. Thus $B_\theta(\mathfrak{g}_1^0, \mathfrak{g}_2^0) = 0$. Similarly, $B(\mathfrak{g}_1^0, \mathfrak{g}_2^0) = 0$. Q.E.D.

Lemma 4.10. *Let* $e \in \mathfrak{g}^\phi$ *and* $f \in \mathfrak{g}^{-\phi}$, *and suppose* $B(e, f) = 0$, *or equivalently,* $B_\theta(f, \theta e) = 0$ *or* $B_\theta(e, \theta f) = 0$. *Then* $[e, f] \in \mathfrak{m}$.

Proof. Since $[e, f] \in \mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$ and since $\mathfrak{m}$ is the B-orthogonal complement of $\mathfrak{a}$ in $\mathfrak{g}^0$, it is sufficient to show that $B([e, f], \mathfrak{a}) = 0$. But if $h \in \mathfrak{a}$, then

$$B([e, f], h) = -B(e, [h, f]) = \phi(h)B(e, f) = 0,$$

and so the lemma is proved. Q.E.D.

Lemma 4.11. *We have* $\mathfrak{g}_2^0 \subset \mathfrak{m}$.

Proof. Every element in $\mathfrak{g}_2^0$ is of the form $[e_\phi, f]$, where $f \in \mathfrak{g}_2^{-1}$. Since $e_\phi \in \mathfrak{g}_1$, Lemma 4.9 implies that $B(e_\phi, f) = 0$. But then $[e_\phi, f] \in \mathfrak{m}$ by Lemma 4.10. Q.E.D.

Lemma 4.12. *We have* $\mathfrak{g}_1^0 = kh_\phi \oplus (\mathfrak{g}_1^0 \cap \mathfrak{m})$.

Proof. It is sufficient to show that $\mathfrak{g}_1^0 \subset kh_\phi + \mathfrak{m}$. But $\mathfrak{g}_1^0 = [e_\phi, \mathfrak{g}_1^{-1}]$ and $\mathfrak{g}_1^{-1} \subset \mathfrak{g}^{-\phi} = kf_\phi \oplus f_\phi^\perp$, where $f_\phi^\perp$ is the B_θ-orthogonal complement of f_ϕ in $\mathfrak{g}^{-\phi}$. In fact, $B_\theta(f_\phi, f_\phi) = B_\theta(e_\phi, e_\phi) \neq 0$. Hence $\mathfrak{g}_1^0 \subset kh_\phi + [e_\phi, f_\phi^\perp]$, and $[e_\phi, f_\phi^\perp] \subset \mathfrak{m}$ by Lemma 4.10. Q.E.D.

Lemma 4.13. *We have* $\mathfrak{g}_0^0 = \operatorname{Ker} \phi \oplus (\mathfrak{g}_0^0 \cap \mathfrak{m})$.

Proof. Since $\mathfrak{g}_0^0 = \mathfrak{g}_0$ is the centralizer of $\mathfrak{u}_\phi$ in $\mathfrak{g}_\phi$, $\mathfrak{g}_0^0$ is stable under θ, and so $\mathfrak{g}_0^0 = (\mathfrak{g}_0^0 \cap \mathfrak{a}) \oplus (\mathfrak{g}_0^0 \cap \mathfrak{m})$. But the centralizer of $\mathfrak{u}_\phi$ 'n $\mathfrak{a}$ is clearly $\operatorname{Ker} \phi$, and so $\mathfrak{g}_0^0 \cap \mathfrak{a} = \operatorname{Ker} \phi$. Q.E.D.

Let $\mathfrak{m}_i = \mathfrak{g}_i \cap \mathfrak{m} = \mathfrak{g}_i^0 \cap \mathfrak{m}$ $(i = 1, 2, 3)$, and note that $\mathfrak{m}_0$ is the centralizer of e_ϕ in $\mathfrak{m}$ and hence coincides with the subalgebra $\mathfrak{m}_0$ in the statement of Lemma 4.7. The next lemma summarizes the last three:

Lemma 4.14. *We have* $\mathfrak{g}_2^0 = \mathfrak{m}_2$, $\mathfrak{g}_1^0 = kh_\phi \oplus \mathfrak{m}_1$ *and* $\mathfrak{g}_0^0 = \operatorname{Ker} \phi \oplus \mathfrak{m}_0$. *In particular,* $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

For all $x \in \mathfrak{g}$, define $x^* = [e_\phi, x]$. Write x^{**} instead of $(x^*)^*$. Also, define $x_* = [f_\phi, x]$, and write x_{**} for $(x_*)_*$.

Recall the following standard fact about the representation theory of the three-dimensional simple Lie algebra $\mathfrak{u}_\phi$: Let π be a finite-dimensional irreducible representation of $\mathfrak{u}_\phi$ on the space V and let $v \in V$ be a nonzero eigenvector for $\pi(h_\phi)$. Let p be the smallest nonnegative integer j such that $\pi(f_\phi)^{j+1}(v) = 0$ and q the smallest nonnegative integer j such that $\pi(e_\phi)^{j+1}(v) = 0$. Then $\pi(f_\phi)\pi(e_\phi)(v) = (p+1)qv$ and $\pi(e_\phi)\pi(f_\phi)(v) = (q+1)pv$. This implies:

Lemma 4.15. *For all* $x \in \mathfrak{m}_2$, $(x^{**})_* = 4x^*$, $(x^*)_* = 6x$, $(x_{**})^* = 4x_*$ *and* $(x_*)^* = 6x$.

Lemma 4.16. *Let* $x, y \in \mathfrak{m}_2$. *Then* $[x, y^{**}] = [x^{**}, y] = (2/3)[y^*, x^*]$.

Proof. By Lemma 4.15,

$$[x, y^{**}] = (1/6)[(x^*)_*, y^{**}] = (1/6)[(y^{**})_*, x^*] = (2/3)[y^*, x^*].$$

Hence also

$$[x^{**}, y] = -[y, x^{**}] = -(2/3)[x^*, y^*] = (2/3)[y^*, x^*]. \quad \text{Q.E.D.}$$

Lemma 4.17. *For all* $x, y \in \mathfrak{m}_2$, $[x, y]^{**} = (2/3)[x^*, y^*]$.

Proof. We have

$$[x, y]^{**} = [x^{**}, y] + 2[x^*, y^*] + [x, y^{**}] = (2/3)[x^*, y^*]$$

by Lemma 4.16. Q.E.D.

For all $x \in \mathfrak{g}_\phi$, let x_i $(i = 0, 1, 2)$ be the component of x in $\mathfrak{g}_i$ with respect to the decomposition $\mathfrak{g}_\phi = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Lemma 4.18. *For all* $x, y \in \mathfrak{m}_2$, $[x, y]_1 = 0$.

Proof. By Lemma 4.17, $[x, y]^{**} = (2/3)[x^*, y^*]$, so that

$$\begin{aligned}([x, y]^{**})_* &= (2/3)[(x^*)_*, y^*] + (2/3)[x^*, (y^*)_*]\\ &= 4[x, y^*] + 4[x^*, y] = 4[x, y]^*,\end{aligned}$$

using Lemma 4.15. But $[x, y]^{**} = ([x, y]_2)^{**}$, and $(([x, y]_2)^{**})_* = 4([x, y]_2)^*$ by Lemma 4.15. Hence $[x, y]^* = ([x, y]_2)^*$. But since $[x, y]^* = ([x, y]_1)^* + ([x, y]_2)^*$, we get $([x, y]_1)^* = 0$, and so $[x, y]_1 = 0$. Q.E.D.

Lemma 4.19. *For all* $x, y \in \mathfrak{m}_2$, $[[x, y]_0, y^{**}] = -2[[x, y]_2, y^{**}]$.

Proof. By Lemma 4.16, we have

$$[[x, y]_2, y^{**}] = [([x, y]_2)^{**}, y] = [[x, y]^{**}, y] = -[[x, y^{**}], y]$$

(Lemmas 4.16 and 4.17)

$$= -[[x, y], y^{**}] - [x, [y^{**}, y]] = -[[x, y], y^{**}]$$

(Lemma 4.16)

$$= -[[x, y]_0, y^{**}] - [[x, y]_2, y^{**}],$$

by Lemma 4.18. The lemma now follows. Q.E.D.

If $x \in \mathfrak{m}$, note that $x_* = -\theta x^*$ and $x_{**} = \theta x^{**}$.

Lemma 4.20. *Let* $x, y \in \mathfrak{m}_2$, *and suppose* $B_\theta(x^{**}, y^{**}) = 0$. *Then*

$[x^{**}, y_{**}] \in \mathfrak{m}$, $[x^{**}, y_{**}] = -[y^{**}, x_{**}]$, *and* $[x^{**}, y_{**}]_1 = 0$.

Proof. By Lemma 4.10 applied to $\mathfrak{g}^{2\phi}$ in place of $\mathfrak{g}^{\phi}$, $e = x^{**}$ and $f = y_{**} = \theta y^{**}$, we have $[x^{**}, y_{**}] \in \mathfrak{m}$. Thus

$$[x^{**}, y_{**}] = \theta[x^{**}, y_{**}] = [\theta x^{**}, \theta y_{**}] = [x_{**}, y^{**}] = -[y^{**}, x_{**}],$$

proving the second assertion.

To prove the last, first note that $(y_{**})^* = 4y_*$, by Lemma 4.15. Hence

$$[x^{**}, y_{**}]^* = [x^{**}, (y_{**})^*] = 4[x^{**}, y_*]$$

$$= 4[x^{**}, y]_* - 4[(x^{**})_*, y] = 4[x^{**}, y]_* - 16[x^*, y]$$

(again by Lemma 4.15)

$$= -4([x, y]^{**})_* - 16[x^*, y],$$

by Lemmas 4.16 and 4.17. Thus

$$([x^{**}, y_{**}]_1)^* = -16[x^*, y]_1.$$

Hence by the second assertion, we also have

$$([x^{**}, y_{**}]_1)^* = -([y^{**}, x_{**}]_1)^* = 16[y^*, x]_1 = -16[x, y^*]_1.$$

Thus

$$([x^{**}, y_{**}]_1)^* = -8([x^*, y] + [x, y^*])_1 = -8([x, y]^*)_1 = -8([x, y]_1)^* = 0,$$

by Lemma 4.18. It is finally clear that $[x^{**}, y_{**}]_1 = 0$. Q.E.D.

Lemma 4.21. *Let* $x, y \in \mathfrak{m}_2$, *and suppose* $B_\theta(x^{**}, y^{**}) = 0$. *Then* $[x^*, y_{**}] = -6[x, y]_*$.

Proof. We have $[x_*, y^{**}] = (1/4)[x_{**}, y^{**}]^*$ by Lemma 4.15, and this is $(1/4)[x^{**}, y_{**}]^*$ by Lemma 4.20. But $[x^{**}, y_{**}] \in \mathfrak{m}$ (Lemma 4.20). Thus the last assertion of Lemma 4.20 shows that $[x_*, y^{**}] \in \mathfrak{g}_2^1$. Now

$$[x_*, y^{**}]^* = [(x_*)^*, y^{**}] = 6[x, y^{**}]$$

(by Lemma 4.15)

$$= -6[x, y]^{**}$$

by Lemmas 4.16 and 4.17. But both $[x_*, y^{**}]$ and $[x, y]^*$ are in $\mathfrak{g}_2^1$ by the above and Lemma 4.18. Hence $[x_*, y^{**}] = -6[x, y]^*$, and the lemma follows by applying θ. Q.E.D.

Lemma 4.22. *Let* $x, y \in \mathfrak{m}_2$, *and suppose* $B_\theta(x^{**}, y^{**}) = 0$ *and* $B_\theta(y^{**}, y^{**}) = 1/2(\phi, \phi)$. *Then*

$$[[x, y]_0, y^{**}] = -x^{**}/18.$$

Proof. By Lemma 4.19, $[[x, y]_0, y^{**}] = -2[[x, y]_2, y^{**}]$. But $[x, y]_2 = (1/6)([x, y]_*)^*$, by Lemmas 4.15 and 4.18, and so

$$[[x, y]_0, y^{**}] = -(1/3)[([x, y]_*)^*, y^{**}]$$
$$= -(1/3)[[x, y]_*, y^{**}]^* = (1/18)[[x^*, y_{**}], y^{**}]^*,$$

by Lemma 4.21. Also,

$$B_\theta(y^{**}, y^{**}) = 1/2(\phi, \phi) = 2/(2\phi, 2\phi),$$

and so as in §2 we must have the bracket relations for a three-dimensional simple Lie algebra, say $\mathfrak{u}_{2\phi}$, spanned by $h_{2\phi}$, y^{**} and $-\theta y^{**}$:

$$[h_{2\phi}, y^{**}] = 2y^{**}, \quad [h_{2\phi}, -\theta y^{**}] = 2\theta y^{**} \quad \text{and} \quad [y^{**}, -\theta y^{**}] = h_{2\phi}.$$

But $-\theta y^{**} = -y_{**}$ and $h_{2\phi} = \frac{1}{2} h_\phi$. Thus x^* is an eigenvector for ad $h_{2\phi}$ with eigenvalue 1, and must lie in a two-dimensional irreducible $\mathfrak{u}_{2\phi}$-submodule of $\mathfrak{g}$. Hence applying the discussion preceding Lemma 4.15 to $\mathfrak{u}_{2\phi}$, we get

$$[y^{**}, [-y_{**}, x^*]] = x^*.$$

Thus $[[x, y]_0, y^{**}] = -x^{**}/18$, and the lemma is proved. Q.E.D.

In the notation of Lemma 4.7, a multiple e' of the nonisotropic vector $e_1 \in \mathfrak{g}^{2\phi}$ may be chosen so that $B_\theta(e', e') = 1/2(\phi, \phi)$. Then e' is of the form y^{**} for some $y \in \mathfrak{m}_2$. Let $e'' \in e_1^\perp$. Then $e'' = x^{**}$ for some $x \in \mathfrak{m}_2$, and so by Lemma 4.22, $[-18[x, y]_0, e'] = e''$. Thus there exists $z \in \mathfrak{m}_0$ such that $[z, e_1] = e''$. Lemma 4.7 is finally proved, and hence so is Theorem 4.6. Q.E.D.

5. The structure of $\mathfrak{N}_\phi^\mathfrak{m}$. Continuing to work in the setting of §2, we shall transfer Theorem 4.6 to its "noncommutative analogue", i.e., to the structure theorem for $\mathfrak{N}_\phi^\mathfrak{m}$ (see below).

Retain the notation of §4. In particular, $\phi \in \Sigma$ is fixed. Recall the canonical linear isomorphism $\lambda: S(\mathfrak{g}) \to \mathfrak{G}$. Let $\mathfrak{N}_\phi$ be the universal enveloping algebra of the Lie subalgebra $\mathfrak{n}_\phi = \mathfrak{g}^\phi \oplus \mathfrak{g}^{2\phi}$ of $\mathfrak{g}$ defined in Theorem 4.6, so that $\lambda: S(\mathfrak{n}_\phi) \to \mathfrak{N}_\phi$ is an $\mathfrak{m}$-module isomorphism which restricts to a linear isomorphism from $S(\mathfrak{n}_\phi)^\mathfrak{m}$ to $\mathfrak{N}_\phi^\mathfrak{m}$. We shall now use Theorem 4.6 to give an explicit description of the algebra $\mathfrak{N}_\phi^\mathfrak{m}$. Recall the canonical quadratic $\mathfrak{m}$-invariants $p_\phi \in S^2(\mathfrak{g}^\phi)^\mathfrak{m}$ and (if $2\phi \in \Sigma$) $p_{2\phi} \in S^2(\mathfrak{g}^{2\phi})^\mathfrak{m}$. Define

$$q_\phi = 2\lambda(p_\phi)/(\phi, \phi) \in \mathfrak{N}_\phi^\mathfrak{m},$$

and similarly, if $2\phi \in \Sigma$, define

$$q_{2\phi} = 2\lambda(p_{2\phi})/(2\phi, 2\phi) = \lambda(p_{2\phi})/2(\phi, \phi) \in \mathfrak{N}_\phi^{\mathfrak{m}}.$$

Theorem 5.1. *$\mathfrak{N}_\phi^{\mathfrak{m}}$ is commutative and in fact is a polynomial algebra. More precisely, in the four cases of Theorem 4.6, we have:*

Case 1. $\mathfrak{N}_\phi^{\mathfrak{m}} = \mathfrak{N}_\phi = k[x]$, the polynomial algebra generated by an arbitrary nonzero $x \in \mathfrak{g}^\phi$.

Case 2. $\mathfrak{N}_\phi^{\mathfrak{m}} = k[q_\phi]$, the polynomial algebra generated by q_ϕ.

Case 3. $\mathfrak{N}_\phi^{\mathfrak{m}} = k[q_\phi, y]$, where y is an arbitrary nonzero element of $\mathfrak{g}^{2\phi}$; this is the polynomial algebra in the indicated generators.

Case 4. $\mathfrak{N}_\phi^{\mathfrak{m}} = k[q_\phi, q_{2\phi}]$, the polynomial algebra generated by q_ϕ and $q_{2\phi}$.

Proof. Cases 1 and 2 follow immediately from the corresponding cases of Theorem 4.6, together with the fact that $\lambda: S(\mathfrak{n}_\phi) \to \mathfrak{N}_\phi$ is an algebra isomorphism since $\mathfrak{n}_\phi$ is abelian.

Since $\lambda(S(\mathfrak{n}_\phi)^{\mathfrak{m}}) = \mathfrak{N}_\phi^{\mathfrak{m}}$, Theorem 4.6 shows that the elements q_ϕ and y in Case 3 and q_ϕ and $q_{2\phi}$ in Case 4 lie in $\mathfrak{N}_\phi^{\mathfrak{m}}$. Also, since $\mathfrak{g}^{2\phi}$ is central in $\mathfrak{n}_\phi$, we see that q_ϕ commutes with y in Case 3 and $q_{2\phi}$ in Case 4.

Denote the usual filtration of the enveloping algebra $\mathfrak{N}_\phi$ by $\mathfrak{N}_0 \subset \mathfrak{N}_1 \subset \mathfrak{N}_2 \subset \cdots$, so that $\mathfrak{N}_0 = k \cdot 1$ and $\mathfrak{N}_1 = k \cdot 1 \oplus \mathfrak{n}_\phi$, and for each $r \in \mathbf{Z}_+$ let $\pi_r: \mathfrak{N}_r \to \mathfrak{N}_r/\mathfrak{N}_{r-1}$ be the canonical map. (Here we take $\mathfrak{N}_{-1} = 0$.) We also have the usual grading $S(\mathfrak{n}_\phi) = \coprod_{r=0}^\infty S^r(\mathfrak{n}_\phi)$ of $S(\mathfrak{n}_\phi)$. For each $r \in \mathbf{Z}_+$, let $\sigma_r: S^r(\mathfrak{n}_\phi) \to \mathfrak{N}_r/\mathfrak{N}_{r-1}$ be the canonical map, so that σ_r is a linear isomorphism by the Poincaré-Birkhoff-Witt theorem (see [2, Proposition 2.3.6]).

Now suppose that we are in Case 3. We claim that q_ϕ and y are algebraically independent. In fact, if not, then for some $r \in \mathbf{Z}_+$, there is an equation

$$\sum_{j=0}^{r} \sum_{i=0}^{[j/2]} a_{ij} q_\phi^i y^{j-2i} = 0,$$

where the $a_{ij} \in k$, and some $a_{ir} \neq 0$ ($i = 0, \ldots, [r/2]$); $[\cdot]$ denotes the "greatest integer" function. Thus $\sum_{i=0}^{[r/2]} a_{ir} q_\phi^i y^{r-2i} \in \mathfrak{N}_{p-1}$, so that

$$\pi_r\left(\sum_{i=0}^{[r/2]} a_{ir} q_\phi^i y^{r-2i}\right) = 0.$$

Consider the element

$$s = \sum_{i=0}^{[r/2]} a_{ir}\left(\frac{2}{(\phi, \phi)} p_\phi\right)^i y^{r-2i} \in S^r(\mathfrak{n}_\phi).$$

Then

$$\sigma_r(s) = \pi_r(\lambda(s)) = \pi_r\left(\sum_{i=0}^{[r/2]} a_{ir}\lambda\left(\frac{2}{(\phi,\phi)}p_\phi\right)^i \lambda(y)^{r-2i}\right)$$
$$= \pi_r\left(\sum_{i=0}^{[r/2]} a_{ir}q_\phi^i y^{r-2i}\right) = 0,$$

and so $s = 0$. But this is a contradiction, since p_ϕ and y are algebraically independent in $S(\mathfrak{n}_\phi)$, and the claim is established. A similar argument shows that in Case 4, q_ϕ and $q_{2\phi}$ are algebraically independent.

All that remains is to show that q_ϕ and y generate $\mathfrak{N}_\phi^\mathfrak{m}$ in Case 3 and that q_ϕ and $q_{2\phi}$ generate $\mathfrak{N}_\phi^\mathfrak{m}$ in Case 4. We shall carry out the argument only for Case 3; Case 4 is similar. Assume inductively that q_ϕ and y generate $\mathfrak{N}_\phi^\mathfrak{m} \cap \mathfrak{N}_r$, where $r \in \mathbf{Z}_+$. (This is trivially true for $r = 0$.) Now

$$\mathfrak{N}_t = \lambda\left(\coprod_{i=0}^{t} S^i(\mathfrak{n}_\phi)\right)$$

for all $t \in \mathbf{Z}_+$. Let

$$z \in \mathfrak{N}_\phi^\mathfrak{m} \cap \mathfrak{N}_{r+1} = \lambda\left(S(\mathfrak{n}_\phi)^\mathfrak{m} \cap \coprod_{i=0}^{r+1} S^i(\mathfrak{n}_\phi)\right).$$

Then z is of the form

$$z = \lambda\left(\sum_{j=0}^{r+1}\sum_{i=0}^{[j/2]} a_{ij}\left(\frac{2}{(\phi,\phi)}p_\phi\right)^i y^{j-2i}\right)$$

($a_{ij} \in k$) by Theorem 4.6. But

$$z - \sum_{j=0}^{r+1}\sum_{i=0}^{[j/2]} a_{ij}q_\phi^i y^{j-2i} \in \mathfrak{N}_\phi^\mathfrak{m} \cap \mathfrak{N}_r,$$

and so the induction hypothesis implies that z can be expressed as a polynomial in q_ϕ and y. This completes the proof of Theorem 5.1. Q.E.D.

6. The case $2\alpha \notin \Sigma$. In this section, we compute certain commutators in the universal enveloping algebra $\mathfrak{G}$ of $\mathfrak{g}$, and then we use these to determine certain conical vectors in the twisted induced $\mathfrak{g}$-modules X^ν, where $\nu \in \mathfrak{a}^*$ (see §2). Specifically, we prove our main results (Theorems 10.1 and 10.2) in the special case in which twice the relevant restricted root is not a restricted root (see Theorems 6.17 and 6.18). But the first part of the section, through Lemma 6.4, is valid in general, and this will be important in §8.

Maintain the hypotheses and notation of the last section. For convenience assume for awhile (through Corollary 6.10) that k is algebraically closed.

Continue to fix $\phi \in \Sigma$, and choose h_ϕ, e_ϕ, f_ϕ, and u_ϕ as in §2. Applying the constructions of the beginnings of §§4 and 5 to $-\phi$ in place of ϕ, we have canonical elements $p_{-\phi} \in S^2(\mathfrak{g}^{-\phi})^{\mathfrak{m}}$ and $q_{-\phi} = 2\lambda(p_{-\phi})/(\phi, \phi) \in \mathfrak{N}^{\mathfrak{m}}_{-\phi}$. Our goal now is to compute the commutator $[e_\phi, q_{-\phi}]$ in $\mathfrak{G}$.

Since $B_\theta(e_\phi, e_\phi) = 2/(\phi, \phi)$, it is clear that $B_\theta(f_\phi, f_\phi) = 2(\phi, \phi)$ also. Using the notation of the proof of Lemma 4.7, we recall from Lemma 4.9 that the decomposition $\mathfrak{g}_\phi = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is B_θ-orthogonal, and hence so is the decomposition $\mathfrak{g}^{-\phi} = \mathfrak{g}_1^{-1} \oplus \mathfrak{g}_2^{-1}$. Set $f_1 = f_\phi$. Since B_θ is nonsingular on $\mathfrak{g}^{-\phi}$ and k is algebraically closed, we may complete f_1 to a B_θ-orthogonal basis $\{f_1, \ldots, f_n\}$ of $\mathfrak{g}^{-\phi}$ such that $B_\theta(f_i, f_i) = 2/(\phi, \phi)$ for all $i = 1, \ldots, n$. But since $f_1 \in \mathfrak{g}_1^{-1}$, we may also assume that $f_1, \ldots, f_r \in \mathfrak{g}_1^{-1}$ and that $f_{r+1}, \ldots, f_n \in \mathfrak{g}_2^{-1}$. Here $n = \dim \mathfrak{g}^\phi = \dim \mathfrak{g}^{-\phi}$ and $r = \dim \mathfrak{g}_1^{-1}$. Note that $\dim \mathfrak{g}^{2\phi} = \dim \mathfrak{g}_2^{-1} = n - r$, and hence that $\mathfrak{g}^{2\phi} \neq 0$ if and only if $r < n$.

The canonical element $p_{-\phi} \in S^2(\mathfrak{g}^{-\phi})$ is equal to the sum of the squares of the elements of any B_θ-orthonormal basis of $\mathfrak{g}^{-\phi}$, and so

$$p_{-\phi} = \frac{(\phi, \phi)}{2} \sum_{i=1}^{n} f_i^2.$$

Thus

$$q_{-\phi} = \frac{2}{(\phi, \phi)} \lambda(p_{-\phi}) = \sum_{i=1}^{n} f_i^2 \in \mathfrak{N}_{-\phi}.$$

To compute $[e_\phi, q_{-\phi}]$, we first note that

$$\begin{aligned}
[e_\phi, q_{-\phi}] &= \sum_{i=1}^{n} [e_\phi, f_i^2] = \sum_{i=1}^{n} ([e_\phi, f_i]f_i + f_i[e_\phi, f_i]) \\
&= \sum_{i=1}^{n} ([[e_\phi, f_i], f_i] + 2f_i[e_\phi, f_i]) \\
&= \sum_{i=1}^{n} ([f_i, [f_i, e_\phi]] + 2f_i[e_\phi, f_i]).
\end{aligned}$$

Lemma 6.1. $[f_1, [f_1, e_\phi]] = -2f_1$.

Proof. This follows immediately from the bracket relations for h_ϕ, e_ϕ and $f_1 = f_\phi$. Q.E.D.

Lemma 6.2. *For all* $i = 2, \ldots, n$, $[e_\phi, f_i] \in \mathfrak{m}$.

Proof. Apply Lemma 4.10. Q.E.D.

Lemma 6.3. *For all* $i = 2, \dots, r$, $[f_i, [f_i, e_\phi]] = 2f_1$, *and for all* $i = r + 1, \dots, n$, $[f_i, [f_i, e_\phi]] = 6f_1$.

Proof. Let $i = 2, \dots, n$. Then

$$[e_\phi, [f_i, [f_i, e_\phi]]] = -[e_\phi, [f_i, [e_\phi, f_i]]] = -[f_i, [e_\phi, [e_\phi, f_i]]].$$

But $[e_\phi, f_i] \in \mathfrak{m}$ (Lemma 6.2), so that

$$\theta[e_\phi, [e_\phi, f_i]] = [\theta e_\phi, [e_\phi, f_i]] = -[f_1, [e_\phi, f_i]].$$

Now we can apply the standard representation theory of the three-dimensional simple Lie algebra $\mathfrak{u}_\phi$. If $2 \le i \le r$, then $f_i \in \mathfrak{g}_1^{-1}$, and so $[f_1, [e_\phi, f_i]] = 2f_i$, and if $r + 1 \le i \le n$, then $f_i \in \mathfrak{g}_2^{-1}$, and so $[f_1, [e_\phi, f_i]] = 6f_i$. Hence $[e_\phi, [e_\phi, f_i]] = -2\theta f_i$ or $-6\theta f_i$, respectively, and so

$$[e_\phi, [f_i, [f_i, e_\phi]]] = 2[f_i, \theta f_i] \quad \text{or} \quad 6[f_i, \theta f_i],$$

respectively. But $[f_i, \theta f_i] = -B_\theta(f_i, f_i)x_{-\phi}$ (see §2), and this is just h_ϕ. Thus

$$[e_\phi, [f_i, [f_i, e_\phi]]] = 2h_\phi \quad \text{or} \quad 6h_\phi,$$

respectively. But $[f_i, [f_i, e_\phi]] \in \mathfrak{g}^{-\phi}$, and so has eigenvalue -2 for ad h_ϕ. Since $[e_\phi, [f_i, [f_i, e_\phi]]]$ is a multiple of h_ϕ, the representation theory of $\mathfrak{u}_\phi$ implies that $[f_i, [f_i, e_\phi]]$ must be a multiple of f_1. Since $[e_\phi, f_1] = h_\phi$, the multiple is determined and the lemma follows. Q.E.D.

In view of these lemmas, we have

$$\begin{aligned}[e_\phi, q_{-\phi}] &= -2f_1 + 2f_1h_\phi + 2(r-1)f_1 + 6(n-r)f_1 + 2\sum_{i=2}^{n} f_i[e_\phi, f_i] \\ &= 2\left((3n - 2r - 2)f_1 + f_1h_\phi + \sum_{i=2}^{n} f_i[e_\phi, f_i]\right).\end{aligned}$$

Let

$$\rho_\phi = \tfrac{1}{2}((\dim \mathfrak{g}^{\phi})\phi + (\dim \mathfrak{g}^{2\phi})(2\phi)) \in \mathfrak{a}^*. \tag{1}$$

Then $\rho_\phi = \frac{1}{2}(n + 2(n-r))\phi = \frac{1}{2}(3n - 2r)\phi$, and so $\rho_\phi(h_\phi) = 3n - 2r$. The conclusion is:

Lemma 6.4. *Define* ρ_ϕ *as in* (1). *Then*

$$[e_\phi, q_{-\phi}] = 2\left((\rho_\phi - \phi)(h_\phi)f_\phi + f_\phi h_\phi + \sum_{i=2}^{n} f_i[e_\phi, f_i]\right).$$

We could now use the derivation law to write down an expression for $[e_\phi, q^d_{-\phi}]$, for all $d \in \mathbf{Z}_+$. In order to simplify matters, however, we shall assume at this point that $\mathfrak{g}^{2\phi} = 0$, which implies that $\mathfrak{g}^{-\phi}$ is an abelian Lie subalgebra of $\mathfrak{g}$. The much subtler general situation is deferred to subsequent sections.

Lemma 6.5. *Suppose* $2\phi \notin \Sigma$. *For all* $d \in \mathbf{Z}_+$,

$$[e_\phi, q^d_{-\phi}] = 2dq^{d-1}_{-\phi}\left(f_\phi(h_\phi + (\rho_\phi - d\phi)(h_\phi)) + \sum_{i=2}^{n} f_i[e_\phi, f_i]\right).$$

Proof. From Lemmas 6.4 and 6.2 and the commutativity of $\mathfrak{g}^{-\phi}$, we get

$$\begin{aligned}[e_\phi, q^d_{-\phi}] &= \sum_{j=1}^{d} q^{d-j}_{-\phi}[e_\phi, q_{-\phi}]q^{j-1}_{-\phi} \\ &= 2dq^{d-1}_{-\phi}\left((\rho_\phi - \phi)(h_\phi)f_\phi + \sum_{i=2}^{n} f_i[e_\phi, f_i]\right) \\ &\quad + 2f_\phi \sum_{j=1}^{d} q^{d-j}_{-\phi}h_\phi q^{j-1}_{-\phi}.\end{aligned}$$

But $q_{-\phi}$ is clearly a restricted weight vector for the action of $\mathfrak{a}$ on $\mathfrak{G}$ with restricted weight -2ϕ, and so

$$\begin{aligned}h_\phi q^{j-1}_{-\phi} &= [h_\phi, q^{j-1}_{-\phi}] + q^{j-1}_{-\phi}h_\phi \\ &= -2(j-1)\phi(h_\phi)q^{j-1}_{-\phi} + q^{j-1}_{-\phi}h_\phi = q^{j-1}_{-\phi}(-4(j-1) + h_\phi).\end{aligned}$$

Hence

$$\sum_{i=1}^{d} q^{d-j}_{-\phi}h_\phi q^{j-1}_{-\phi} = \sum_{j=1}^{d} q^{d-1}_{-\phi}(-4(j-1) + h_\phi) = q^{d-1}_{-\phi}(dh_\phi - 2d(d-1)),$$

and so

$$\begin{aligned}[e_\phi, q^d_{-\phi}] &= 2dq^{d-1}_{-\phi}\left((\phi d) - \phi)(h_\phi)f_\phi + \sum_{i=2}^{n} f_i[e_\phi, f_i]\right) \\ &\quad + 2dq^{d-1}_{-\phi}f_\phi h_\phi - 4dq^{d-1}_{-\phi}f_\phi(d-1) \\ &= 2dq^{d-1}_{-\phi}\left(f_\phi(\rho_\phi(h_\phi) - 2 - 2(d-1) + h_\phi) + \sum_{i=2}^{n} f_i[e_\phi, f_i\right) \\ &= 2dq^{d-1}_{-\phi}\left(f_\phi(h_\phi + (\rho_\phi - d\phi)(h_\phi)) + \sum_{i=2}^{n} f_i[e_\phi, f_i]\right).\text{Q.E.D.}\end{aligned}$$

The following result is now immediate:

Corollary 6.6. *Suppose* $\phi \in \Sigma_+$ *and* $2\phi \notin \Sigma$. *Let* X *be a* $\mathfrak{g}$*-module and* $x \in X$ *a conical restricted weight vector with restricted weight* $\mu \in \mathfrak{a}^*$. *Then for all* $d \in \mathbf{Z}_+$,

$$e_\phi \cdot (q_{-\phi}^d \cdot x) = 2d((\mu + \rho_\phi - d\phi)(h_\phi)) f_\phi q_{-\phi}^{d-1} \cdot x.$$

If $\dim \mathfrak{g}^\phi = 1$ (in which case $\mathfrak{g}^{\pm 2\phi} = 0$ automatically), we also have the following lemma and corollary:

Lemma 6.7. *Suppose* $\dim \mathfrak{g}^\phi = 1$. *Then for all* $d \in \mathbf{Z}_+$,

$$[e_\phi, f_\phi^d] = d f_\phi^{d-1}(h_\phi + (\rho_\phi - d\phi/2)(h_\phi)).$$

Proof. Since $[e_\phi, f_\phi] = h_\phi$, we have

$$[e_\phi, f_\phi^d] = \sum_{j=1}^{d} f_\phi^{d-j} h_\phi f_\phi^{j-1}.$$

But

$$h_\phi f_\phi^{j-1} = [h_\phi, f_\phi^{j-1}] + f_\phi^{j-1} h_\phi = f_\phi^{j-1}(-2(j-1) + h_\phi),$$

so that

$$[e_\phi, f_\phi^d] = f_\phi^{d-1}(dh_\phi - d(d-1)) = d f_\phi^{d-1}(h_\phi + (\rho_\phi - d\phi/2)(h_\phi)),$$

since $\rho_\phi(h_\phi) = \frac{1}{2}\phi(h_\phi) = 1$. Q.E.D.

Corollary 6.8. *Suppose* $\phi \in \Sigma_+$ *and* $\dim \mathfrak{g}^\phi = 1$. *Let* X *be a* $\mathfrak{g}$*-module and* $x \in X$ *an* $\mathfrak{n}$*-invariant restricted weight vector with restricted weight* $\mu \in \mathfrak{a}^*$. *Then for all* $d \in \mathbf{Z}_+$,

$$e_\phi \cdot (f_\phi^d \cdot x) = d((\mu + \rho_\phi - d\phi/2)(h_\phi)) f_\phi^{d-1} \cdot x.$$

Corollaries 6.6 and 6.8 imply the following two results. These have the benefit of being true even if k is not algebraically closed, as the field extension technique shows; we also use the fact that the B_θ-nonisotropic vectors in $\mathfrak{g}^\phi$ span $\mathfrak{g}^\phi$.

Corollary 6.9. *Suppose* $\phi \in \Sigma_+$ *and* $2\phi \notin \Sigma_+$. *Let* X *be a* $\mathfrak{g}$*-module and* $x \in X$ *a conical restricted weight vector with restricted weight* $\mu \in \mathfrak{a}^*$. *Then for all* $e_0 \in \mathfrak{g}^\phi$ *and* $d \in \mathbf{Z}_+$,

$$e_0 \cdot (q^d_{-\phi} \cdot x) = -2d((\mu + \rho_\phi - d\phi)(h_\phi))(\theta e_0)q^{d-1}_{-\phi} \cdot x.$$

Corollary 6.10. *Suppose $\phi \in \Sigma_+$ and* $\dim \mathfrak{g}^\phi = 1$. *Let X be a $\mathfrak{g}$-module and $x \in X$ an $\mathfrak{n}$-invariant restricted weight vector with restricted weight $\mu \in \mathfrak{a}^*$. Then for all $e_0 \in \mathfrak{g}^\phi$ and $d \in \mathbf{Z}_+$,*

$$e_0 \cdot ((\theta e_0)^d \cdot x) = -\tfrac{1}{2} B_\theta(e_0, e_0)(\phi, \phi)d((\mu + \rho_\phi - d\phi/2)(h_\phi))(\theta e_0)^{d-1} \cdot x.$$

Assume for the rest of this section that k is an arbitrary field of characteristic zero—not necessarily algebraically closed. We are now ready to prove the following basic result:

Lemma 6.11. *Suppose $\phi \in \Sigma_+$ and $2\phi \notin \Sigma_+$. Let $\nu \in \mathfrak{a}^*$, and let x_0 be the canonical generator of the twisted induced $\mathfrak{g}$-module $X^\nu = V^{\nu-\rho}$ (see §2). Set $Y = (\mathfrak{N}_{-\phi} \cdot x_0)^{\mathfrak{m}\oplus\mathfrak{n}_\phi}$ (see §5 for the definitions of $\mathfrak{N}_{-\phi}$ and $\mathfrak{n}_\phi$), and define $h'_\phi \in \mathfrak{a}$ to be h_ϕ if $\dim \mathfrak{g}^\phi > 1$ and $2h_\phi$ if $\dim \mathfrak{g}^\phi = 1$. If $(\nu-\rho+\rho_\phi)(h'_\phi)$ is not a positive even integer, then Y is the span of x_0. Suppose $(\nu - \rho + \rho_\phi)(h'_\phi) = 2l$, l a positive integer. Then Y is two-dimensional, with basis $\{x_0, f^l \cdot x_0\}$, where $f = q_{-\phi}$ if $\dim \mathfrak{g}^\phi > 1$ and f is a nonzero element of $\mathfrak{g}^{-\phi}$ if $\dim \mathfrak{g}^\phi = 1$. In this case, $f^l \cdot x_0$ is a restricted weight vector in X^ν with restricted weight $s_\phi(\nu - \rho + \rho_\phi) - \rho_\phi$ (recall from §2 that s_ϕ is the Weyl reflection with respect to ϕ).*

Proof. Since the map $\omega: \mathfrak{N}^- \to X^\nu$ which takes $y \in \mathfrak{N}^-$ to $y \cdot x_0$ is an $\mathfrak{m}$-module isomorphism (see §2), we see that

$$(\mathfrak{N}_{-\phi} \cdot x_0)^\mathfrak{m} = (\omega(\mathfrak{N}_{-\phi}))^\mathfrak{m} = \mathfrak{N}^\mathfrak{m}_{-\phi} \cdot x_0.$$

But by Theorem 5.1 (Cases 1 and 2), $\mathfrak{N}^\mathfrak{m}_{-\phi}$ is the polynomial algebra $k[f]$, where f is as in the statement of the lemma. Hence

$$(\mathfrak{N}_{-\phi} \cdot x_0)^\mathfrak{m} = k[f] \cdot x_0.$$

Let $u \in k[f]$, so that $u = \sum_{d=0}^\infty a_d f^d$ ($a_d \in k$, and only finitely many $a_d \neq 0$), and let e_0 be a B_θ-nonisotropic vector in $\mathfrak{g}^\phi$; if $\dim \mathfrak{g}^\phi = 1$, take $e_0 = \theta f$. Suppose $\dim \mathfrak{g}^\phi > 1$. Then by Corollary 6.9,

$$e_0 \cdot (u \cdot x_0) = -2 \sum_{d=0}^\infty a_d d((\nu - \rho + \rho_\phi - d\phi)(h_\phi))(\theta e_0)q^{d-1}_{-\phi} \cdot x_0,$$

and this expression is zero if and only if $a_d d((\nu - \rho + \rho_\phi - d\phi)(h_\phi)) = 0$ for all d. But this is the case if and only if $a_d = 0$ for all $d > 0$ such that $(\nu - \rho + \rho_\phi)(h_\phi) \neq 2d$. The lemma for $\dim \mathfrak{g}^\phi > 1$ now follows from Corollary 4.3; the last assertion of the lemma is clear since $q^l_{-\phi} \cdot x_0$ has restricted weight $\nu - \rho - 2l\phi$, and

$$s_\phi(\nu - \rho + \rho_\phi) = (\nu - \rho + \rho_\phi) - (\nu - \rho + \rho_\phi)(h_\phi)\phi = \nu - \rho + \rho_\phi - 2l\phi.$$

The case $\dim \mathfrak{g}^\phi = 1$ is similar, using Corollary 6.10. Q.E.D.

Remark. Note that Lemma 6.11 holds when ϕ is not necessarily a simple restricted root, and even when $\frac{1}{2}\phi$ is a restricted root.

The situation in Lemma 6.11 simplifies nicely when $\dim \mathfrak{a} = 1$; the next result is an immediate consequence of the lemma:

Theorem 6.12. *Suppose* $\dim \mathfrak{a} = 1$ *and* $\phi \in \Sigma_+$ *is the only positive root. Let* $\nu \in \mathfrak{a}^*$. *Then the conical space* Y *of the twisted induced* $\mathfrak{g}$*-module* X^ν *is either one- or two-dimensional. Define* $h'_\phi \in \mathfrak{a}$ *to be* h_ϕ *if* $\dim \mathfrak{g}^\phi > 1$ *and* $2h_\phi$ *if* $\dim \mathfrak{g}^\phi = 1$, *and let* x_0 *be the canonical generator of* X^ν. *If* $\nu(h'_\phi)$ *is not a positive even integer, then* Y *is the span of* x_0. *Suppose* $\nu(h'_\phi) = 2l$, l *a positive integer. Then* $\dim Y = 2$, *and* Y *has basis* $\{x_0, f^l \cdot x_0\}$, *where* $f = q_{-\phi}$ *if* $\dim \mathfrak{g}^\phi > 1$ *and* f *is a nonzero element of* $\mathfrak{g}^{-\phi}$ *if* $\dim \mathfrak{g}^\phi = 1$. *In this case,* $f^l \cdot x_0$ *is a restricted weight vector in* X^ν *with restricted weight* $s_\phi\nu - \rho$.

Lemma 6.11 also gives some interesting information about the conical space of X^ν even when $\dim \mathfrak{a}$ is arbitrary. To see this, we need some general facts.

Lemma 6.13. *Let* $\Pi \subset \Sigma_+$ *be the set of simple restricted roots. Then the subalgebra* $\mathfrak{n}$ *of* $\mathfrak{g}$ *is generated by the subspaces* $\mathfrak{g}^\alpha$ *as* α *ranges through* Π.

Proof. We may, and do, assume that k is algebraically closed. For all $\psi \in \Sigma$, define the *order* $o(\psi)$ of ψ to be the integer $\Sigma\, n_\alpha$ $(\alpha \in \Pi)$, where the integers n_α are defined by the condition $\psi = \Sigma\, n_\alpha \alpha (\alpha \in \Pi)$. Then $\psi \in \Sigma_+$ if and only if $o(\psi) > 0$, and $\psi \in \Pi$ if and only if $o(\psi) = 1$. We shall show by induction on $o(\psi)$ $(\psi \in \Sigma_+)$ that $\mathfrak{g}^\psi$ lies in the space generated by the $\mathfrak{g}^\alpha$ $(\alpha \in \Pi)$. This is clearly true if $o(\psi) = 1$, so assume it is true for $\sigma(\psi) = m$ $(m \geq 1)$, and let $\psi' \in \Sigma_+$ have order $m + 1$. Then the standard theory of root systems shows that there exists $\alpha \in \Pi$ such that the scalar product $(\psi', \alpha) > 0$, and hence $\psi = \psi' - \alpha$ is a positive restricted root of order m. Define the subspace V of $\mathfrak{g}$ by $V = \amalg_{j=-\infty}^{\infty} \mathfrak{g}^{\psi+n\alpha}$, and construct as in §2 (taking α for ϕ) a subalgebra $\mathfrak{u}_\alpha$ of $\mathfrak{g}$ spanned by h_α, e_α and f_α. Then V is a $\mathfrak{u}_\alpha$-submodule of $\mathfrak{g}$, and ad h_α has eigenvalue $\psi(h_\alpha) + 2n$ on the subspace $\mathfrak{g}^{\psi+n\alpha}$ of V; in particular, $\mathfrak{g}^{\psi+n\alpha}$ is exactly the $(\psi(h_\alpha) + 2n)$-eigenspace for ad h_α in V. But

$$\psi(h_\alpha) + 2 = (\psi + \alpha)(h_\alpha) = \psi'(h_\alpha) > 0,$$

and so the integer $\psi(h_\alpha) \geq -1$. Hence ad h_α has eigenvalue ≥ -1 on $\mathfrak{g}^\psi$, and so by the representation theory of the three-dimensional simple Lie algebra $\mathfrak{u}_\alpha$, we see that $[e_\alpha, \mathfrak{g}^\psi] = \mathfrak{g}^{\psi+\alpha} = \mathfrak{g}^{\psi'}$, and so $[\mathfrak{g}^\alpha, \mathfrak{g}^\psi] = \mathfrak{g}^{\psi'}$. In view of the induction hypothesis, we are finished. Q.E.D.

Remark. The above proof is of course similar to the proof of Lemma 4.2.

Lemma 6.14. *Let* $\alpha \in \Pi$ (*see Lemma* 6.13). *Then* $s_\alpha\rho - \rho = s_\alpha\rho_\alpha - \rho_\alpha$, *and* $\rho(h_\alpha) = \rho_\alpha(h_\alpha)$.

Proof. The first assertion is proved in [7(b), Lemma 4.16]. It follows that

$$\rho - \rho_\alpha = s_\alpha(\rho - \rho_\alpha) = (\rho - \rho_\alpha) - (\rho - \rho_\alpha)(h_\alpha)\alpha,$$

and so $(\rho - \rho_\alpha)(h_\alpha) = 0$. Q.E.D.

Lemma 6.15. $\mathfrak{N}^-$ *is a direct sum of restricted weight spaces (with respect to the natural action of* $\mathfrak{a}$ *on* $\mathfrak{G}$) *with restricted weights consisting of those elements of* $\mathfrak{a}^*$ *of the form* $-\Sigma\, n_\beta\beta$, *where* β *ranges through* Π *and* $n_\beta \in \mathbf{Z}_+$. *Let* $\alpha \in \Pi$, *and suppose* $y \in \mathfrak{N}^-$ *is a restricted weight vector with restricted weight of the form* $c\alpha$ $(c \in k)$. *Then* $y \in \mathfrak{N}_{-\alpha}$ *and* $c \in -\mathbf{Z}_+$.

Proof. Let $\Sigma_+^1 = \{\psi \in \Sigma_+ | \tfrac{1}{2}\psi \notin \Sigma_+\}$. Then $\mathfrak{n}^- = \amalg\, \mathfrak{n}_{-\psi}$ as ψ ranges through Σ_+^1. Let $\psi_1, \psi_2, \ldots, \psi_p$ be the elements of Σ_+^1. Then the multiplication map in $\mathfrak{G}$ induces a linear isomorphism

$$\mathfrak{N}^- \simeq \mathfrak{N}_{-\psi_1} \otimes \mathfrak{N}_{-\psi_2} \otimes \cdots \otimes \mathfrak{N}_{-\psi_p}.$$

The lemma now follows easily. Q.E.D.

Lemma 6.16. *Let* $\alpha \in \Pi$, $\nu \in \mathfrak{a}^*$ *and* x_0 *the canonical generator of the twisted induced module* X^ν. *The sum of the restricted weight spaces of* X^ν *with restricted weights of the form* $\nu - \rho + c\alpha$ $(c \in k)$ *is exactly* $\mathfrak{N}_{-\alpha} \cdot x_0$.

Proof. This is clear from Lemma 6.15 and the fact that the linear isomorphism $\omega: \mathfrak{N}^- \to X^\nu$ which takes y to $y \cdot x_0$ raises restricted weights by $\nu - \rho$; i.e., if $y \in \mathfrak{N}^-$ is a restricted weight vector with restricted weight $\mu \in \mathfrak{a}^*$, then $\omega(y)$ is a restricted weight vector with restricted weight $\nu - \rho + \mu$. Q.E.D.

We now have the following generalization of Theorem 6.12:

Theorem 6.17. *Let* α *be a simple restricted root, and suppose* $2\alpha \notin \Sigma$. *Let* $\nu \in \mathfrak{a}^*$, *and let* Y *be the subspace of the twisted induced* $\mathfrak{g}$-*module* X^ν *spanned by the conical restricted weight vectors with restricted weights of* *the form $\nu - \rho + c\alpha$ $(c \in k)$. Then Y is either one- or two-dimensional. Define $h'_\alpha \in \mathfrak{a}$ to be h_α if $\dim \mathfrak{g}^\alpha > 1$ and $2h_\alpha$ if $\dim \mathfrak{g}^\alpha = 1$, and let x_0 be the canonical generator of X^ν. If $\nu(h'_\alpha)$ is not a positive even integer, then Y is the span of x_0. Suppose $\nu(h'_\alpha) = 2l$, l a positive integer. Then $\dim Y = 2$, and Y has basis $\{x_0, f^l \cdot x_0\}$, where $f = q_{-\alpha}$ if $\dim \mathfrak{g}^\alpha > 1$ and f is a nonzero element of $\mathfrak{g}^{-\alpha}$ if $\dim \mathfrak{g}^\alpha = 1$. In this case, $f^l \cdot x_0$ is a restricted weight vector in X^ν with restricted weight $s_\alpha \nu - \rho$.*

Proof. Since the conical space of X^ν is clearly $\mathfrak{a}$-stable and hence the direct sum of its intersections with the restricted weight spaces of X^ν, $Y = (\mathfrak{N}_{-\alpha} \cdot x_0)^{\mathfrak{m}\oplus\mathfrak{n}}$ by Lemma 6.16. Let $y \in (\mathfrak{N}_{-\alpha} \cdot x_0)^{\mathfrak{m}\oplus\mathfrak{n}_\alpha}$, so that $y = u \cdot x_0$, where $u \in \mathfrak{N}_{-\alpha}$. Let β be a simple restricted root not equal to α. Then $\beta - \alpha$ is not a restricted root and is not zero, so that $[\mathfrak{g}^\beta, \mathfrak{n}_{-\alpha}] = [\mathfrak{g}^\beta, \mathfrak{g}^{-\alpha}] = 0$. Hence $[\mathfrak{g}^\beta, u] = 0$ in $\mathcal{G}$, and so

$$\mathfrak{g}^\beta \cdot (u \cdot x_0) = u \cdot (\mathfrak{g}^\beta \cdot x_0) = 0.$$

Lemma 6.13 now shows that $y \in Y$. Thus $Y = (\mathfrak{N}_{-\alpha} \cdot x_0)^{\mathfrak{m}\oplus\mathfrak{n}_\alpha}$, and the theorem now follows from Lemmas 6.11 and 6.14. Q.E.D.

Remark. In the notation of Theorem 6.17, the assertion that $\nu(h'_\alpha)$ be a nonnegative even integer (possibly zero) is equivalent to the existence in X^ν of an $\mathfrak{m}$-invariant restricted weight vector with restricted weight $s_\alpha\nu - \rho = \nu - \rho - \nu(h_\alpha)\alpha$ (use Lemma 6.16). In this case, the $\mathfrak{m}$-invariant restricted weight vectors with restricted weight $s_\alpha \nu - \rho$ span a one-dimensional space and are conical vectors. Note also that if $\nu \in \mathfrak{a}^*$ is arbitrary and if f and x_0 are defined as in Theorem 6.17, then $f^m \cdot x_0$ (m a positive integer) is $\mathfrak{n}$-invariant if and only if its restricted weight is $s_\alpha\nu - \rho$.

We can reformulate our conclusions as follows:

Theorem 6.18. *Let α be a simple restricted root such that $2\alpha \notin \Sigma$. Let $\mu, \nu \in \mathfrak{a}^*$, and suppose that $\mu - \nu$ is of the form $c\alpha$ $(c \in k)$. (If $\dim \mathfrak{a} = 1$, then this is automatic.) Then $\mathrm{Hom}_\mathfrak{g}(X^\mu, X^\nu)$ is at most one-dimensional, and $\dim \mathrm{Hom}_\mathfrak{g}(X^\mu, X^\nu) = 1$ if and only if either $\mu = \nu$, or else $\mu = s_\alpha \nu$ and $\nu(h'_\alpha)$ is a nonnegative even integer, where $h'_\alpha = h_\alpha$ if $\dim \mathfrak{g}^\alpha > 1$ and $h'_\alpha = 2h_\alpha$ if $\dim \mathfrak{g}^\alpha = 1$. Also, $\dim \mathrm{Hom}_\mathfrak{g}(x^\mu, X^\nu) = 1$ if and only if X^μ is isomorphic to a $\mathfrak{g}$-submodule of X^ν.*

Proof. Recall from §2 that $\mathrm{Hom}_\mathfrak{g}(X^\mu, X^\nu)$ is isomorphic to the intersection Z of the conical space of X^ν with the restricted weight space for $\mu - \rho$. If $\mu = \nu$, then clearly $\dim Z = 1$. Suppose $\mu = s_\alpha \nu$ and $\nu(h'_\alpha)$ is a nonnegative even integer. Then the above remark implies that $\dim Z = 1$. Conversely, suppose $Z \neq 0$, so that X^ν contains a conical restricted weight vector x

with restricted weight $\mu-\rho$. Since $\mu=\nu+c\alpha$, $\mu-\rho=\nu-\rho+c\alpha$, and so $x\in Y$, in the notation of Theorem 6.17. If $\mu\neq\nu$, then x is not a multiple of x_0 (again in the notation of Theorem 6.17), so that $\nu(h'_\alpha)$ is a positive even integer and $\mu-\rho=s_\alpha\nu-\rho$, i.e., $\mu=s_\alpha\nu$, by Theorem 6.17. The last assertion of the theorem follows from the fact that any nonzero $\mathfrak{g}$-module map from X^μ into X^ν is injective (see §2). Q.E.D.

7. The fundamental commutation relation in $\mathfrak{N}_{-\phi}$. We shall continue to use the notation of §6, with k algebraically closed. But in this section, we explicitly assume that $\mathfrak{g}^{2\phi}\neq 0$, i.e., that $2\phi\in\Sigma$. We have the canonical elements $p_{-2\phi}\in S^2(\mathfrak{g}^{-2\phi})^{\mathfrak{m}}$ and $q_{-2\phi}=\lambda(p_{-2\phi})/2(\phi,\phi)\in\mathfrak{N}^{\mathfrak{m}}_{-\phi}$ (see §5).

It is clearly important to compute the commutator $[e_\phi, q_{-2\phi}]$ in $\mathcal{G}$. This will easily turn out to be essentially $[f_\phi, q_{-\phi}]$, and we have to know to what extent this element commutes with $q_{-\phi}$. In particular, we want to compute $[[f_\phi, q_{-\phi}], q_{-\phi}]$. Lemma 6.4 also points out the importance of this commutator, since we need it in principle to simplify the commutator $[e_\phi, q^d_{-\phi}]$. It will turn out that $[[f_\phi, q_{-\phi}], q_{-\phi}]$ is essentially $f_\phi q_{-2\phi}$, and this is what we call the fundamental commutation relation in $\mathfrak{N}_{-\phi}$, the main result of this section. Because of this, we know how to compute the further commutators $[\cdots[[f_\phi, q_{-\phi}], q_{-\phi}]\cdots q_{-\phi}]$. The abstract algebraic setting in the next section will reveal a more precise reason for calling our relation "fundamental". The point will be that the fundamental relation and the trivial relation $f_\phi q_{-2\phi}=q_{-2\phi}f_\phi$ are in a sense *all* the relations involving f_ϕ, $q_{-\phi}$ and $q_{-2\phi}$.

Lemma 7.1. *The map* $\operatorname{ad} f_\phi\colon \mathfrak{g}_2^{-1}\to\mathfrak{g}_2^{-2}$ *is an isometry from* $4B_\theta|\mathfrak{g}_2^{-1}\times\mathfrak{g}_2^{-1}$ *to* $B_\theta|\mathfrak{g}_2^{-2}\times\mathfrak{g}_2^{-2}$.

Proof. Let $x, y\in\mathfrak{g}_2^{-1}$. Then

$$B_\theta([f_\phi,x],[f_\phi,y])=-B([f_\phi,x],\theta[f_\phi,y])=B([f_\phi,x],[e_\phi,\theta y])$$

$$=-B([e_\phi,[f_\phi,x]],\theta y)=-4B(x,\theta y)$$

(by Lemma 4.15)

$$=4B_\theta(x,y).\quad \text{Q.E.D.}$$

Recall from §6 the B_θ-orthogonal basis $\{f_1,\dots,f_n\}$ of $\mathfrak{g}^{-\phi}$.

Lemma 7.2. *We have*

$$q_{-2\phi}=\frac{1}{16}\sum_{i=1}^n[f_\phi,f_i]^2=\frac{1}{16}\sum_{i=r+1}^n[f_\phi,f_i]^2.$$

Proof. By Lemma 7.1, $\{[f_\phi,f_{r+1}],\dots,[f_\phi,f_n]\}$ is a B_θ-orthogonal basis of $\mathfrak{g}_2^{-2} = \mathfrak{g}^{-2\phi}$ such that each $B_\theta([f_\phi, f_i], [f_\phi, f_i]) = 8/(\phi, \phi)$. Since $q_{-2\phi}$ is $1/2(\phi, \phi)$ times the sum of the squares of the elements of any B_θ-orthonormal basis of $\mathfrak{g}^{-2\phi}$, we must have $q_{-2\phi} = (1/16) \sum_{i=r+1}^n [f_\phi, f_i]^2$. But $[f_\phi, f_j] = 0$ if $j = 1, \ldots, r$ and so the lemma follows. Q.E.D.

Lemma 7.3. *We have*

$$[e_\phi, q_{-2\phi}] = \frac{1}{4}[f_\phi, q_{-\phi}] = \frac{1}{2} \sum_{i=1}^n f_i[f_\phi, f_i]$$

$$= \frac{1}{2} \sum_{i=r+1}^n f_i[f_\phi, f_i].$$

Proof. By Lemma 7.2,

$$[e_\phi, q_{-2\phi}] = \frac{1}{16} \sum_{i=r+1}^n [e_\phi, [f_\phi, f_i]^2]$$

$$= \frac{1}{16} \sum_{i=r+1}^n ([e_\phi, [f_\phi, f_i]][f_\phi, f_i]$$

$$+ [f_\phi, f_i][e_\phi, [f_\phi, f_i]])$$

$$= \frac{1}{4} \sum_{i=r+1}^n (f_i[f_\phi, f_i] + [f_\phi, f_i]f_i)$$

(by Lemma 4.15)

$$= \frac{1}{2} \sum_{i=r+1}^n f_i[f_\phi, f_i]$$

(since $[f_\phi, f_i] \in \mathfrak{g}^{-2\phi}$, which is central in $\mathfrak{N}_{-\phi}$)

$$= \frac{1}{2} \sum_{i=1}^n f_i[f_\phi, f_i].$$

On the other hand, $q_{-\phi} = \sum_{i=1}^n f_i^2$, so that

$$[f_\phi, q_{-\phi}] = \sum_{i=1}^n [f_\phi, f_i^2]$$

$$= \sum_{i=1}^n ([f_\phi, f_i]f_i + f_i[f_\phi, f_i])$$

$$= 2 \sum_{i=1}^n f_i[f_\phi, f_i]. \quad \text{Q.E.D.}$$

Theorem 7.4. (*The fundamental commutation relation in* $\mathfrak{N}_{-\phi}$.) *We have*

$$[[f_\phi, q_{-\phi}], q_{-\phi}] = -64 f_\phi q_{-2\phi}.$$

More generally, suppose the field k *is arbitrary of characteristic zero, and let* $f \in \mathfrak{g}^{-\phi}$. *Then*

$$[[f, q_{-\phi}], q_{-\phi}] = -64 f q_{-2\phi}.$$

Proof. It is clearly sufficient to prove the first assertion. But by Lemma 7.3,

$$[[f_\phi, q_{-\phi}], q_{-\phi}] = 4[[e_\phi, q_{-2\phi}], q_{-\phi}]$$

$$= 4[[e_\phi, q_{-\phi}], q_{-2\phi}] = 8 f_\phi [h_\phi, q_{-2\phi}],$$

by Lemma 6.4, and this is just $-64 f_\phi q_{-2\phi}$. Q.E.D.

8. The transfer principles. Here we assume that $\mathfrak{g}^{2\phi} \neq 0$, as in §7. But we take k to be an arbitrary field of characteristic zero.

If we attempt to compute directly the conical vectors in the twisted induced modules X^ν ($\nu \in \mathfrak{a}^*$), we are confronted with monumental difficulties (cf. the remark at the end of this section). Trying to avoid these problems, we discovered a metamathematical "transfer principle" (Theorem 8.6) which enables us essentially to transfer certain theorems about conical vectors in modules over one semisimple symmetric Lie algebra to theorems about conical vectors in modules over any other semisimple symmetric Lie algebra. This reduces the problem of computing certain conical vectors to any one special case of semisimple symmetric Lie algebra (in which twice the relevant simple restricted root is a restricted root). The proof of this "transfer principle for conical vectors" is based on another metamathematical result (Theorem 8.4) which states that certain kinds of algebraic identities in $\mathfrak{N}_{-\phi}$ can be transferred from one semisimple symmetric Lie algebra to another. The starting point for the proof of this theorem is the "fundamental commutation relation" of the last section.

Let $P = k[w, x, y, z]$, the polynomial algebra in four indeterminates, and define a P-module structure on $\mathfrak{N}_{-\phi}$ by the correspondences

$$w \mapsto \text{left multiplication by } q_{-\phi},$$

$$x \mapsto \text{left multiplication by } q_{-2\phi},$$

$$y \mapsto \text{right multiplication by } q_{-\phi},$$

$$z \mapsto \text{right multiplication by } q_{-2\phi}.$$

This P-module structure is well defined because $[q_{-\phi}, q_{-2\phi}] = 0$ in $\mathfrak{N}_{-\phi}$.

Theorem 8.1. *Let* f *be an arbitrary* B_θ*-nonisotropic element of* $\mathfrak{g}^{-\phi}$, *and let* P^f *denote the annihilator of* f *in* P *under the above module action.*

Then the ideal P^f *is generated by* $x - z$ *and* $w^2 - 2wy + y^2 + 64x$, *that is,*

$$P^f = P(x - z) + P(w^2 - 2wy + y^2 + 64x).$$

Proof. Since $q_{-2\phi}$ is central in $\mathfrak{N}_{-\phi}$, it is clear that $x - z \in P^f$, and so $P(x - z) \subset P^f$. The fundamental commutation relation, Theorem 7.4, implies immediately that $w^2 - 2wy + y^2 + 64x \in P^f$, and hence the ideal generated by this element is contained in P^f. What we must show now is that these two ideals generate P^f.

Let $a \in P^f$, $a \neq 0$, and regard P as $k[x, y, z][w]$. Since the leading coefficient 1 of $w^2 - 2wy + y^2 + 64x$ is a unit in $k[x, y, z]$, the Euclidean algorithm implies the existence of s, $t \in k[x, y, z][w]$, where t is a polynomial of degree at most 1 in w, such that

$$a = s(w^2 - 2wy + y^2 + 64x) + t.$$

Here t is of the form $u + wv$, where $u, v \in k[x, y, z]$. Since $a \in P^f$, $t \in P^f$. Also, there exist polynomials $u', v' \in k[y, z]$ such that

$$u \equiv u' \pmod{P(x - z)} \quad \text{and} \quad v \equiv v' \pmod{P(x - z)}.$$

Hence

$$t \equiv u' + wv' \pmod{P(x - z)},$$

and so

$$a \equiv u' + wv' \pmod{P(x - z) + P(w^2 - 2wy + y^2 + 64x)}.$$

In particular, $u' + wv' \in P^f$. Write $u' = u'(y, z)$ and $v' = v'(y, z)$. Then by the definition of the module action of P on $\mathfrak{N}_{-\phi}$, we have

$$fu'(q_{-\phi}, q_{-2\phi}) + q_{-\phi} f v'(q_{-\phi}, q_{-2\phi}) = 0,$$

and so

$$f(u'(q_{-\phi}, q_{-2\phi}) + q_{-\phi} v'(q_{-\phi}, q_{-2\phi})) - [f, q_{-\phi}] v'(q_{-\phi}, q_{-2\phi}) = 0.$$

Set $\alpha(y, z) = u'(y, z) + yv'(y, z)$ and $\beta(y, z) = -v'(y, z)$ ($\alpha, \beta \in k[y, z]$). Then

$$f\alpha(q_{-\phi}, q_{-2\phi}) + [f, q_{-\phi}]\beta(q_{-\phi}, q_{-2\phi}) = 0.$$

It is sufficient to show that $\alpha = \beta = 0$, since then we will have $u' = v' = 0$, and so $a \in P(x - z) + P(w^2 - 2wy + y^2 + 64x)$.

As in the proof of Theorem 5.1, let $\mathfrak{N}_0 \subset \mathfrak{N}_1 \subset \mathfrak{N}_2 \subset \cdots$ be the usual filtration of $\mathfrak{N}_{-\phi}$, and for each $r \in \mathbf{Z}_+$, let $\pi_r : \mathfrak{N}_r \to \mathfrak{N}_r/\mathfrak{N}_{r-1}$ be the canonical map. (Here $\mathfrak{N}_{-1} = 0$.) Also, let $\sigma_r : S^r(\mathfrak{n}_{-\phi}) \to \mathfrak{N}_r/\mathfrak{N}_{r-1}$ be the natural map, so that σ_r is a linear isomorphism by the Poincaré-Birkhoff-Witt theorem.

Write

$$\alpha(y, z) = \sum_{j=0}^{c} \sum_{i=0}^{j} a_{ij} y^i z^{j-i}$$

and

$$\beta(y, z) = \sum_{j=0}^{d} \sum_{i=0}^{j} b_{ij} y^i z^{j-i}$$

with $c, d \in \mathbf{Z}_+$ and $a_{ij}, b_{ij} \in k$. If $\alpha \neq 0$, we may assume that some $a_{ic} \neq 0$ $(i = 0, \ldots, c)$, and if $\beta \neq 0$, we may also assume that some $b_{id} \neq 0$ $(i = 0, \ldots, d)$. Also, if $\alpha = 0$, take $c = 0$ and if $\beta = 0$, take $d = 0$.

Now we claim that $[f, q_{-\phi}] \in \mathfrak{N}_2$ and $[f, q_{-\phi}] \notin \mathfrak{N}_1$. In fact, it is sufficient to prove this when k is algebraically closed. But then a suitable multiple of f may be taken as the f_ϕ of §7, and the claim follows from Lemma 7.3. In particular, $f\alpha(q_{-\phi}, q_{-2\phi}) \in \mathfrak{N}_{2c+1}$ and $[f, q_{-\phi}]\beta(q_{-\phi}, q_{-2\phi}) \in \mathfrak{N}_{2d+2}$; recall that the sum of these two terms is zero. Either $2c + 1 > 2d + 2$ or $2c + 1 < 2d + 2$. Suppose the first inequality holds. Then

$$\pi_{2c+1}(f\alpha(q_{-\phi}, q_{-2\phi})) = 0,$$

so that

$$\pi_{2c+1}\left(f \sum_{i=0}^{c} a_{ic}\, q_{-\phi}^{i} q_{-2\phi}^{c-i}\right) = 0.$$

Let

$$p'_{-\phi} = \frac{2}{(\phi, \phi)}\, p_{-\phi} \quad \text{and} \quad p'_{-2\phi} = \frac{1}{2(\phi, \phi)}\, p_{-2\phi},$$

and set

$$s = f \sum_{i=0}^{c} a_{ic} (p'_{-\phi})^i (p'_{-2\phi})^{c-i} \in S^{2c+1}(\mathfrak{n}_{-\phi}).$$

Then

$$\begin{aligned}\sigma_{2c+1}(s) = \pi_{2c+1}(\lambda(s)) &= \pi_{2c+1}\left(\lambda(f) \sum_{i=0}^{c} a_{ic} \lambda(p'_{-\phi})^i \lambda(p'_{-2\phi})^{c-i}\right) \\ &= \pi_{2c+1}\left(f \sum_{i=0}^{c} a_{ic}\, q_{-\phi}^{i} q_{-2\phi}^{c-i}\right) = 0.\end{aligned}$$

Hence $s = 0$, and so each $a_{ic} = 0$ $(i = 0, \ldots, c)$. This is only possible if $\alpha = 0$. But then $c = 0$, and the inequality $2c + 1 > 2d + 2$ cannot hold. Hence we may assume that $2c + 1 < 2d + 2$. In this case,

$$\pi_{2d+2}([f, q_{-\phi}]\beta(q_{-\phi}, q_{-2\phi})) = 0,$$

and so

$$\pi_{2d+2}\left([f, q_{-\phi}] \sum_{i=0}^{d} b_{id} q^{i}_{-\phi} q^{d-i}_{-2\phi}\right) = 0.$$

Since $[f, q_{-\phi}] \notin \mathfrak{N}_1$ (see above), there exists a nonzero element $g \in S^2(\mathfrak{n}_{-\phi})$ such that $\lambda(g) \equiv [f, q_{-\phi}] \pmod{\mathfrak{N}_1}$. Set

$$h = g \sum_{i=0}^{d} b_{id}(p'_{-\phi})^i (p'_{-2\phi})^{d-i} \in S^{2d+2}(\mathfrak{n}_{-\phi}).$$

Then

$$\sigma_{2d+2}(h) = \pi_{2d+2}(\lambda(h)) = \pi_{2d+2}\left(\lambda(g) \sum_{i=0}^{d} b_{id}\lambda(p'_{-\phi})^i \lambda(p'_{-2\phi})^{d-i}\right)$$

$$= \pi_{2d+2}\left([f, q_{-\phi}] \sum_{i=0}^{d} b_{id} q^{i}_{-\phi} q^{d-i}_{-2\phi}\right) = 0.$$

Hence $h = 0$. But $g \neq 0$, so that each $b_{id} = 0$ $(i = 0, \dots, d)$. This proves that $\beta = 0$, and so $d = 0$. Since $2c + 1 < 2d + 2$, we also have $c = 0$. Thus α is a scalar, and the equation $f\alpha = 0$ shows that $\alpha = 0$. We have proved that $\alpha = \beta = 0$, and hence the theorem. Q.E.D.

Suppose now that $\dim \mathfrak{g}^{2\phi} = 1$, and suppose there exists an element $r_{-2\phi} \in \mathfrak{g}^{-2\phi}$ such that $r^2_{-2\phi} = q_{-2\phi}$ in $\mathfrak{N}_{-\phi}$. (Such an element exists if k is algebraically closed, but otherwise, it might not exist.) Define a new P-module structure on $\mathfrak{N}_{-\phi}$ by the correspondences

$w \mapsto$ left multiplication by $q_{-\phi}$,

$x \mapsto$ left multiplication by $r_{-2\phi}$,

$y \mapsto$ right multiplication by $q_{-\phi}$,

$z \mapsto$ right multiplication by $r_{-2\phi}$.

This P-module structure is well defined since $[q_{-\phi}, r_{-2\phi}] = 0$ in $\mathfrak{N}_{-\phi}$.

Theorem 8.2. *Under the above hypotheses, let $f \in \mathfrak{g}^{-\phi}$ be B_θ-nonisotropic, and let P_f be the annihilator of f in P under the new module action. Then*

$$P_f = P(x - z) + P(w^2 - 2wy + y^2 + 64x^2).$$

Proof. The first part of the proof of Theorem 8.1 carries over to the present situation and shows that is sufficient to prove the following: Let

$\alpha(y, z)$, $\beta(y, z) \in k[y, z]$, and suppose

$$f\alpha(q_{-\phi}, r_{-2\phi}) + [f, q_{-\phi}]\beta(q_{-\phi}, r_{-2\phi}) = 0.$$

Then $\alpha = \beta = 0$.

It is clearly sufficient to assume that k is algebraically closed and that f is the element f_ϕ of §§6 and 7. But then by Lemma 7.3, $[f_\phi, q_{-\phi}] = 2f_n[f_\phi, f_n]$ (where f_n is as in that lemma; see §6), since $\dim \mathfrak{g}^{2\phi} = 1$. By Lemma 7.2, $[f_\phi, f_n]^2 = 16q_{-2\phi}$ in $\mathfrak{N}_{-\phi}$, and since $[f_\phi, f_n] \in \mathfrak{g}^{-2\phi}$, we must have $4r_{-2\phi} = \pm[f_\phi, f_n]$. Changing the sign of f_n if necessary, we may assume that $4r_{-2\phi} = [f_\phi, f_n]$. Setting $\alpha'(y, z) = \alpha(y, z)$ and $\beta'(y, z) = 8z\beta(y, z)$ in $k[y, z]$, we have

$$f_\phi\alpha'(q_{-\phi}, r_{-2\phi}) + f_n\beta'(q_{-\phi}, r_{-2\phi}) = 0, \tag{2}$$

and it is sufficient to show that $\alpha' = \beta' = 0$.

Now $[e_\phi, f_n] \in \mathfrak{m}$ (where e_ϕ is as in §6), by Lemma 6.2, and so

$$[e_\phi, f_n] \cdot \alpha'(q_{-\phi}, r_{-2\phi}) = [e_\phi, f_n] \cdot \beta'(q_{-\phi}, r_{-2\phi}) = 0$$

in $\mathfrak{N}_{-\phi}$, since $q_{-\phi}, r_{-2\phi} \in \mathfrak{N}^{\mathfrak{m}}_{-\phi}$. Also, $[[e_\phi, f_n], f_\phi] = -6f_n$ by Lemma 4.15 and $[[e_\phi, f_n], f_n] = 6f_\phi$ by Lemma 6.3. Hence the application of $[e_\phi, f_n]$ to (*) gives

$$f_n\alpha'(q_{-\phi}, r_{-2\phi}) - f_\phi\beta'(q_{-\phi}, r_{-2\phi}) = 0. \tag{3}$$

Abbreviate $\alpha'(q_{-\phi}, r_{-2\phi})$ by α_0 and $\beta'(q_{-\phi}, r_{-2\phi})$ by β_0. Multiplying (2) on the right by α_0, multiplying (3) on the right by $-\beta_0$, and adding the two results, we get $f_\phi(\alpha_0^2 + \beta_0^2) = 0$. Since $\mathfrak{G}$ has no zero divisors,

$$(\alpha_0 + (-1)^{1/2}\beta_0)(\alpha_0 - (-1)^{1/2}\beta_0) = \alpha_0^2 + \beta_0^2 = 0,$$

and so $\alpha_0 = \pm(-1)^{1/2}\beta_0$. Thus (*) implies that $\alpha_0 = \beta_0 = 0$. The fact that $\alpha'(y, z) = \beta'(y, z) = 0$ now follows from Theorem 5.1, Case 3. Q.E.D.

Now assume the original hypotheses of this section, so that $\mathfrak{g}^{2\phi} \neq 0$. The following consequence of the last two theorems is immediate:

Corollary 8.3. *Let Q be the polynomial algebra in two variables over k, and let $a_i, b_i \in Q$ $(i = 1, \ldots, r, r \in \mathbf{Z}_+)$. Let f be a B_θ-nonisotropic element of $\mathfrak{g}^{-\phi}$. Then*

$$\sum_{i=1}^{r} a_i(q_{-\phi}, q_{-2\phi})fb_i(q_{-\phi}, q_{-2\phi}) = 0 \tag{4}$$

in $\mathfrak{N}_{-\phi}$ if and only if

$$\sum_{i=1}^{r} a_i \otimes b_i \in P(x-z) + P(w^2 - 2wy + y^2 + 64x),$$

where we identify P *with* $Q \otimes Q$ *in the natural way. Suppose in addition that* $\dim \mathfrak{g}^{2\phi} = 1$ *and that there exists an element* $r_{-2\phi} \in \mathfrak{g}^{-2\phi}$ *such that* $r_{-2\phi}^2 = q_{-2\phi}$. *Then*

$$\sum_{i=1}^{r} a_i(q_{-\phi}, r_{-2\phi}) f b_i(q_{-\phi}, r_{-2\phi}) = 0 \tag{5}$$

in $\mathfrak{N}_{-\phi}$ *if and only if*

$$\sum_{i=1}^{r} a_i \otimes b_i \in P(x-z) + P(w^2 - 2wy + y^2 + 64x^2),$$

where we again identify P *with* $Q \otimes Q$.

This corollary proves:

Theorem 8.4. (*The transfer principle for* $\mathfrak{N}_{-\phi}$.) *Let* Q *be the polynomial algebra in two variables over* k, *and let* $a_i, b_i \in Q$ $(i = 1, \ldots, r, r \in \mathbf{Z}_+)$. *Let* $(\mathfrak{g}, \theta)$ *be a semisimple symmetric Lie algebra over* k *with symmetric decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{a}$ *a splitting Cartan subspace of* $\mathfrak{p}$, $\Sigma \subset \mathfrak{a}^*$ *the corresponding system of restricted roots,* $\phi \in \Sigma$ *such that* $2\phi \in \Sigma$, $\mathfrak{N}_{-\phi}$ *the universal enveloping algebra of the Lie subalgebra* $\mathfrak{n}_{-\phi} = \mathfrak{g}^{-\phi} \oplus \mathfrak{g}^{-2\phi}$ *of* $\mathfrak{g}$, $\lambda: S(\mathfrak{n}_{-\phi}) \to \mathfrak{N}_{-\phi}$ *the canonical linear isomorphism,* B *the Killing form of* $\mathfrak{g}$, B_θ *the symmetric bilinear form on* $\mathfrak{g}$ *defined by the condition* $B_\theta(x, y) = -B(x, \theta y)$ *for all* $x, y \in \mathfrak{g}$, f *a* B_θ*-nonisotropic vector in* $\mathfrak{g}^{-\phi}$, $p_{-\phi} \in S^2(\mathfrak{g}^{-\phi})$ *and* $p_{-2\phi} \in S^2(\mathfrak{g}^{-2\phi})$ *the canonical elements defined by* B_θ, *and* $q_{-\phi} = 2\lambda(p_{-\phi})/(\phi, \phi)$ *and* $q_{-2\phi} = \lambda(p_{-2\phi})/2(\phi, \phi) \in \mathfrak{N}_{-\phi}$. *Then the truth or falsity of equation* (4) *in* $\mathfrak{N}_{-\phi}$ *depends only on* a_i *and* b_i $(i = 1, \ldots, r)$ *and not on* $\mathfrak{g}, \theta, \mathfrak{a}, \phi$ *or* f. *Moreover, suppose in addition that* $\dim \mathfrak{g}^{2\phi} = 1$ *and that there exists an element* $r_{-2\phi} \in \mathfrak{g}^{-2\phi}$ *such that* $r_{-2\phi}^2 = q_{-2\phi}$. *Then the truth or falsity of equation* (5) *in* $\mathfrak{N}_{-\phi}$ *depends only on* a_i *and* b_i $(i = 1, \ldots, r)$, *and not on* $\mathfrak{g}, \theta, \mathfrak{a}, \phi, f$ *or* $r_{-2\phi}$.

In order to apply this theorem to conical vectors, we need:

Lemma 8.5. *Suppose* ϕ *and* $2\phi \in \Sigma_+$, *and let* V *be a* $\mathfrak{g}$*-module,* $v \in V^{\mathfrak{m} \oplus \mathfrak{n}_\phi}$ *a restricted weight vector with restricted weight* $\mu \in \mathfrak{a}^*$, $e_0 \in \mathfrak{g}^\phi$ *and* $i, j \in \mathbf{Z}_+$. *Then*

$$e_0 \cdot (q_{-2\phi}^j q_{-\phi}^i \cdot v) = y_{ij} \cdot v,$$

where $y_{ij} \in \mathfrak{N}_{-\phi}$ *is given by the formula*

$$y_{ij} = -\frac{1}{4}j[\theta e_0, q_{-\phi}]q_{-2\phi}^{j-1}q_{-\phi}^{i}$$

$$- \sum_{m=1}^{i} 2((\mu + \rho_\phi)(h_\phi) + 2 - 4m)q_{-2\phi}^{j}q_{-\phi}^{i-m}(\theta e_0)q_{-\phi}^{m-1},$$

where ρ_ϕ *is as in Lemma 6.4. Moreover, suppose in addition that* $\dim \mathfrak{g}^{2\phi} = 1$ *and that there exists an element* $r_{-2\phi} \in \mathfrak{g}^{-2\phi}$ *such that* $r_{-2\phi}^2 = q_{-2\phi}$. *Then*

$$r_{-2\phi}e_0 \cdot (r_{-2\phi}^{j}q_{-\phi}^{i} \cdot v) = y'_{ij} \cdot v,$$

where $y'_{ij} \in \mathfrak{N}_{-\phi}$ *is given by the formula*

$$y'_{ij} = -\frac{1}{8}j[\theta e_0, q_{-\phi}]r_{-2\phi}^{j-1}q_{-\phi}^{i}$$

$$- \sum_{m=1}^{i} 2((\mu + \rho_\phi)(h_\phi) + 2 - 4m)r_{-2\phi}^{j+1}q_{-\phi}^{i-m}(\theta e_0)q_{-\phi}^{m-1}.$$

Proof. We may assume that k is algebraically closed and that $e_0 = e_\phi$, so that $\theta e_0 = -f_\phi$. To prove the first assertion, note that

$$e_\phi \cdot (q_{-2\phi}^{j}q_{-\phi}^{i} \cdot v) = \sum_{l=1}^{j} q_{-2\phi}^{j-l}[e_\phi, q_{-2\phi}]q_{-2\phi}^{l-1}q_{-\phi}^{i} \cdot v$$

$$+ \sum_{m=1}^{i} q_{-2\phi}^{j}q_{-\phi}^{i-m}[e_\phi, q_{-\phi}]q_{-\phi}^{m-1} \cdot v.$$

By Lemma 7.3, the first term on the right is $\frac{1}{4}j[f_\phi, q_{-\phi}]q_{-2\phi}^{j-1}q_{-\phi}^{i} \cdot v$. To handle the second term, use Lemma 6.4. Since $v \in V^{\mathfrak{m}}$, Lemma 6.2 shows that the second term is

$$\sum_{m=1}^{i} 2q_{-2\phi}^{j}q_{-\phi}^{i-m}((\rho_\phi - \phi)(h_\phi)f_\phi + f_\phi h_\phi)q_{-\phi}^{m-1} \cdot v.$$

But it was shown in the proof of Lemma 6.5 that $h_\phi q_{-\phi}^{m-1} = q_{-\phi}^{m-1}(h_\phi - 4(m-1))$. Thus the term becomes

$$\sum_{m=1}^{i} 2((\mu + \rho_\phi)(h_\phi) + 2 - 4m)q_{-2\phi}^{j}q_{-\phi}^{i-m}f_\phi q_{-\phi}^{m-1} \cdot v,$$

and this proves the first assertion of the lemma.

Now suppose that $\dim \mathfrak{g}^{2\phi} = 1$ and that $r_{-2\phi}^2 = q_{-2\phi}$ ($r_{-2\phi} \in \mathfrak{g}^{-2\phi}$). Then

$$r_{-2\phi}e_\phi \cdot (r_{-2\phi}^j q_{-\phi}^i \cdot v) = \sum_{l=1}^{j} r_{-2\phi}^{j-l+1}[e_\phi, r_{-2\phi}]r_{-2\phi}^{l-1}q_{-\phi}^i \cdot v$$
$$+ \sum_{m=1}^{i} r_{-2\phi}^{j+1}q_{-\phi}^{i-m}[e_\phi, q_{-\phi}]q_{-\phi}^{m-1} \cdot v.$$

The second term is treated exactly as in the first part of the proof, and all that remains is to show that the first term is $(1/8)j[f_\phi, q_{-\phi}]r_{-2\phi}^{j-1}q_{-\phi}^i \cdot v$. But $[f_\phi, q_{-\phi}] = 2f_n[f_\phi, f_n]$ and $[f_\phi, f_n] = \pm 4r_{-2\phi}$ as in the proof of Theorem 8.2, and so

$$[f_\phi, q_{-\phi}] = \pm 8f_n r_{-2\phi}$$

and

$$[e_\phi, r_{-2\phi}] = \pm\frac{1}{4}[e_\phi, [f_\phi, f_n]] = \pm f_n,$$

by Lemma 4.15. Thus the two indicated terms are equal, and the lemma is proved. Q.E.D.

We can now prove:

Theorem 8.6. (*The transfer principle for conical vectors.*) *Let Q be the polynomial algebra in two variables over k, and let $a_0 \in Q$. Also, let $c_0 \in k$. In continuation of the notation of Theorem 8.4, let Σ_+ be a positive system in Σ, $\alpha \in \Sigma_+$ a simple restricted root such that $2\alpha \in \Sigma$, $h_\alpha \in \mathfrak{a}$ as defined in §2, $\nu \in \mathfrak{a}^*$ such that $\nu(h_\alpha) = c_0$, X^ν the twisted induced $\mathfrak{g}$-module (see §2) and $x_0 \in X^\nu$ the canonical generator. Then the truth or falsity of the assertion "$a_0(q_{-\alpha}, q_{-2\alpha}) \cdot x_0$ is a conical vector in X^ν" depends only on a_0 and c_0, and not on $\mathfrak{g}$, θ, $\mathfrak{a}$, Σ_+, α or ν (except that $\nu(h_\alpha) = c_0$). Moreover, suppose in addition that $\dim \mathfrak{g}^{2\alpha} = 1$ and that there exists an element $r_{-2\alpha} \in \mathfrak{g}^{-2\alpha}$ such that $r_{-2\alpha}^2 = q_{-2\alpha}$. Then the truth or falsity of the assertion "$a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$ is a conical vector in X^ν" depends only on a_0 and c_0, and not on $\mathfrak{g}$, θ, $\mathfrak{a}$, Σ_+, α, $r_{-2\alpha}$ or ν (where $\nu(h_\alpha) = c_0$).*

Proof. Write $Q = k[x, y]$ and $a_0 = \sum_{i,j=0}^t b_{ij}x^i y^j$ ($t \in \mathbb{Z}_+$ and $b_{ij} \in k$) and assume $a_0 \neq 0$. In view of Theorem 5.1 (Cases 3 and 4), $a_0(q_{-\alpha}, q_{-2\alpha}) \cdot x_0$ is a nonzero $\mathfrak{m}$-invariant vector in X^ν. Let e_0 be a B_θ-nonisotropic vector in $\mathfrak{g}^\alpha$. Then by Corollary 4.3 and Lemma 6.13 (see the proof of Theorem 6.17), $e_0 \cdot (a_0(q_{-\alpha}, q_{-2\alpha}) \cdot x_0) = 0$ if and only if $a_0(q_{-\alpha}, q_{-2\alpha}) \cdot x_0$ is conical. But by Lemma 8.5, this is the case if and only if

$$\sum_{i,j=0}^{t} b_{ij}y_{ij} = 0 \quad \text{in } \mathfrak{N}_{-\alpha}, \tag{6}$$

where $y_{ij} \in \mathfrak{N}_{-\alpha}$ is as in Lemma 8.5, with ϕ replaced by α and μ by $\nu-\rho$ ($\rho = ½\Sigma(\dim \mathfrak{g}^{\psi})\psi$, $\psi \in \Sigma_+$). But $(\nu - \rho + \rho_\alpha)(h_\alpha) = \nu(h_\alpha) = c_0$ by Lemma 6.14, so that

$$y_{ij} = -\frac{1}{4} j(\theta e_0) q_{-2\alpha}^{j-1} q_{-\alpha}^{i+1} + \frac{1}{4} j q_{-\alpha}(\theta e_0) q_{-2\alpha}^{j-1} q_{-\alpha}^{i}$$

$$- \sum_{m=1}^{i} 2(c_0 + 2 - 4m) q_{-2\alpha}^{j} q_{-\alpha}^{i-m} (\theta e_0) q_{-\alpha}^{m-1}.$$

Since θe_0 is a B_θ-nonisotropic vector in $\mathfrak{g}^{-\alpha}$, (6) is an equation of the form treated in Theorem 8.4, with ϕ replaced by α, and with the a_i and b_i in Theorem 8.4 dependent only on a_0 and c_0. That theorem now implies the first assertion of the present one.

Now assume that $\dim \mathfrak{g}^{2\alpha} = 1$ and that $r_{-2\alpha}^2 = q_{-2\alpha}$ $(r_{-2\alpha} \in \mathfrak{g}^{-2\alpha})$, and let $a_0 \neq 0$ and e_0 be as above. By Case 3 of Theorem 5.1, $a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$ is a nonzero $\mathfrak{m}$-invariant vector in X^ν. Also, since $r_{-2\alpha}$ is a nonzero element of $\mathfrak{N}^-$, $r_{-2\alpha} e_0 \cdot (a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0) = 0$ if and only if

$$e_0 \cdot (a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0) = 0,$$

and this is true if and only if $a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$ is conical, as above. Combining the last parts of Lemma 8.5 and Theorem 8.4 as above, we get the last assertion of the theorem. Q.E.D.

Remark. Of course, the above proof in principle provides an explicit reformulation of the assertion "$a_0(q_{-\alpha}, q_{-2\alpha}) \cdot x_0$ is a conical vector in X^ν" in terms of a_0 and c_0 alone, and similarly for $a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$, under the extra hypotheses. But these reformulations are much too complicated to be useful in determining directly the conical vectors in the induced modules X^ν. Instead, we shall compute the conical vectors for a special $\mathfrak{g}$ (see §9), and then use Theorem 8.6 to obtain them for general $\mathfrak{g}$. The determination of the conical vectors in the special case is not trivial, but at least it *can* be done.

9. **A special case.** Following the plan indicated by Theorem 8.6, we shall determine all the conical vectors in all the twisted induced modules X^ν ($\nu \in \mathfrak{a}^*$) for a special semisimple symmetric Lie algebra $(\mathfrak{g}, \theta)$. Here $(\mathfrak{g}, \theta)$ will have essentially the same structure as the real semisimple Lie algebra $\mathfrak{su}(2, 1)$. Our methods will be special; in fact, one of our main points is that it is too difficult to compute directly the conical vectors in general (cf. §8). We are grateful to L. Corwin and N. Wallach for their help in carrying out this special case (see the introduction).

Assume k is algebraically closed. Let $\mathfrak{g} = \mathfrak{sl}(3, k)$, the simple Lie algebra of all traceless 3×3 matrices over k. Let $i = (-1)^{1/2}$, and let $\mathfrak{k} \subset \mathfrak{g}$ and $\mathfrak{p} \subset \mathfrak{g}$ be the spaces of matrices

$$\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & ia_{21} \\ -a_{13} & -ia_{12} & a_{11} \end{pmatrix}\right\} \quad \text{and} \quad \left\{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & 0 & -ib_{21} \\ b_{13} & ib_{12} & -b_{11} \end{pmatrix}\right\},$$

respectively, where $a_{ij}, b_{ij} \in k$ and $2a_{11} + a_{22} = 0$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, so that the linear automorphism θ of $\mathfrak{g}$ which is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$ is a Lie algebra automorphism. Thus $(\mathfrak{g}, \theta)$ is a semisimple symmetric Lie algebra with symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

For all $l, m = 1, 2, 3$, let E_{lm} denote the 3×3 matrix which is 1 in the (l, m)-entry and 0 in all other entries. Let $\mathfrak{a}$ be the one-dimensional subspace of $\mathfrak{p}$ spanned by the matrix $h = 2(E_{11} - E_{33})$. Then $\mathfrak{a}$ is a splitting Cartan subspace of $\mathfrak{p}$. Let α be the linear functional on $\mathfrak{a}$ which is 2 on h. Then the set Σ of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is $\{\pm\alpha, \pm 2\alpha\}$, $\mathfrak{g}^0$ is the set of traceless diagonal matrices, $\mathfrak{g}^\alpha$ is the span of E_{12} and E_{23}, $\mathfrak{g}^{-\alpha}$ is the span of E_{21} and E_{32}, $\mathfrak{g}^{2\alpha}$ is the span of E_{13}, and $\mathfrak{g}^{-2\alpha}$ is the span of E_{31}. Also, let h' be the matrix $E_{11} - 2E_{22} + E_{33}$. Then the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$ is the span of h', and $\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$.

Let Σ_+ be the positive system in Σ consisting of α and 2α. Then α is the unique simple restricted root. Since $\alpha(h) = 2$, $h = h_\alpha$ as defined in §2.

The Killing form B of $\mathfrak{g}$ is given by the formula $B(x, y) = 6 \operatorname{tr} xy$. Thus on $\mathfrak{g}^{-\alpha}$, the form $B_\theta(x, y) = -B(x, \theta y)$ is given by the formula

$$B_\theta(aE_{21} + bE_{32}, cE_{21} + dE_{32}) = -6i(ad + bc)$$

$(a, b, c, d \in k)$, and on $\mathfrak{g}^{-2\alpha}$, B_θ is given by

$$B_\theta(aE_{31}, bE_{31}) = 6ab$$

$(a, b \in k)$. Hence $\{(12)^{-1/2}(E_{21} + iE_{32}), (12)^{-1/2}(iE_{21} + E_{32})\}$ is a B_θ-orthonormal basis of $\mathfrak{g}^{-\alpha}$, and $\{6^{-1/2}E_{31}\}$ is a B_θ-orthonormal basis of $\mathfrak{g}^{-2\alpha}$. Since the canonical elements $p_{-\alpha} \in S^2(\mathfrak{g}^{-\alpha})^{\mathfrak{m}}$ and $p_{-2\alpha} \in S^2(\mathfrak{g}^{-2\alpha})^{\mathfrak{m}}$ (see §4) are the sums of the squares of the members of B_θ-orthonormal bases of $\mathfrak{g}^{-\alpha}$ and $\mathfrak{g}^{-2\alpha}$, respectively, we have

$$p_{-\alpha} = (i/3)E_{21}E_{32} \quad \text{and} \quad p_{-2\alpha} = (1/6)E_{31}^2.$$

The element $x_\alpha \in \mathfrak{a}$ (see §2) is $(1/12)(E_{11} - E_{33})$, so that $(\alpha, \alpha) =$

$B(x_\alpha, x_\alpha) = 1/12$. Hence

$$q_{-\alpha} = 24\lambda(p_{-\alpha}) = 4i(E_{21}E_{32} + E_{32}E_{21}) = 8iE_{21}E_{32} + 4iE_{31} \in \mathfrak{N}^{\mathfrak{m}}_{-\alpha}$$

and

$$q_{-2\alpha} = 6\lambda(p_{-2\alpha}) = E_{31}^2 \in \mathfrak{N}^{\mathfrak{m}}_{-\alpha},$$

in the notation of §5. We may choose $r_{-2\alpha} = E_{31} \in \mathfrak{g}^{-2\alpha}$ (see Theorem 8.6), since $\dim \mathfrak{g}^{2\alpha} = 1$ and $E_{31}^2 = q_{-2\alpha}$. By Theorem 5.1 (Case 3), $\mathfrak{N}^{\mathfrak{m}}_{-\alpha}$ is the polynomial algebra $k[q_{-\alpha}, r_{-2\alpha}]$.

Let $\nu \in \mathfrak{a}^*$. We want to determine the conical vectors in the twisted induced $\mathfrak{g}$-module $X^\nu = V^{\nu-\rho}$ induced from the subalgebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$, where $\rho = 2\alpha \in \mathfrak{a}^*$ and $\mathfrak{n} = \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha}$ (see §2). Let x_0 be the canonical generator of X^ν. Then

$$(X^\nu)^{\mathfrak{m}} = \mathfrak{N}^{\mathfrak{m}}_{-\alpha} \cdot x_0 = k[q_{-\alpha}, r_{-2\alpha}] \cdot x_0.$$

Thus we must determine the polynomials a_0 in two variables over k such that $\mathfrak{n} \cdot (a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0) = 0$.

It is hard to guess what conical vectors should look like, but once we know, it is relatively easy to prove that they are in fact conical (in the present special case):

Lemma 9.1. *Suppose $\nu(h_\alpha) = 2l$, l a positive integer, and let*

$$x = (q_{-\alpha} - 4i(l-1)r_{-2\alpha})(q_{-\alpha} - 4i(l-3)r_{-2\alpha}) \cdots$$
$$(q_{-\alpha} + 4i(l-3)r_{-2\alpha})(q_{-\alpha} + 4i(l-1)r_{-2\alpha}) \cdot x_0$$

in X^ν. Then x is a conical vector.

Proof. Since $E_{13} = [E_{12}, E_{23}]$, $\mathfrak{g}^\alpha$ generates $\mathfrak{n}$, and so it is sufficient to show that $E_{12} \cdot x = E_{23} \cdot x = 0$. By straightforward computation, using the matrix product relation $E_{\alpha\beta}E_{\gamma\delta} = E_{\alpha\delta}$ if $\beta = \gamma$ and $= 0$ if $\beta \neq \gamma$ ($\alpha, \beta, \gamma, \delta = 1, 2, 3$), we have the following commutation relations in the universal enveloping algebra of $\mathfrak{g}$:

$$[E_{12}, q_{-\alpha}] = 4iE_{32} + 2iE_{32}h_\alpha + 4iE_{32}h', \quad [E_{12}, r_{-2\alpha}] = -E_{32},$$

$$[E_{23}, q_{-\alpha}] = 4iE_{21} + 2iE_{21}h_\alpha - 4iE_{21}h', \quad [E_{23}, r_{-2\alpha}] = E_{21}.$$

Let a be any one of the factors $q_{-\alpha} + 4ijr_{-2\alpha}$ ($j = -(l-1), -(l-3), \cdots, l-1$) appearing in the expression for x in the statement of the lemma. Then $[h_\alpha, a] = -4a$ and $[h', a] = 0$. Also $h_\alpha \cdot x_0 = (\nu - \rho)(h_\alpha)x_0 = (2l-4)x_0$ and $h' \cdot x_0 = 0$. The above commutation relations thus give

$$E_{12}\cdot x=[E_{12},(q_{-\alpha}-4i(l-1)r_{-2\alpha})](q_{-\alpha}-4i(l-3)r_{-2\alpha})$$
$$\cdots(q_{-\alpha}+4i(l-1)r_{-2\alpha})\cdot x_0$$
$$+(q_{-\alpha}-4i(l-1)r_{-2\alpha})[E_{12},(q_{-\alpha}-4i(l-3)r_{-2\alpha})]$$
$$\cdots(q_{-\alpha}+4i(l-1)r_{-2\alpha})\cdot x_0+\cdots$$
$$=(4iE_{32}+2iE_{32}h_\alpha+4iE_{32}h'+4i(l-1)E_{32})(q_{-\alpha}-4i(l-3)r_{-2\alpha})$$
$$\cdots(q_{-\alpha}+4i(l-1)r_{-2\alpha})\cdot x_0$$
$$+(q_{-\alpha}-4i(l-1)r_{-2\alpha})(4iE_{32}+2iE_{32}h_\alpha+4iE_{32}h'+4i(l-3)E_{32})$$
$$\cdots(q_{-\alpha}+4i(l-1)r_{-2\alpha})\cdot x_0+\cdots$$
$$=(4iE_{32}+2iE_{32}(-4(l-1)+2l-4)+4i(l-1)E_{32})(q_{-\alpha}-4i(l-3)r_{-2\alpha})$$
$$\cdots(q_{-\alpha}+4i(l-1)r_{-2\alpha})\cdot x_0$$
$$+(q_{-\alpha}-4i(l-1)r_{-2\alpha})(4iE_{32}+2iE_{32}(-4(l-2)+2l-4)+4i(l-3)E_{32})$$
$$\cdots(q_{-\alpha}+4i(l-1)r_{-2\alpha})\cdot x_0+\cdots$$
$$=0+0+\cdots=0.$$

A similar computation shows that $E_{23}\cdot x_0=0$. However, x must be written in the "opposite order," as

$$(q_{-\alpha}+4i(l-1)r_{-2\alpha})(q_{-\alpha}+4i(l-3)r_{-2\alpha})$$
$$\cdots(q_{-\alpha}-4i(l-3)r_{-2\alpha})(q_{-\alpha}-4i(l-1)r_{-2\alpha})\cdot x_0,$$

to make the computation exactly parallel to the above one. Q.E.D.

Remark. Because of the flexibility allowed in writing the expression for x in either order in the above proof, we could prove easily that x is conical without appealing to the difficult commutation relations in $\mathfrak{N}^-$. This flexibility is lost for Lie algebras $\mathfrak{g}$ in which the double root space $\mathfrak{g}^{2\alpha}$ is more than one-dimensional, since the "square root" $r_{-2\alpha}$ of $q_{-2\alpha}$ does not exist.

Now we turn to the uniqueness of the conical vectors.

Lemma 9.2. *Let $a_0(y, z)$ be a polynomial in two variables over k. Then $a_0(q_{-\alpha}, r_{-2\alpha})\cdot x_0$ is a conical restricted weight vector in X^ν if and only if either a_0 is a nonzero scalar or else $\nu(h_\alpha)=2l$, where l is a positive integer, and a_0 is a nonzero multiple of*

$$a_l = (y - 4i(l-1)z)(y - 4i(l-3)z) \cdots (y + 4i(l-3)z)(y + 4i(l-1)z).$$

If l is even, then

$$a_l = \prod_{j=1;\, j \text{ odd}}^{l-1} (y^2 + 16j^2z^2),$$

and if l is odd,

$$a_l = y \prod_{j=2;\, j \text{ even}}^{l-1} (y^2 + 16j^2z^2).$$

Proof. Let $\mathfrak{h} = \mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and the elements y of $\mathfrak{h}$ can be written $y = y_1E_{11} + y_2E_{22} + y_3E_{33}$, where $y_i \in k$ and $y_1 + y_2 + y_3 = 0$. Define $\lambda_1, \lambda_2, \lambda_3 \in \mathfrak{h}^*$ by the formulas

$$\lambda_1(y) = y_1 - y_2, \quad \lambda_2(y) = y_2 - y_3 \quad \text{and} \quad \lambda_3(y) = y_1 - y_3.$$

Then the set R of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is $\{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3\}$. Denoting the root spaces for $\mathfrak{g}$ with respect to $\mathfrak{h}$ by $\mathfrak{g}^{\pm\lambda_i}$, we have $\mathfrak{g}^{\lambda_1} = kE_{12}$, $\mathfrak{g}^{\lambda_2} = kE_{23}$, $\mathfrak{g}^{\lambda_3} = kE_{13}$, $\mathfrak{g}^{-\lambda_1} = kE_{21}$, $\mathfrak{g}^{-\lambda_2} = kE_{32}$ and $\mathfrak{g}^{-\lambda_3} = kE_{31}$. Let $R_+ = \{\lambda_1, \lambda_2, \lambda_3\}$, so that R_+ is a positive system in R. Then the previously defined subalgebra $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$ is the same as the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \coprod \mathfrak{g}^\lambda\ (\lambda \in R_+)$, and $\mathfrak{n} = \coprod \mathfrak{g}^\lambda\ (\lambda \in R_+)$. Let $\rho' \in \mathfrak{h}^*$ be the linear functional which is the previously defined ρ on $\mathfrak{a}$ and 0 on $\mathfrak{m}$. Then $\rho' = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)$, i.e., ρ' is half the sum of the positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Also, define $\nu' \in \mathfrak{h}^*$ by $\nu' = \nu$ on $\mathfrak{a}$ and $\nu' = 0$ on $\mathfrak{m}$. Then the previously defined twisted induced $\mathfrak{g}$-module X^ν is the same as the Verma module associated with ν', in the sense of [2, §7.1.4]. That is, X^ν is the $\mathfrak{g}$-module induced by the character of $\mathfrak{b}$ which is $\nu' - \rho'$ on $\mathfrak{h}$ and 0 on $\mathfrak{n}$.

In order to describe the Weyl group W_R of $\mathfrak{g}$ with respect to $\mathfrak{h}$, let $\mathfrak{h}_1$ be the space of all (not necessarily traceless) 3×3 diagonal matrices and let $\mu_1, \mu_2, \mu_3 \in \mathfrak{h}_1^*$ be the basis of $\mathfrak{h}_1^*$ dual to the basis E_{11}, E_{22}, E_{33} of $\mathfrak{h}_1$. Now $\mathfrak{h}^*$ may be identified with the space of k-linear combinations of μ_1, μ_2 and μ_3, modulo the subspace $k(\mu_1 + \mu_2 + \mu_3)$. Then W_R is the group of automorphisms of $\mathfrak{h}^*$ induced by the six permutations of μ_1, μ_2 and μ_3.

Let $\nu_1 \in \mathfrak{a}^*$, and define $\nu_1' \in \mathfrak{h}^*$ to be ν_1 on $\mathfrak{a}$ and 0 on $\mathfrak{m}$. Then $x_1 \in X^\nu$ is a conical vector with restricted weight ν_1 if and only if x_1 is a (nonzero) $\mathfrak{n}$-invariant vector with weight ν_1' for the action of $\mathfrak{h}$ on X^ν. But there exists a nonzero $\mathfrak{n}$-invariant vector in X^ν with weight $\nu_2 \in \mathfrak{h}^*$ only if there exists $w \in W_R$ such that $\nu_2 + \rho' = w\nu'$ and $\nu' - (\nu_2 + \rho')$ is a nonnegative integral linear combination of the elements of R_+, by [2, Proposition

7.6.2]. Moreover, the $\mathfrak{n}$-invariant vectors in X^ν with weight ν_2 form at most a one-dimensional space, by a theorem of Verma [2, Théorème 7.6.6]. Let Z be the intersection of the conical space of X^ν with the restricted weight space corresponding to ν_1. It follows that if $Z \neq 0$, then $\dim Z = 1$, and in this case, either $\nu_1 = \nu - \rho$, or else $\nu_1 = -\nu - \rho$ and $\nu = l\alpha$ (i.e., $\nu(h_\alpha) = 2l$), where l is a nonnegative integer. Now apply Lemma 9.1. (If $l = 0$, then $\nu = 0$, $\nu_1 = -\rho$ and Z is the span of x_0.) Q.E.D.

10. Conclusions. We are now ready to combine the results of §§5, 6, 8 and 9 to remove the hypothesis "$2\alpha \notin \Sigma$" from Theorems 6.17 and 6.18.

Let $(\mathfrak{g}, \theta)$ be a semisimple symmetric Lie algebra over the field k of characteristic zero, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the symmetric decomposition of $(\mathfrak{g}, \theta)$, $\mathfrak{a}$ a splitting Cartan subspace of $\mathfrak{p}$, $\Sigma \subset \mathfrak{a}^*$ the corresponding restricted root system, $\Sigma_+ \subset \Sigma$ a positive system, and $\rho \in \mathfrak{a}^*$ as defined in §2. For every $\phi \in \Sigma$, define $h'_\phi \in \mathfrak{a}$ to be h_ϕ if $\dim \mathfrak{g}^\phi > 1$ (see §2) and $2h_\phi$ if $\dim \mathfrak{g}^\phi = 1$. Let s_ϕ be the Weyl reflection with respect to ϕ (see §2). Also, let q_ϕ and $q_{2\phi}$ be the elements of the universal enveloping algebra of $\mathfrak{g}$ defined in §5; if $2\phi \notin \Sigma$, take $q_{2\phi} = 0$.

Here are our main results, which generalize Theorems 6.17 and 6.18:

Theorem 10.1. *Let $\alpha \in \Sigma_+$ be a simple restricted root and $\nu \in \mathfrak{a}^*$. Let Y be the subspace of the twisted induced $\mathfrak{g}$-module X^ν spanned by the conical restricted weight vectors with restricted weights of the form $\nu - \rho + c\alpha$ $(c \in k)$; if* $\dim \mathfrak{a} = 1$, *then Y is the conical space of X^ν. Then* $\dim Y$ *is either 1 or 2. If $\nu(h'_\alpha)$ is not a positive even integer, then Y is the span of x_0, the canonical generator of X^ν. Suppose $\nu(h'_\alpha) = 2l$, l a positive integer. Then* $\dim Y = 2$. *Define the element ζ_l in the universal enveloping algebra of $\mathfrak{g}$ as follows: If* $\dim \mathfrak{g}^\alpha > 1$ *and l is even,*

$$\zeta_l = \prod_{j=1;\, j\ odd}^{l-1} (q_{-\alpha}^2 + 16j^2 q_{-2\alpha});$$

if $\dim \mathfrak{g}^\alpha > 1$ *and l is odd,*

$$\zeta_l = q_{-\alpha} \prod_{j=2;\, j\ even}^{l-1} (q_{-\alpha}^2 + 16j^2 q_{-2\alpha});$$

and if $\dim \mathfrak{g}^\alpha = 1$, *$\zeta_l = f^l$, where f is a nonzero element of $\mathfrak{g}^{-\alpha}$. Then Y has basis $\{x_0, \zeta_l \cdot x_0\}$, and $\zeta_l \cdot x_0$ is a conical restricted weight vector in X^ν with restricted weight $s_\alpha\nu - \rho$.*

Theorem 10.2. *Let α be a simple restricted root, let $\mu, \nu \in \mathfrak{a}^*$, and suppose that $\mu - \nu$ is of the form $c\alpha$ $(c \in k)$. (If* $\dim \mathfrak{a} = 1$, *then this is automa-*

tic.) Then $\mathrm{Hom}_{\mathfrak{g}}(X^\mu, X^\nu)$ *is at most one-dimensional, and* $\dim \mathrm{Hom}_{\mathfrak{g}}(X^\mu, X^\nu) = 1$ *if and only if either* $\mu = \nu$, *or else* $\mu = s_\alpha \nu$ *and* $\nu(h'_\alpha)$ *is a nonnegative even integer. Moreover,* $\dim \mathrm{Hom}_{\mathfrak{g}}(X^\mu, X^\nu) = 1$ *if and only if* X^μ *is isomorphic to a* $\mathfrak{g}$*-submodule of* X^ν.

Proof. Theorem 10.2 follows from Theorem 10.1, just as in the proof of Theorem 6.18. To prove Theorem 10.1, note first that the case $2\alpha \notin \Sigma$ is covered in Theorem 6.17. Suppose that $2\alpha \in \Sigma$. It is clearly sufficient to assume now that k is algebraically closed. By Lemma 6.16, $Y = (\mathfrak{N}^{\mathfrak{m}}_{-\alpha} \cdot x_0)^{\mathfrak{n}}$. Moreover, $\mathfrak{N}^{\mathfrak{m}}_{-\alpha}$ is the polynomial algebra $k[q_{-\alpha}, q_{-2\alpha}]$ if $\dim \mathfrak{g}^{2\alpha} > 1$ and $\mathfrak{N}^{\mathfrak{m}}_{-\alpha}$ is the polynomial algebra $k[q_{-\alpha}, r_{-2\alpha}]$ if $\dim \mathfrak{g}^{2\alpha} = 1$, by Theorem 5.1; here $r_{-2\alpha} \in \mathfrak{g}^{-2\alpha}$ and $r^2_{-2\alpha} = q_{-2\alpha}$ (such an element exists since k is algebraically closed). Hence Y is the set of $\mathfrak{m} \oplus \mathfrak{n}$-invariants in X^ν of the form $a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$ if $\dim \mathfrak{g}^{2\alpha} = 1$ and of the form $a_0(q_{-\alpha}, q_{-2\alpha}) \cdot x_0$ if $\dim \mathfrak{g}^{2\alpha} > 1$, where a_0 ranges through the polynomials in two variables over k. The stage is set for the application of the transfer principle for conical vectors (Theorem 8.6). Suppose that $\dim \mathfrak{g}^{2\alpha} = 1$, and that $a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$ is a conical vector. If $\nu(h_\alpha)$ is not a positive even integer, then a_0 is a nonzero scalar, by the last part of Theorem 8.6, combined with Lemma 9.2. Suppose now that $\nu(h_\alpha) = 2l$, where l is a positive integer. Then the same two results show that $a_0(q_{-\alpha}, r_{-2\alpha}) \cdot x_0$ is a (nonzero) linear combination of x_0 and $\zeta_l \cdot x_0$, in the notation of the theorem. Conversely, $\zeta_l \cdot x_0$ is, in fact, a conical vector, again by Theorem 8.6 and Lemma 9.2 (or Lemma 9.1). This proves the present theorem in case $\dim \mathfrak{g}^{2\alpha} = 1$. If $\dim \mathfrak{g}^{2\alpha} > 1$, the theorem follows from the same argument, this time using the first part of Theorem 8.6. Note that since the polynomials a_l in Lemma 9.2 are polynomials in y and z^2, the space Y has the same description whether $\dim \mathfrak{g}^{2\alpha} = 1$ or $\dim \mathfrak{g}^{2\alpha} > 1$. Q.E.D.

Remark. (Cf. the Remark following Theorem 6.17.) In the notation of Theorem 10.1, $\nu(h'_\alpha)$ is a nonnegative even integer if and only if X^ν contains an $\mathfrak{m}$-invariant restricted weight vector with restricted weight $s_\alpha\nu - \rho$, or equivalently, a conical restricted weight vector with restricted weight $s_\alpha\nu - \rho$. But in general not every $\mathfrak{m}$-invariant restricted weight vector with restricted weight $s_\alpha\nu - \rho$ is conical.

Remark. If $\dim \mathfrak{a} = 1$ and $\dim \mathfrak{g}^\alpha > 1$, then $\nu(h'_\alpha) = \nu(h_\alpha)$ is a nonnegative even integer if and only if ν is a nonnegative integral multiple of the unique simple restricted root α.

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DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520