TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 208, 1975

UNIQUENESS AND α-CAPACITY ON THE GROUP 2^{ω} (1)

BY

WILLIAM R. WADE

ABSTRACT. We introduce a class of Walsh series $\mathcal{J}_\alpha^+$ for each $0 < \alpha < 1$ and show that a necessary and sufficient condition that a closed set $E \subseteq 2^\omega$ be a set of uniqueness for $\mathcal{J}_\alpha^+$ is that the α-capacity of E be zero.

1. Introduction. A Walsh series $S \equiv \sum_{k=0}^\infty a_k \psi_k$ is said to *belong to the class* $\mathfrak{A}$ if

$$\lim_{n\to\infty} 2^{-n} S_{2^n}(x) = 0 \quad \text{for all } x \in 2^\omega; \tag{1}$$

where

$$S_N(x) \equiv \sum_{k=0}^{N-1} a_k \psi_k(x)$$

for $N = 0, 1, \ldots$.

Let $0 \le \alpha \le 1$ and for each positive integer k set $[k] = 2^n$ where n is the nonnegative integer determined by $2^n \le k < 2^{n+1}$. A Walsh series $S \equiv \sum_{k=0}^\infty a_k \psi_k$ is said to *belong to the class* $\mathcal{J}_\alpha$ if

$$\sum_{k=1}^\infty a_k^2 [k]^{\alpha-1} < \infty. \tag{2}$$

The Walsh series S is said to *belong to the class* $\mathcal{J}_\alpha^+$ if in addition to (2) there exist integers $0 < n_1 < n_2 < \cdots$ such that

$$S_{2^{n_j}}(x) \ge 0 \quad \text{a.e.} \tag{3}$$

If $0 < \alpha < 1$ then $\mathcal{J}_\alpha \subseteq \mathfrak{A}$, since by Schwarz's inequality

$$[2^{-n} S_{2^n}(x)]^2 \le 2^{-n\alpha} \sum_{k=0}^{2^n-1} a_k^2 [k]^{\alpha-1}.$$

Received by the editors February 5, 1974 and, in revised form, May 2, 1974.

AMS (MOS) subject classifications (1970). Primary 42A56, 42A48; Secondary 42A62.

Key words and phrases. Haar functions, Walsh functions, sets of uniqueness, α-capacity.

(1) This research was partially sponsored by a University of Tennessee Faculty Research Grant.

Let $\mathcal{B}$ be a certain class of Walsh series. A subset E of the group 2^ω is said to be a *set of uniqueness for* $\mathcal{B}$ if $S \in \mathcal{B}$ and $\lim_{n\to\infty} S_{2^n}(x) = 0$ for $x \in 2^\omega \sim E$ imply that S is the zero series.

For each Borel set $E \subseteq 2^\omega$ let $\mathfrak{M}(E)$ denote the set of all nonnegative Borel measures concentrated on E with total variation 1. Let $0 < \alpha < 1$. We associate with each measure $\mu \in \mathfrak{M}(E)$ a *potential function*

$$\bar{U}_\mu(x) = \int_{2^\omega} K_\alpha(x-y)\, d\mu(y), \tag{4}$$

where K_α is the nonnegative, lower semicontinuous, integrable function $\{x\}^{-\alpha}$ introduced in [6]. Let

$$W_\alpha(E) \equiv \inf\{W_\alpha^\mu(E):\ \mu \in \mathfrak{M}(E)\},$$

where for each $\mu \in \mathfrak{M}(E)$,

$$W_\alpha^\mu(E) = \|\bar{U}_\mu\|_\infty. \tag{5}$$

Then E is said to be of *α-capacity zero* if $W_\alpha(E) = +\infty$.

Crittenden and Shapiro [3] have shown that a Borel set $E \subseteq 2^\omega$ is a set of uniqueness for $\mathfrak{A}$ if and only if E is countable. For each $\alpha \in (0, 1)$ we shall show that a closed set $E \subseteq 2^\omega$ is a set of uniqueness for $\mathfrak{T}_\alpha^+$ if and only if the α-capacity of E is zero. For a large class of null Walsh series which is contained in $\mathfrak{T}_\alpha^+$ see [8].

This author is indebted to Professor Victor L. Shapiro who first posed this problem in 1964 with $\mathfrak{T}_\alpha$ in place of $\mathfrak{T}_\alpha^+$. The analysis presented here would also solve the original problem if a group 2^ω analogue of Frostman's maximal principle were known. For this connection and a theorem concerning the trigonometric analogue of this problem see [1].

2. **Fundamental lemmas.** We begin this section quoting two results which are straightforward modifications of Theorem 2.9 and Lemma 3.2 in [6].

Lemma 1. *Let $E_1, E_2, \ldots$ be a nested sequence of closed subsets of 2^ω such that $E \equiv \bigcap_{n=1}^\infty E_n$ is a set of α-capacity zero. Then* $\lim_{n\to\infty} W_\alpha(E_n) = +\infty$.

Lemma 2. *Given a set $E \subseteq 2^\omega$ of positive α-capacity there is a measure $\mu \in \mathfrak{M}(E)$ such that its potential function is in $L^\infty(2^\omega)$ and satisfies*

$$\bar{U}_\mu(x) \geq W_\alpha(E) \tag{6}$$

for almost every $x \in E$.

The first lemma we prove is

Lemma 3. *If* $S = \sum_{k=0}^{\infty} a_k \psi_k \in \mathfrak{A}$ *and* c, d *are dyadic rationals in* $[0, 1]$, *then there is a Walsh series* $T \in \mathfrak{A}$ *and an integer* N *such that* $n > N$ *implies*

$$T_{2^n}(x) = S_{2^n}(x) \quad \text{for } x \in [c, d) \tag{7}$$

and

$$T_{2^n}(x) \equiv 0 \quad \text{for } x \notin [c, d).$$

To establish this result we define $j \circ k$ for each pair j, k of nonnegative integers by $\psi_{j \circ k} \equiv \psi_j \psi_k$. Let $P(x) = \sum_{k=0}^{M} \beta_k \psi_k(x)$ be the Walsh polynomial which is equal to 1 for $x \in [c, d)$ and equal to 0 elsewhere. Let $T = \sum_{k=0}^{\infty} \gamma_k \psi_k$ where

$$\gamma_k = \sum_{j=0}^{M} \beta_j a_{k \circ j}. \tag{8}$$

Then by Šneĭder [11, p. 285],

$$T_{2^n}(x) \equiv P(x) S_{2^n}(x), \qquad x \in 2^{\omega}, \tag{9}$$

when $2^n > M$. In particular, the choice of P forces T to have the desired properties.

Fine [4] has shown that a Walsh series S which converges to zero on an interval I with dyadic rational endpoints necessarily converges uniformly on I. It turns out that the 2^nth partial sums of S eventually *vanish* on I. In fact:

Lemma 4. *Let* F *be a closed subset of* $[0, 1]$ *and* $S = \sum_{k=0}^{\infty} a_k \psi_k \in \mathfrak{A}$. *Suppose further that* $\lim_{n \to \infty} S_{2^n}(x) = 0$ *a.e.* $x \in [0, 1] \sim F$ *and that* $\limsup_{n \to \infty} |S_{2^n}(x)| < \infty$ *for all but countably many* $x \in [0, 1] \sim F$. *Then for any interval* $(c, d) \subseteq [0, 1] \sim F$ *with dyadic rational endpoints there is an integer* N *such that* $n \geq N$ *and* $x \in (c, d)$ *imply* $S_{2^n}(x) \equiv 0$.

To prove Lemma 4 let T be the Walsh series corresponding to S and (c, d) given by Lemma 3. The conclusion of Lemma 3 and the hypotheses of Lemma 4 show us that T is a Walsh series, belonging to $\mathfrak{A}$, whose 2^nth partial sums converge to zero almost everywhere, are pointwise bounded off a countable set and satisfy (7) for n greater than some integer N. T is necessarily the zero series by the main theorem in [12]. Hence $S_{2^n}(x) \equiv 0$ for $x \in (c, d)$ and $n > M$ by (7).

3. The characterization.

Theorem. *Let $\alpha \in (0, 1)$ and E be a closed subset of the group 2^ω. Then a necessary and sufficient condition that E be a set of uniqueness for $\mathcal{T}_\alpha^+$ is that the α-capacity of E be zero.*

Necessity. Suppose the α-capacity of E is not zero. Then by definition there is at least one measure $\mu \in \mathfrak{M}(E)$ such that $W_\alpha^\mu(E) < \infty$. Let $d_0, d_1, \ldots$ represent the Walsh-Fourier-Stieltjes coefficients of μ and set $S = \Sigma_{k=0}^\infty d_k \psi_k$. S is not the zero series since $d_0 = \|\mu\| = 1$. Also, $\lim_{n\to\infty} S_{2n}(x) = 0$ for $x \in [0, 1] \sim E$ since μ is supported on E [3, p. 563]. Furthermore $S_{2n} \geq 0$ since $S_{2n} = D_{2n} * \mu$ and $D_{2n} \geq 0$. Hence it suffices to show

Lemma 5. *Let $0 < \alpha < 1$, E be a closed subset of the group 2^ω, and $\mu \in \mathfrak{M}(E)$. Then there is a positive constant B depending only on α such that*

$$\sum_{k=0}^\infty d_k^2 [k]^{\alpha - 1} \leq W_\alpha^\mu(E) \cdot B \tag{10}$$

where $d_0, d_1, \ldots$ are the Walsh-Fourier-Stieltjes coefficients of μ.

Let $b_0, b_1, \ldots$ represent the Walsh-Fourier coefficients of K_α. Harper [6] has shown that there is a positive constant B depending only on α such that

$$Bb_k = [k]^{\alpha-1}, \qquad k = 1, 2, \ldots. \tag{11}$$

For convenience let us define $[0]^{\alpha-1}$ so that (11) holds with $k = 0$.

To prove (10) we may suppose that $W_\alpha^\mu(E) < \infty$. In this case it is known that $\Sigma_{k=0}^\infty d_k^2 b_k = \int_{2^\omega} \overline{U}_\mu(x)\, d_\mu(x)$. Combining this with (11) we have

$$B^{-1} \sum_{k=0}^\infty d_k^2 [k]^{\alpha-1} = \int_{2^\omega} \overline{U}_\mu(x)\, d_\mu(x) \leq W_\alpha^\mu(E).$$

Sufficiency. Suppose E is of α-capacity zero. Let $S = \Sigma_{k=0}^\infty a_k \psi_k$ be a Walsh series belonging to $\mathcal{T}_\alpha^+$ such that $\lim_{n\to\infty} S_{2n}(x) = 0$ for $x \in 2^\omega \sim E$. We must show $a_k = 0$ for $k = 0, 1, \ldots$.

We first show that $a_0 = 0$. Let $\lambda\colon 2^\omega \to [0, 1]$ be defined by $\lambda(x_1, x_2, \ldots) = \Sigma_{k=1}^\infty x_k 2^{-k}$. Since λ is continuous [4] and E is compact, $\lambda(E)$ is necessarily closed in $[0, 1]$. Let $[0, 1] \sim \lambda(E) = \bigcup_{k=1}^\infty I_k$ where $I_1, I_2, \ldots$ is a sequence of open intervals with dyadic rational endpoints. Finally define a sequence of closed sets $E_1 \supset E_2 \supset \cdots$ in the group 2^ω by

$$E_N = 2^\omega \sim \lambda^{-1}\left(\bigcup_{k=1}^{N} I_k\right).$$

Now as a Walsh series on $[0, 1]$, $\lim_{n\to\infty} S_{2^n}(x) = 0$ for $x \notin \lambda(E)$. Hence N applications of Lemma 4 allow us to conclude that for n sufficiently large, $S_{2^n}(x) \equiv 0$ for $x \in \bigcup_{k=1}^N I_k$. As a series on the group 2^ω, this means

(12) $$S_{2^n}(x) \equiv 0 \quad \text{for } x \in 2^\omega \sim E_N.$$

Let $0 < n_1 < n_2 < \cdots$ satisfy (3). Since $a_0 \equiv \int_{2^\omega} S_{2^{n_j}}(x)\,dx$ we conclude that $a_0 \geq 0$. By (12),

(13) $$|a_0| = \int_{E_N} S_{2^{n_j}}(x)\,dx$$

for j sufficiently large. Use Lemma 2 to choose an equilibrium measure $\mu \in \mathfrak{M}(E)$ satisfying (6). Then by (13) and (3)

$$|a_0| \leq \frac{1}{W_\alpha(E_N)} \int_{E_N} S_{2^{n_j}} \bar{U}_\mu(x)\,dx$$

for j sufficiently large. Hence by (12) and Parseval we have

$$|a_0| \leq \frac{1}{W_\alpha(E_N)} \sum_{k=0}^{2^{n_j}-1} a_k b_k d_k$$

for j sufficiently large, where $b_0, b_1, \ldots$ are the Walsh-Fourier coefficients of K_α and $d_0, d_1, \ldots$ are the Walsh-Fourier-Stieltjes coefficients of μ. Applying Schwarz's inequality we have

$$a_0^2 \leq W_\alpha^{-2}(E_N) \sum_{k=0}^{2^{n_j}-1} b_k^2 d_k^2 [k]^{(1-\alpha)} \cdot \sum_{k=0}^{2^{n_j}-1} a_k^2 [k]^{(\alpha-1)}.$$

But $S \in \mathcal{T}_\alpha$ so by (11) and Lemma 5 we conclude that

(14) $$a_0^2 \leq \text{const } W_\alpha^{-1}(E_N).$$

Observe that $\lambda^{-1} \circ \lambda(E) = \bigcap_{N=1}^\infty E_N$. Now $\lambda^{-1} \circ \lambda(E) \sim E$ is at most countable and the α-capacity of E is zero, so the α-capacity of $\bigcap_{N=1}^\infty E_N$ must also be zero. Hence by Lemma 1, $\lim_{N\to\infty} W_\alpha(E_N) = +\infty$, which by (14) implies that $a_0 = 0$.

For future reference, let us call what we have just proved a lemma.

Lemma 6. *Let $S \in \mathcal{T}_\alpha^+$ and E be a closed set of α-capacity zero. If $\lim_{n\to\infty} S_{2^n}(x) = 0$ for $x \in 2^\omega \sim E$, then the constant term of S is zero.*

Suppose for some $m \geq 0$ that $a_k = 0$ for $k = 0, 1, \ldots, 2^m - 1$. We shall show that

$$a_k = 0 \quad \text{for } k = 2^m, 2^m + 1, \ldots, 2^{m+1} - 1 \tag{15}$$

thereby finishing the proof of the Theorem by induction.

To prove (15) fix an integer $l \in [2^m, 2^{m+1})$ and set

$$P(x) = D_{2^{m+1}}(l \cdot 2^{m+1} \dotplus x). \tag{16}$$

It is easy to see that $P(x) = \sum_{j=0}^{2^{m+1}-1} \beta_j^{(l)} \psi_j(x)$ where $\beta_j^{(l)} = \pm 1$ and that the matrix

$$A \equiv (\beta_j^{(l)}: j = 2^m, \ldots, 2^{m+1} - 1 \text{ and } l = 2^m, \ldots, 2^{m+1} - 1)$$

is nonsingular. For a similar result concerning Haar polynomials see [7].

Now set $T = \sum_{k=0}^{\infty} \gamma_k \psi_k$ where

$$\gamma_k = \sum_{j=0}^{2^{m+1}-1} \beta_j^{(l)} a_{k \circ j}. \tag{17}$$

A routine computation shows that $T \in \mathcal{T}_a$. As in the proof of Lemma 3,

$$T_{2^n}(x) = P(x) S_{2^n}(x), \qquad x \in 2^\omega,$$

for n sufficiently large. In particular $\lim_{n\to\infty} T_{2^n}(x) = 0$ for $x \in 2^\omega \sim E$, and $T_{2^{n_j}}(x) \geq 0$ a.e., $j = 1, 2, \ldots$. Hence $\gamma_0 = 0$ by Lemma 6. By the inductive hypotheses and (17) we conclude $0 = \sum_{j=2^m}^{2^{m+1}-1} \beta_j^{(l)} a_j$. This identity holds for each $l = 2^m, 2^m + 1, \ldots, 2^{m+1} - 1$ so we finally arrive at the matrix equation

$$A \cdot \begin{bmatrix} a_{2^m} \\ \cdot \\ \cdot \\ \cdot \\ a_{2^{m+1}-1} \end{bmatrix} = 0.$$

Since the matrix A is nonsingular (15) is established as required.

BIBLIOGRAPHY

1. A. Broman, *On two classes of trigonometric series*, Ph. D. Dissertation, University of Upsala, 1947.

2. L. Carleson, *Selected problems on exceptional sets*, Van Nostrand Math. Studies, no. 13, Van Nostrand, Princeton, N. J., 1967. MR 37 #1576.

3. R. B. Crittenden and V. L. Shapiro, *Sets of uniqueness in the group* 2^{ω}, Ann. of Math. (2) **81** (1965), 550–564. MR **31** #3783.

4. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372–414. MR **11**, 352.

5. B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. **103** (1960), 139–215. MR **22** #8232.

6. L. H. Harper, *Capacities of sets and harmonic analysis on the group* 2^{ω}, Trans. Amer. Math. Soc. **126** (1967), 303–315. MR **34** #6445.

7. J. R. McLaughlin and J. J. Price, *Comparison of Haar series with gaps with trigonometric series*, Pacific J. Math. **28** (1969), 623–627. MR **39** #3219.

8. F. Shipp, *Über Walsh-Fourier Reihen mit nicht-negativen Partialsummen*, Ann. Univ. Sci. Budapest Sect. Math. **12** (1969), 43–48.

9. V. L. Shapiro, *$U(\epsilon)$-sets for Walsh series*, Proc. Amer. Math. Soc. **16** (1965), 867–870. MR **31** #5035.

10. P. Sjölin, *An inequality of Paley and convergence a. e. of Walsh-Fourier series*, Ark. Mat. **7** (1969), 551–570. MR **39** #3222.

11. A. A. Šneĭder, *On the uniqueness of expansions for Walsh functions*, Mat. Sb. **24** (66) (1949), 279–300. (Russian) MR **11**, 352.

12. W. R. Wade, *A uniqueness theorem for Haar and Walsh series*, Trans. Amer. Math. Soc. **141** (1969), 187–194. MR **39** #4587.

13. ———, *Uniqueness of Haar series which are $(C, 1)$ summable to Denjoy integrable functions*, Trans. Amer. Math. Soc. **176** (1973), 489–498. MR 47 #704.

14. J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math. **55** (1923), 5–14.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916