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THE UNIQUENESS OF THE ONE-DIMENSIONAL PARABOSON FIELD

BY

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ABSTRACT. A paraboson analog of the one-dimensional boson field is discussed and a uniqueness result similar to a result of Putnam is obtained. It is shown that the paraboson operators must be unbounded.

Introduction. Green first introduced the parafermion and paraboson field algebras in 1953 [2]. Much is known about the representation of the parafermion field operators but there has been little mathematical work concerning representations in the paraboson case due to the unbounded nature of the operators. No bounded relations corresponding to the Weyl relations for bosons have been found so the paraboson operators must be treated in unbounded form. In one dimension the paraboson relations reduce to the single relation

$$C^*C^2 - C^2C^* = 2C,$$

where C is a creation operator and its adjoint C^* is an annihilation operator. This relation cannot be taken literally but must be given a suitable interpretation. We will prove a uniqueness result parallel to the ones proved by Putnam and Tillmann for bosons. As a corollary to the proof it will be shown that there are no bounded operators satisfying the paraboson relations. This is clear when the polar decomposition of C yields a self-adjoint operator with discrete spectrum, an assumption which is usually made but is not necessarily satisfied.

Uniqueness of the paraboson field. Let C be a closed, densely defined linear operator on a Hilbert space K satisfying

$$C^*C = CC^* + I. \tag{1}$$

Putnam [5] proved that C^*C then has purely discrete spectrum consisting of the nonnegative integers. Tillmann [8, p. 263] also showed that $\{C, K\}$ is unitarily equivalent to a direct sum of one-dimensional free boson fields. See also [6, Theorem 4.4.1].

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We will extend this result to the paraboson case but first we will need an analog of the one-dimensional free boson field.

DEFINITION. A one-dimensional free paraboson field of order $p > 0$ is a pair $\{C, K\}$, where K is a Hilbert space with orthonormal basis $\{e_j: j = 0, 1, 2, \ldots\}$ and C is the closed, densely defined operator on K such that $Ce_j = \gamma_j e_{j+1}$ where

$$\gamma_j = \begin{cases} \sqrt{j+1}, & j \text{ even}, \\ \sqrt{j+p}, & j \text{ odd}. \end{cases}$$

Note that when $p = 1$, $\gamma_j = \sqrt{j+1}$ and this gives the free boson field. p can be any nonnegative number and, unlike the parafermion case, need not be an integer.

We are now able to state our main result.

THEOREM. *Let C be a closed, densely defined linear operator on a Hilbert space K such that CC^* and C^*C commute,*

$$C^*C^2 = C^2C^* + 2C, \tag{2}$$

and either $CC^* + C^*C$ has discrete spectrum or $\{C, K\}$ is completely reducible. Then both of the latter two conditions hold and $\{C, K\}$ is unitarily equivalent to a direct sum of free paraboson fields.

The relation (1) implies that CC^* and C^*C commute while (2) only implies that formally their commutator vanishes. We will need that the spectral projections of CC^* and C^*C commute (which can easily be checked in the case of the free paraboson field) so we are forced to assume the commutativity directly. It is not sufficient to assume that the commutator of CC^* and C^*C vanishes on a dense invariant domain. See, for example, [4, pp. 603–606]. In the boson case the complete reducibility is automatic while for parabosons it is not. This is because the operator $n = \frac{1}{2}(CC^* + C^*C)$ which plays the part of the number operator may have continuous spectrum in the paraboson case when the representation is reducible.

The theorem will be proved in two parts. We first handle the discrete spectrum case and then show that when n has a partially continuous spectrum the representation is reducible.

Let $n = \frac{1}{2}(CC^* + C^*C)$. n is self-adjoint and nonnegative since CC^* and C^*C are nonnegative commuting self-adjoint operators. Let $n = \int \lambda E(d\lambda)$ be the spectral resolution of n. Define

$$D = \{w \in K: E(\Delta)w = w, \text{ for some bounded interval } \Delta\}.$$

D is dense in K and n is essentially self-adjoint on D. Since CC^* and C^*C commute with n, D is invariant under these operators. Thus D is a subset of the domains of C, C^*, CC^*, C^*C, C^2C^*, CC^*C and C^*CC^*. (2) implies that D is also a subset of the domain of C^*C^2. From these domain conditions and (2) it follows that for $w \in D$,

$$[C^*, C^2]w = 2Cw, \qquad [n, C]w = Cw.$$

Let $\sigma_d(n)$ be the discrete spectrum of n and assume $\sigma_d(n) \neq \emptyset$. Choose $q \in \sigma_d(n)$ such that $q - 1 \notin \sigma_d(n)$. Choose $v \in K$ such that $nv = qv$, $v \neq 0$. Then $v \in D$ and $nCv = C(n + 1)v = (q + 1)Cv$, so $Cv \in D$. Similarly, $C^kv \in D$ and $nC^kv = (q + k)C^kv$. Let M be the set of finite linear combinations of C^kv. $M \subseteq D$ so $M \subseteq \mathrm{Dom}(C) \cap \mathrm{Dom}(C^*)$. Clearly $CM \subseteq M$. We will show that $C^*M \subseteq M$. To see this let $w \in D$.

$$\begin{aligned}\langle nw, C^*v\rangle &= \langle Cnw, v\rangle = \langle (n-1)Cw, v\rangle \\ &= \langle Cw, (n-1)v\rangle = \langle Cw, (q-1)v\rangle \\ &= \langle w, (q-1)C^*v\rangle.\end{aligned}$$

Since n is essentially self-adjoint on D, $C^*v \in \mathrm{Dom}(n)$ and $nC^*v = (q-1)C^*v$. This implies that $C^*v = 0$ since $(q - 1) \notin \sigma_d(n)$.

$$C^*Cv = C^*Cv + CC^*v = 2nv = 2qv \in M.$$

If $k > 1$,

$$\begin{aligned}&C^*C^kv = C^*C^2(C^{k-2}v) = C^2C^*C^{k-2}v + 2C^{k-1}v,\\ (3)\qquad &C^*C^kv = C^{2j}C^*C^{k-2j}v + 2jC^{k-1}v \quad \text{if } j \leqslant \tfrac{1}{2}k.\end{aligned}$$

Therefore $C^*C^kv = \beta_kC^{k-1}v$ where

$$\beta_k = \begin{cases} k, & k \text{ even},\\ k - 1 + 2q, & k \text{ odd},\end{cases}$$

Let K_1 be the closure of M, and let $K_2 = M^\perp$ so that $K = K_1 \oplus K_2$. Let $v_j = \alpha_jC^jv$, where α_j is chosen to make v_j a unit vector, i.e.

$$\alpha_0 = \|v\|^{-1}, \qquad \alpha_j = \|v\|^{-1}\left[\prod_{k=1}^{j} \beta_k\right]^{-1/2}, \qquad j \geqslant 1.$$

Then $v_j \in K_1$, and if $j > k$,

$$\langle v_j, v_k\rangle = \alpha_j\alpha_k\langle C^jv, C^kv\rangle = \alpha_j\alpha_k\langle C^{j-k-1}v, C^{*(k+1)}C^kv\rangle = 0,$$

since $C^{*(k+1)}C^kv$ is proportional to C^*v. Therefore $\langle v_j, v_k\rangle = \delta_{jk}$. Simple calculations yield

$$Cv_j = \sqrt{\beta_{j+1}}\, v_{j+1}; \qquad C^*v_j = \sqrt{\beta_j}\, v_{j-1}, \quad \text{if } j \geq 1;$$

$$C^*Cv_j = \beta_{j+1}v_j; \qquad CC^*v_j = \beta_j v_j, \quad \text{if } j \geq 1.$$

Let $T = \sqrt{C^*C}$. $T^2 v_j = \beta_{j+1} v_j$ so $Tv_j = \sqrt{\beta_{j+1}}\, v_j$. If P_j projects on the one-dimensional space spanned by v_j, P_j commutes with all of the spectral projections of T. Therefore $P = \sum_{j=1}^{\infty} P_j$ does also. Thus $PT \subseteq TP$, so if $w \in \mathrm{Dom}(T)$ then $Pw \in \mathrm{Dom}(T)$. Since $\mathrm{Dom}(T) = \mathrm{Dom}(C)$, if $w \in \mathrm{Dom}(C)$ then $Pw \in \mathrm{Dom}(C)$. P projects onto K_1 so

(4) $$\mathrm{Dom}(C) = \mathrm{Dom}(C) \cap K_1 \oplus \mathrm{Dom}(C) \cap K_2.$$

A similar argument applied to $S = \sqrt{CC^*}$ yields

(5) $$\mathrm{Dom}(C^*) = \mathrm{Dom}(C^*) \cap K_1 \oplus \mathrm{Dom}(C^*) \cap K_2.$$

It is now a simple matter to prove the following four statements:

(i) $w \in \mathrm{Dom}(C) \cap K_2$ implies $Cw \in K_2$,
(ii) $w \in \mathrm{Dom}(C^*) \cap K_2$ implies $C^*w \in K_2$,
(iii) $w \in \mathrm{Dom}(C) \cap K_1$ implies $Cw \in K_1$,
(iv) $w \in \mathrm{Dom}(C^*) \cap K_1$ implies $C^*w \in K_1$.

We will prove (ii) and (iii). (i) and (iv) are done similarly. If $w \in \mathrm{Dom}(C^*) \cap K_2$ and $x \in M$, $\langle C^*w, x \rangle = \langle w, Cx \rangle = 0$ so $C^*w \in M^\perp = K_2$ which gives (ii). If $w \in \mathrm{Dom}(C) \cap K_1$ and $x \in \mathrm{Dom}(C^*) \cap K_2$, $\langle Cw, x \rangle = \langle w, C^*x \rangle = 0$ by (ii) so $Cw \in (\mathrm{Dom}(C^*) \cap K_2)^\perp = K_2^\perp = K_1$ since $\mathrm{Dom}(C^*) \cap K_2$ is dense in K_2 by (5). For $i = 1, 2$ define C_i, a linear operator on K_i, as the restriction of C to K_i. We will next show that C_i^* is the restriction of C^* to K_i.

Let $w \in \mathrm{Dom}(C^*) \cap K_i$. If $x \in \mathrm{Dom}(C_i)$ then

$$\langle C_i x, w \rangle = \langle Cx, w \rangle = \langle x, C^*w \rangle,$$

so $w \in \mathrm{Dom}(C_i^*)$ and $C_i^* w = C^*w$. Thus $C^*|_{K_i} \subseteq C_i^*$. Now assume $w \in \mathrm{Dom}(C_i^*)$. If $x \in \mathrm{Dom}(C)$, $x = x_1 + x_2$, $x_1 \in \mathrm{Dom}(C) \cap K_1$, $x_2 \in \mathrm{Dom}(C) \cap K_2$.

$$\langle Cx, w \rangle = \langle Cx_1, w \rangle + \langle Cx_2, w \rangle = \langle Cx_i, w \rangle = \langle C_i x_i, w \rangle$$

$$= \langle x_i, C_i^* w \rangle = \langle x_1, C_i^* w \rangle + \langle x_2, C_i^* w \rangle = \langle x, C_i^* w \rangle,$$

so $w \in \mathrm{Dom}(C^*)$ and $C^*w = C_i^* w$. Thus $C_i^* \subseteq C^*|_{K_i}$. This shows that $C = C_1 \oplus C_2$ where C_1 and C_2 each satisfy the conditions of the theorem and

$$C_1 v_j = \sqrt{\beta_{j+1}}\, v_{j+1} = \gamma_j v_{j+1} \text{ if } p = 2q.$$

By transfinite induction, we may write

$$K = \left(\sum_\mu \oplus K_\mu\right) \oplus K_0, \qquad C = \left(\sum_\mu \oplus C_\mu\right) \oplus C_0,$$

where for each μ, $\{C_\mu, K_\mu\}$ is a free paraboson field and $\{C_0, K_0\}$ satisfies the conditions of the theorem but $\frac{1}{2}(C_0C_0^* + C_0^*C_0)$ has no point spectrum.

The theorem is now a consequence of the following statement: If $\sigma_d(n) = \emptyset$ then $\{C, K\}$ is reducible. To see that this is true suppose $\sigma_d(n) = \emptyset$. If Δ is a bounded set of real numbers and $w \in E(\Delta)K$, then $w \in \mathrm{Dom}(n)$ and

$$\begin{aligned}\langle Cw, Cw\rangle + \langle C^*w, C^*w\rangle &= \langle w, C^*Cw\rangle + \langle w, CC^*w\rangle \\ &= \langle w, 2nw\rangle \leqslant 2\|\Delta\|\ \|w\|^2\end{aligned}$$

where $\|\Delta\| = \sup \Delta$. Thus, n, C and C^* are bounded on $E(\Delta)K$. Now assume Δ is a bounded open interval, $\Delta = (a, b)$, and suppose $E(\Delta)w = w$. We will show that

$$E(\Delta + 1)Cw = Cw, \tag{6}$$

$$E(\Delta - 1)C^*w = C^*w, \tag{7}$$

where $\Delta + k = \{d + k: d \in \Delta\}$.

Let k be a nonnegative integer and let $\delta = (b - a)/2k$. Assume k is sufficiently large so that $\delta < 1$.

Let $\Delta_j = (a + 2j\delta, a + 2(j + 1)\delta)$ for $j = 0, 1, \ldots, k - 1$. Let $w_j = E(\Delta_j)w$ so $w = \Sigma_{j=0}^{k-1} w_j$ and $E(\Delta_j)w_j = w_j$. Let $\theta_j = a + 2(j + \frac{1}{2})\delta$, the midpoint of Δ_j, and let

$$z = nw - \sum_{j=0}^{k-1} \theta_j w_j = \sum_{j=0}^{k-1} (n - \theta_j)w_j.$$

$$\begin{aligned}\|z\|^2 &= \left\| \sum_{j=0}^{k-1} (n - \theta_j)w_j \right\|^2 = \sum_{j=0}^{k-1} \|(n - \theta_j)w_j\|^2 \\ &\leqslant \delta^2 \sum_{j=0}^{k-1} \|w_j\|^2 = \delta^2 \|w\|^2\end{aligned}$$

where we have used that the w_j's are orthogonal.

Since $E(\Delta)z = z$,

$$\|Cz\|^2 \leqslant 2\|\Delta\|\ \|z\|^2 \leqslant 2\|\Delta\|\delta^2\|w\|^2.$$

$$Cz = \sum_{j=0}^{k-1} C(n - \theta_j)w_j = \sum_{j=0}^{k-1} (n - \theta_j - 1)Cw_j.$$

This is a sum of orthogonal terms since

$$\langle C(n - \theta_j)w_j, C(n - \theta_{j'})w_{j'}\rangle = \langle (n - \theta_j)w_j, C^*C(n - \theta_{j'})w_{j'}\rangle$$

and $(n - \theta_j)w_j \in E(\Delta_j)K$ while $C^*C(n - \theta_{j'})w_{j'} \in E(\Delta_{j'})K$ since C^*C commutes with n.

$$\sum_{j=0}^{k-1} \|(n - \theta_j - 1)Cw_j\|^2 = \|Cz\|^2 \leqslant 2\|\Delta\|\delta^2\|w\|^2.$$

Let $\epsilon_j = \|(n - \theta_j - 1)Cw_j\|$, so $\Sigma_{j=0}^{k-1}\epsilon_j^2 \leqslant 2\|\Delta\|\delta^2\|w\|^2$. Let $\Delta_j^0 = (\theta_j - \sqrt[4]{\delta}, \theta_j + \sqrt[4]{\delta})$ so $\Delta_j \subseteq \Delta_j^0$. Let $y_j = E(\Delta_j^0 + 1)Cw_j - Cw_j$. Then

$$\begin{aligned}\|y_j\|^2 &= \|E((\Delta_j^0 + 1)')Cw_j\|^2 \\ &\leqslant \left\|E((\Delta_j^0 + 1)')\left(\frac{n - \theta_j - 1}{\sqrt[4]{\delta}}\right)Cw_j\right\|^2 \\ &\leqslant \left\|\left(\frac{n - \theta_j - 1}{\sqrt[4]{\delta}}\right)Cw_j\right\|^2 \leqslant \frac{\epsilon_j^2}{\sqrt{\delta}}\end{aligned}$$

where $'$ denotes the complement. Let $\Delta^0 = \bigcup_{j=0}^{k-1}\Delta_j^0$ and $y_j^0 = E(\Delta^0 + 1)Cw_j - Cw_j$. Then $\|y_j^0\|^2 \leqslant \|y_j\|^2 \leqslant \epsilon_j^2\sqrt{\delta}$. Let $y = \Sigma_{j=0}^{k-1}y_j^0 = E(\Delta^0 + 1)Cw - Cw$.

$$\|y\|^2 \leqslant k\sum_{j=0}^{k-1}\|y_j^0\|^2 \leqslant \frac{k}{\sqrt{\delta}}\sum_{j=0}^{k-1}\epsilon_j^2 \leqslant \|\Delta\|^2\|w\|^2\sqrt{\delta}.$$

As $k \longrightarrow \infty$, $\delta \longrightarrow 0$, $\Delta^0 + 1 \longrightarrow \Delta + 1$ so $\|y\| \longrightarrow 0$, $E(\Delta^0 + 1)Cw \longrightarrow Cw$, $E(\Delta^0 + 1) \longrightarrow E(\Delta + 1)$ strongly and thus $E(\Delta + 1)Cw = Cw$. This establishes (6).

To see that (7) holds, let Δ_1 be a bounded interval disjoint from $\Delta \dot{-} 1$ and assume $E(\Delta_1)y = y$,

$$\langle C^*w, y\rangle = \langle w, Cy\rangle = \langle E(\Delta)w, E(\Delta_1 + 1)Cy\rangle = 0.$$

Therefore, $C^*w \in (E((\Delta - 1)')K)^\perp$ so $E(\Delta - 1)C^*w = C^*w$.

Let $q = \inf \sigma(n)$. Choose ϵ, $0 < \epsilon < 1$, such that $E((q + \epsilon, q + 1)) \neq 0$. Let $\Delta = (q, q + \epsilon)$. Let $M = \{w \in K\colon \Sigma_{k=0}^m E(\Delta + k)w = w \text{ for some } m\} \subseteq D$. Let $w \in M$. Define $w_k = E(\Delta + k)w$ so $w = \Sigma_{k=0}^m w_k$. $Cw = \Sigma_{k=0}^m Cw_k$ and so $Cw \in M$. $C^*w = \Sigma_{k=0}^m C^*w_k$ and $E(\Delta + k - 1)C^*w_k = C^*w_k$. Since $E(\Delta - 1) = 0$, $C^*w \in M$. Thus $CM \subseteq M$ and $C^*M \subseteq M$. Let K_1 be the closure of M, $K_2 = M^\perp$ and $P = \Sigma_{k=0}^\infty E(\Delta + k)$ so that P projects on K_1 and is a spectral projection of n. Since $T = \sqrt{C^*C}$ and $S = \sqrt{CC^*}$ commute with n, $PT \subseteq TP$ and $PS \subseteq SP$ so, as before,

$$\operatorname{Dom}(C) = \operatorname{Dom}(C) \cap K_1 \oplus \operatorname{Dom}(C) \cap K_2,$$

$$\operatorname{Dom}(C^*) = \operatorname{Dom}(C^*) \cap K_1 \oplus \operatorname{Dom}(C^*) \cap K_2,$$

and $C = C_1 \oplus C_2$. $K_1 \neq \{0\}$ since $E(\Delta) \neq 0$ by the definition of q. $K_2 \neq \{0\}$ since $E((q + \epsilon, q + 1)) \neq 0$. Thus $\{C, K\}$ is reducible.

This completes the proof of the theorem.

COROLLARY. *There are no bounded representations of Green's paraboson relations.*

PROOF. If C were any bounded creator from a representation of the paraboson relations, it would have to satisfy (2). CC^* and C^*C commute since their commutator is formally zero and they are bounded. We may assume that $0 \notin \sigma_d(n)$ since C and C^* act trivially on $E(\{0\})$. Let M be an even integer greater than $\sup \sigma(n)$. By (6), $C^M = 0$. Choose $q \in \sigma(n)$ such that $q > 0$ and $q - ½ < \inf \sigma(n)$. Let $\Delta = (q - ½, q + ½)$. If $E(\Delta)w = w$ then $C^*w = 0$ since $E(\Delta - 1)C^*w = C^*w$ and $E(\Delta - 1) = 0$. Suppose $E(\Delta)w = w$ and $\|w\| = 1$. By (3),

$$C^*C^k w = \begin{cases} kC^{k-1}w, & k \text{ even},\\ (k-1)C^{k-1}w + C^{k-1}C^*Cw, & k \text{ odd}.\end{cases}$$

Let $v = nw - qw$. $C^*Cw = 2nw = 2qw + 2v$.

$$C^*C^k w = \begin{cases} kC^{k-1}w, & k \text{ even},\\ (k-1+2q)C^{k-1}w + 2C^{k-1}v, & k \text{ odd}.\end{cases}$$

$$\begin{aligned}\|C^k w\|^2 &= \langle C^k w, C^k w\rangle\\ &= \langle C^{k-1}w, C^*C^k w\rangle\\ &= \begin{cases} k\|C^{k-1}w\|^2, & k \text{ even},\\ (k-1+2q)\|C^{k-1}w\|^2 + 2\langle C^{k-1}w, C^{k-1}v\rangle, & k \text{ odd}.\end{cases}\end{aligned}$$

If k is even and $2 \leqslant k \leqslant M$,

$$\begin{aligned}\|C^k w\|^2 &= k(k-2+2q)\|C^{k-2}w\|^2 + 2k\langle C^{k-2}w, C^{k-2}v\rangle\\ &\geqslant 2qk\|C^{k-2}w\|^2 - 2M\|C\|^{2k-4}\|v\|.\end{aligned}$$

Since $\|C\|^2 = \|C^*C\| \leqslant 2\|n\| \leqslant 2M$,

$$\begin{aligned}\|C^k w\|^2 &\geqslant 2qk\|C^{k-2}w\|^2 - (2M)^{k-1}\|v\|\\ &\geqslant 2qk(2q(k-2)\|C^{k-4}w\|^2 - (2M)^{k-3}\|v\|) - (2M)^{k-1}\|v\|\\ &\geqslant (2q)^{k/2}k(k-2)\cdots(2)\|w\|^2 - \|v\|((2M)^{k-1} + q(2M)^{k-2} + \cdots).\end{aligned}$$

$$\|C^M w\|^2 \geqslant (2q)^{M/2}M(M-2)\cdots(2) - \|v\|P(M)$$

where $P(M)$ is some fixed positive function of M. Thus

$$\|v\| \geqslant \frac{(2q)^{M/2} M(M-2)\cdots(2)}{P(M)} = Q.$$

We can now find a sequence, w_j, with $E(\Delta)w_j = w_j$, $\|w_j\| = 1$ and $nw_j \longrightarrow qw_j$. We then have $\|v_j\| \geqslant Q$ but $v_j \longrightarrow 0$. This contradiction completes the proof of the corollary.

If H is a complex Hilbert space, a quantum field over H is a collection $\{K, C, \Gamma, v\}$ where K is a complex Hilbert space, C is a complex linear function from H into the closed densely defined operators on K, Γ is a continuous representation of the unitary group of H on K and v is a vector in K which is cyclic for the algebra generated by the $C(z)$ and $C^*(z)$ such that

$$\Gamma(U)C(z)\Gamma(U)^{-1} = C(Uz), \qquad \Gamma(U)v = v,$$

for all unitaries U on H and all $z \in H$. The quantum field is said to be positive (or have positive energy) if when A is a nonnegative self-adjoint operator on H, the operator $d\Gamma(A)$ defined by $\Gamma(e^{itA}) = e^{itd\Gamma}(A)$ is also nonnegative. In this case we write $d\Gamma \geqslant 0$. The standard parafermion fields are examples of positive energy quantum fields [7] as are the free boson fields [1, Corollary 2, Theorem 4, Theorem 8]. The one-dimensional free paraboson field can also be given this structure as follows.

Let H be a one-dimensional Hilbert space. H is just the field of complex numbers. Let $\{C, K\}$ be the one-dimensional free paraboson field. For $\alpha \in H$, define $C(\alpha) = \alpha C$. A representation Γ of the unitary group $\mathrm{U}(H)$ of H on K can then be defined which intertwines with $C(\alpha)$. $\mathrm{U}(H)$ is just the set of complex numbers with absolute value 1. Define $\Gamma(\beta)$ for $\beta \in \mathrm{U}(H)$ by $\Gamma(\beta)e_j = \beta^j e_j$. Γ is then a representation and $\Gamma(\beta)C(\alpha)\Gamma(\beta)^{-1}e_j = C(\beta\alpha)e_j$. If we define $v = e_0$, we get $\Gamma(\beta)v = v$. For any self-adjoint operator A on H, A is just multiplication by some real number a and $d\Gamma(A)e_j = jae_j$. Hence, $d\Gamma(A) \geqslant 0$ when $A \geqslant 0$. This implies that $\{K, C, \Gamma, v\}$ is a positive quantum field over H.

The order of the paraboson field can be any nonnegative number p. If p is an integer then the field can be obtained from the skew product of p free boson fields by Green's ansatz [3, p. 1157]. When p is not an integer the free paraboson field of order p cannot be obtained from boson fields. This gives the first example of a positive quantum field not derivable from boson or fermion fields for which the creation operators satisfy simple relations.

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