TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 209, 1975

GLOBAL DIMENSION OF DIFFERENTIAL OPERATOR RINGS. II

BY

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ABSTRACT. The aim of this paper is to find the global homological dimension of the ring of linear differential operators $R[\theta_1, \ldots, \theta_u]$ over a differential ring R with u commuting derivations. When R is a commutative noetherian ring with finite global dimension, the main theorem of this paper (Theorem 21) shows that the global dimension of $R[\theta_1, \ldots, \theta_u]$ is the maximum of k and $q + u$, where q is the supremum of the ranks of all maximal ideals M of R for which R/M has positive characteristic, and k is the supremum of the sums rank(P) + diff dim(P) for all prime ideals P of R such that R/P has characteristic zero. [The value diff dim(P) is an invariant measuring the differentiability of P in a manner defined in §3.] In case we are considering only a single derivation on R, this theorem leads to the result that the global dimension of $R[\theta]$ is the supremum of gl dim(R) together with one plus the projective dimensions of the modules R/J, where J is any primary differential ideal of R. One application of these results derives the global dimension of the Weyl algebra in any degree over any commutative noetherian ring with finite global dimension.

1. **Introduction.** As in [5], we reserve the term *differential ring* for a nonzero associative ring R with unit together with a single specified derivation δ on R. In case we have specified a finite collection $\delta_1, \ldots, \delta_u$ of commuting derivations on R, we shall refer to R as a *u-differential ring*. The *ring of differential operators* over a u-differential ring R is additively the group of all polynomials over R in indeterminates $\theta_1, \ldots, \theta_u$, with multiplication subject to the requirements $\theta_i\theta_j = \theta_j\theta_i$ for all i, j, and $\theta_i a = a\theta_i + \delta_i a$ for all i, all $a \in R$. We denote this ring by $R[\theta_1, \ldots, \theta_u]$, or by $R[\theta]$ in the case of a single derivation. The elements of $R[\theta_1, \ldots, \theta_u]$ are normally written as sums of monomials of the form rp, where $r \in R$ and p is a product of powers of the θ_i, although for some arguments it is more convenient to use right-hand coefficients. (Note that when an element of $R[\theta_1, \ldots, \theta_u]$ is written with left-hand coefficients, these coefficients will in general be different from those used to express

Presented to the Society, January 18, 1974 under the title *Homological dimension of differential operator rings*; received by the editors October 1, 1973.

AMS (MOS) subject classifications (1970). Primary 16A60; Secondary 12H05.

Key words and phrases. Global dimension, rings of linear differential operators, differential algebra, Weyl algebras.

the element with right-hand coefficients.) In particular, any element $x \in R[\theta]$ is written as $x = r_0 + r_1\theta + \ldots + r_n\theta^n$ for suitable $r_i \in R$, and when $r_n \neq 0$ we say that n is the *degree* of x and that r_n is the *leading coefficient* of x. Finally, for induction purposes we note that

$$R[\theta_1, \ldots, \theta_u] = R[\theta_1, \ldots, \theta_{u-1}][\theta_u],$$

where δ_u has been implicitly extended to $R[\theta_1, \ldots, \theta_{u-1}]$ by setting $\delta_u\theta_i = 0$ for all i.

The objective of this paper is to derive formulas for the global dimension of $R[\theta_1, \ldots, \theta_u]$, where R is a commutative noetherian u-differential ring with finite global dimension. Basically, the task breaks down into the problems of finding suitable lower bounds and upper bounds for the global dimension of $R[\theta_1, \ldots, \theta_u]$. Since these two problems require relatively different techniques, we allot separate sections of the paper to them. In both cases we also require the techniques of localization: namely ordinary localization of the commutative ring R at a prime ideal, which induces a natural noncommutative localization on the ring $R[\theta_1, \ldots, \theta_u]$.

Our notation for the various homological dimensions involved with a ring S is as follows: r gl dim S denotes the right global dimension of S, and GWD(S) denotes the global weak dimension of S. For any S-module A, we use $\mathrm{pd}_S(A)$ and $\mathrm{wd}_S(A)$ to stand for the respective projective and weak dimensions of A. The reason that weak dimensions are useful is that we shall be dealing mostly with noetherian rings. For if R is a right and left noetherian differential ring, then $R[\theta]$ is right and left noetherian, as observed in [2, p. 68]. By induction, $R[\theta_1, \ldots, \theta_u]$ is right and left noetherian also. Our basic estimates on homological dimensions are given in the following two propositions, which follow automatically by induction from [5, Propositions 2, 3].

PROPOSITION 1. *Let R be any u-differential ring, and set $T = R[\theta_1, \ldots, \theta_u]$. If A is any right T-module, then*

$$\mathrm{pd}_R(A) \leqslant \mathrm{pd}_T(A) \leqslant u + \mathrm{pd}_R(A).$$

PROPOSITION 2. *If R is any u-differential ring with* r gl dim $R < \infty$, *then*

$$\text{r gl dim } R \leqslant \text{r gl dim } R[\theta_1, \ldots, \theta_u] \leqslant u + \text{r gl dim } R.$$

The left-hand inequality in Proposition 2 may fail if r gl dim $R = \infty$, as shown in [5, §2].

We close this section with two propositions which give the basic results on the localization procedures needed later. The first of these is proved in exactly the same manner as [5, Lemma 7].

PROPOSITION 3. *Let R be any commutative u-differential ring, and set $T = R[\theta_1, \ldots, \theta_u]$. If S is any multiplicatively closed subset of R, then the following are true*:

(a) *Each δ_i induces a derivation on R_S according to the rule $\delta_i(r/s) = [(\delta_i r)s - r(\delta_i s)]/s^2$.*

(b) *The natural map $T \longrightarrow R_S[\theta_1, \ldots, \theta_u]$ makes $R_S[\theta_1, \ldots, \theta_u]$ into a flat right and left T-module such that the multiplication map $R_S[\theta_1, \ldots, \theta_u] \otimes_T R_S[\theta_1, \ldots, \theta_u] \longrightarrow R_S[\theta_1, \ldots, \theta_u]$ is an isomorphism.*

(c) r gl dim $R_S[\theta_1, \ldots, \theta_u] \leqslant$ r gl dim T.

PROPOSITION 4. *Let R be a commutative noetherian u-differential ring with* gl dim $R < \infty$. *Then*

$$\text{r gl dim } R[\theta_1, \ldots, \theta_u] = \sup\{\text{r gl dim } R_M[\theta_1, \ldots, \theta_u] \mid M \text{ is a maximal ideal of } R\}.$$

PROOF. Inasmuch as all rings involved in this proposition are right noetherian, it suffices to prove the corresponding statement for global weak dimension. Just as in the proof of [5, Lemma 7], we see that each of the rings $R_M[\theta_1, \ldots, \theta_u]$ is a classical localization of $R[\theta_1, \ldots, \theta_u]$ with respect to the multiplicative set $R \backslash M$. It is easily checked that these localizations satisfy the hypotheses of [13, Proposition 1], from which we obtain the desired result.

2. **Lower bounds.** In this section we set up our basic tool for finding lower bounds for the global dimension of $R[\theta_1, \ldots, \theta_u]$. This is Theorem 7, which allows us to compute the projective dimensions of those $R[\theta_1, \ldots, \theta_u]$-modules which happen to be finitely generated as R-modules. As one consequence, we find that r gl dim $R[\theta_1, \ldots, \theta_u] \geqslant u + \text{rank}(M)$ for any maximal ideal M of R such that R/M has positive characteristic. We begin with two lemmas, the first of which is essentially a special case of [6, Lemma, p. 68].

LEMMA 5. *Let R be any differential ring, and let A be a right $R[\theta]$-module. If E: $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ is an exact sequence of right R-modules with F_R free, then K and F can be made into right $R[\theta]$-modules such that E becomes an exact sequence of $R[\theta]$-modules.*

PROOF. Let f: $K \longrightarrow F$ and g: $F \longrightarrow A$ denote the maps in E. Choosing a decomposition of F as a direct sum of copies of R, and applying δ to each copy of R, we obtain an additive map d: $F \longrightarrow F$ such that $d(xr) = (dx)r + x(\delta r)$ for all $x \in F$, $r \in R$. Define a map h: $F \longrightarrow A$ by the rule $hx = g(dx) + (gx)\theta$, and check that h is an R-homomorphism. Then h lifts to an R-homomorphism k: $F \longrightarrow F$ such that $gk = h$.

Now $d' = k - d$ is an additive endomorphism of F such that $d'(xr) = (d'x)r - x(\delta r)$ for all $x \in F$, $r \in R$, from which we infer that F can be made into a right $R[\theta]$-module by defining $x\theta = d'x$ for all $x \in F$. Computing that now $g(x\theta) = (gx)\theta$ for all $x \in F$, we see that g is an $R[\theta]$-homomorphism. As a consequence, ker g is an $R[\theta]$-submodule of F, hence K can be made into a right $R[\theta]$-module so that f is an $R[\theta]$-homomorphism.

LEMMA 6. *Let R be a semiprime left Goldie differential ring. If J is any essential left ideal of $R[\theta]$, then J contains an element of $R[\theta]$ whose leading coefficient is a regular element of R.*

PROOF. Since R is left Goldie, it must contain a finite direct sum $A_1 \oplus \ldots \oplus A_k$ of nonzero uniform left ideals which is essential in ${}_RR$. The essentiality of J implies that each of the left ideals $R[\theta]A_i$ must contain a nonzero element x_i from J. After multiplying the x_i on the left by suitable powers of θ, we may assume that the x_i all have the same degree, say n. Inasmuch as $R[\theta] = R + \theta R + \theta^2 R + \ldots$, we see that $R[\theta]A_i = A_i + \theta A_i + \theta^2 A_i + \ldots$. Noting that the degree of x_i remains the same when x_i is written with coefficients on the right, we see that $x_i = x_{i0} + \theta x_{i1} + \ldots + \theta^{n-1}x_{i,n-1} + \theta^n a_i$ for some x_{ij}, $a_i \in A_i$, $a_i \neq 0$. Changing back to left-hand coefficients, the leading coefficient of x_i is still a_i, although the other coefficients need not even belong to A_i.

Now Ra_i is a nonzero submodule of the uniform left ideal A_i and hence is essential in A_i, from which we deduce that $Ra_1 \oplus \ldots \oplus Ra_k$ is an essential left ideal of R. Inasmuch as R is a semiprime left Goldie ring, [8, Lemma 7.2.5] says that $Ra_1 \oplus \ldots \oplus Ra_k$ must contain a regular element a of R, say $a = r_1a_1 + \ldots + r_ka_k$. Since each x_i has leading term $a_i\theta^n$, we now conclude that $r_1x_1 + \ldots + r_kx_k$ is an element of J whose leading coefficient is a.

THEOREM 7. *Let R be a semiprime right and left noetherian u-differential ring, and set $T = R[\theta_1, \ldots, \theta_u]$. If A is any nonzero right T-module such that A_R is finitely generated, then* $\mathrm{pd}_T(A) = u + \mathrm{pd}_R(A)$.

PROOF. Each of the rings $T_j = R[\theta_1, \ldots, \theta_j]$ is right and left noetherian, and it is easily checked that each T_j is semiprime as well. Now A is a finitely generated right T_j-module for each j, and we are done if we show that the projective dimension of A over each T_{j+1} is exactly one greater than the projective dimension of A over T_j. Thus it suffices to consider only the 1-differential case: here R is a semiprime right and left noetherian differential ring, A is a nonzero right $R[\theta]$-module such that A_R is finitely generated, and we must prove that $\mathrm{pd}_{R[\theta]}(A) = 1 + \mathrm{pd}_R(A)$.

The case $\mathrm{pd}_R(A) = \infty$ is taken care of by Proposition 1, hence we may assume that $\mathrm{pd}_R(A) = n < \infty$, and we induct on n. As noted above, $R[\theta]$ is a semiprime right and left noetherian ring, hence the maximal right quotient ring Q of $R[\theta]$ coincides with the maximal left quotient ring of $R[\theta]$ (and is a classical right and left quotient ring). Also, $R[\theta]$ is a semiprime right Goldie ring, hence [14, Theorem 1.7] shows that $R[\theta]$ is a right nonsingular ring.

If $n = 0$, then $\mathrm{pd}_{R[\theta]}(A) \leq 1$ by Proposition 1; hence it remains to show that $A_{R[\theta]}$ is not projective. Inasmuch as $A \neq 0$ and all projective right $R[\theta]$-modules are nonsingular, it suffices to show that $A_{R[\theta]}$ is singular. Given any $a \in A$, set $J = \{x \in R[\theta] \mid ax = 0\}$ and note that $R[\theta]/J$ is noetherian as an R-module. Now any nonzero right ideal K of $R[\theta]$ contains elements of arbitrarily high degree, whence K_R cannot be finitely generated. Thus the natural map $K \longrightarrow R[\theta] \longrightarrow R[\theta]/J$ cannot be a monomorphism, i.e., $K \cap J \neq 0$. Therefore J is an essential right ideal of $R[\theta]$ and so A is indeed a singular $R[\theta]$-module.

Next assume that $n = 1$, and choose a positive integer k such that A_R can be generated by k elements. If S denotes the ring of all $k \times k$ matrices over R, then we obtain a Morita equivalence between the category of all right R-modules and the category of all right S-modules, where any right R-module B gets taken to $B \otimes_R R^k$, i.e., to B^k. We intend to use this equivalence to transfer our problem to S-modules, since A^k is a cyclic right S-module. Now δ can be extended to a derivation of S by letting δ act on each entry of any matrix in S, and then $S[\theta]$ may be identified with the ring of all $k \times k$ matrices over $R[\theta]$. With this identification, we get another Morita equivalence between the category of all right $R[\theta]$-modules and the category of all right $S[\theta]$-modules, where any right $R[\theta]$-module B gets taken to B^k. Because of these equivalences, $\mathrm{pd}_S(A^k) = 1$ and $\mathrm{pd}_{R[\theta]}(A) = \mathrm{pd}_{S[\theta]}(A^k)$, hence we may assume without loss of generality that A_R is cyclic.

Therefore we may assume that $A = R/I$ for some right ideal I of R. Inasmuch as A is also a right $R[\theta]$-module, we have $\overline{1}\theta = \overline{\alpha}$ for some $\alpha \in R$. Then $\overline{r}\theta = \overline{(\alpha - \delta)r}$ for all $r \in R$ and consequently $(\alpha - \delta)(I) \subseteq I$. Noting that $R[\theta] = R + (\theta - \alpha)R[\theta]$, we see that $A \cong R[\theta]/J$, where $J = I + (\theta - \alpha)R[\theta]$.

We claim that for any $R[\theta]$-homomorphism $f\colon J \longrightarrow R[\theta]$, $f|_I$ must be left multiplication by some element of $R[\theta]$. Since $R[\theta]$ is a right nonsingular ring, its maximal right quotient ring Q is the injective hull of $R[\theta]_{R[\theta]}$, hence f must be left multiplication by some $t \in Q$. Noting that $t(\theta - \alpha) \in R[\theta]$, we see that $t = x(\theta - \alpha)^{-1}$ for some $x \in R[\theta]$. This element x can be put in the form $x = x_0 + x_1(\theta - \alpha)$ for suitable $x_0 \in R$ and $x_1 \in R[\theta]$, whence $t = x_0(\theta - \alpha)^{-1} + x_1$. If $x_0 = 0$, then f itself is left multiplication by the element $x_1 \in R[\theta]$ and the claim holds, hence we may assume that $x_0 \neq 0$. We have $tJ = fJ \subseteq R[\theta]$, and clearly $x_1 J \subseteq R[\theta]$ as well, whence $x_0(\theta - \alpha)^{-1}J \subseteq R[\theta]$.

Inasmuch as Q is also the maximal left quotient ring of $R[\theta]$, we must have $Kx_0(\theta - \alpha)^{-1} \subseteq R[\theta]$ for some essential left ideal K of $R[\theta]$, and by Lemma 6, K must contain an element y whose leading coefficient is a regular element of R. Now y is clearly a regular element of $R[\theta]$ and so is invertible in Q, hence we obtain $x_0(\theta - \alpha)^{-1} = y^{-1}z$ for some $z \in R[\theta]$, or $yx_0 = z(\theta - \alpha)$. Since $x_0 \neq 0$ we have $z \neq 0$, too, which makes it possible to talk about the degrees of the elements in this last equation. Obviously $\deg[z(\theta - \alpha)] = 1 + \deg(z)$, and since the leading coefficient of y is a regular element we obtain $\deg(yx_0) = \deg(y)$; thus $\deg(y) = 1 + \deg(z)$. Given any $r \in I$, we have $y^{-1}zr = x_0(\theta - \alpha)^{-1}r \in R[\theta]$ (because $r \in J$), whence $zr \in yR[\theta]$. Since $\deg(y) > \deg(z)$, and since $\deg(yw) \geqslant \deg(y)$ for all nonzero $w \in R[\theta]$, this is possible only when $zr = 0$. Thus we obtain $zI = 0$, from which we infer that $x_0(\theta - \alpha)^{-1}I = 0$. It follows that $f|_I$ is just left multiplication by the element $x_1 \in R[\theta]$, as claimed.

As right R-modules, $J = I \oplus (\theta - \alpha)R[\theta]$, from which we see that I can be made into a right $R[\theta]$-module so that the projection $p: J \longrightarrow I$ is an $R[\theta]$-homomorphism. Choose an $R[\theta]$-epimorphism $g: F \longrightarrow I$, where F is a finitely generated free right $R[\theta]$-module. If we assume that $J_{R[\theta]}$ is projective, then p must lift to an $R[\theta]$-homomorphism $h: J \longrightarrow F$ such that $gh = p$. In view of the claim just proved above, we see that $h|_I$ must be left multiplication by some $w \in F$, from which we compute that $(gw)r = r$ for all $r \in I$. Consequently gw is an idempotent and $(gw)R = I$, hence $(R/I)_R$ must be projective. However, this contradicts the assumption that $\mathrm{pd}_R(A) = 1$, and thus $J_{R[\theta]}$ cannot be projective. This gives us $\mathrm{pd}_{R[\theta]}(A) > 1$, so by Proposition 1 we conclude that $\mathrm{pd}_{R[\theta]}(A) = 2$.

Finally, let $n > 1$ and assume the theorem holds for $n - 1$. Choose an exact sequence E: $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ of right R-modules with F_R finitely generated free, and use Lemma 5 to make E into an exact sequence of right $R[\theta]$-modules. Now K is a right $R[\theta]$-module which is finitely generated as an R-module, and $\mathrm{pd}_R(K) = n - 1 > 0$ (so that in particular $K \neq 0$), hence we obtain $\mathrm{pd}_{R[\theta]}(K) = n$ from the induction hypothesis. Inasmuch as $n > 1$ and $\mathrm{pd}_{R[\theta]}(F) \leqslant 1$ by Proposition 1, it now follows from the long exact sequence for Ext that $\mathrm{pd}_{R[\theta]}(A) = n + 1$. □

Using more homological methods, a stronger version of Theorem 7 has been proved in [12, Corollary 1.7(b)].

I am grateful to the referee for pointing out the necessity of condition (b) in the following corollary.

COROLLARY 8. *Let R be a semiprime right and left noetherian u-differential ring, let J be a proper right ideal of R, and let* $\alpha_1, \ldots, \alpha_u \in R$ *such that*

(a) $(\delta_i - \alpha_i)(J) \subseteq J$ *for* $i = 1, \ldots, u$,

(b) $(\delta_i - \alpha_i)(\alpha_j) - (\delta_j - \alpha_j)(\alpha_i) \in J$ *for* $i, j = 1, \ldots, u$.

Then r gl dim $R[\theta_1, \ldots, \theta_u] \geqslant u + \mathrm{pd}_R(R/J)$.

PROOF. Let $A = R/J$, which is a nonzero finitely generated right R-module. Using (a) and (b), we infer that A can be made into a right $R[\theta_1, \ldots, \theta_u]$-module by setting $\overline{r}\theta_i = \overline{(\alpha_i - \delta_i)r}$ for all i and all $r \in R$. (Condition (a) ensures that $x\theta_i$ is well defined, and condition (b) ensures that $x\theta_i\theta_j = x\theta_j\theta_i$. The details are very straightforward.) Consequently, Theorem 7 says that A is a right $R[\theta_1, \ldots, \theta_u]$-module with projective dimension $u + \mathrm{pd}_R(A)$. □

Corollary 8 applies in particular to the case when J is a differential right ideal of R, i.e., $\delta_i(J) \subseteq J$ for all i. In this case, condition (b) is trivially satisfied.

In order to apply Theorem 7 or Corollary 8 in the case when R is a commutative noetherian ring of finite global dimension, we must know that R is semiprime. This is probably well known, as are the other facts in the following proposition, which we include for completeness.

PROPOSITION 9. *Let R be any commutative noetherian ring with* gl dim $R = n < \infty$.

(a) *R is a finite direct product of integral domains, and thus is a semiprime ring.*

(b) *If M is any maximal ideal of R, then* gl dim $R_M = \mathrm{rank}(M) = \mathrm{pd}_R(R/M) \leqslant n$.

(c) *The (classical) Krull dimension of R is n.*

PROOF. (a) For each maximal ideal M of R [9, Part III, Theorem 11] says that gl dim $R_M \leqslant n < \infty$, hence it follows from [9, Part III, Theorem 13] that R_M is a regular local ring. Thus R_M is an integral domain for every maximal ideal M [10, Theorem 164], whence [10, Theorem 168] says that R is a finite direct product of integral domains.

(b) As seen in (a), R_M is a regular local ring. According to [9, Part III, Theorem 12], gl dim R_M is the same as the Krull dimension of R_M, i.e., gl dim $R_M = \mathrm{rank}(M)$. In view of [10, Theorem 176], we also see that the projective dimension of R_M/MR_M over R_M is equal to rank(M). Inasmuch as

R is noetherian, the projective dimension of any finitely generated R-module A is the supremum of the projective dimensions of the R_K-modules A_K, where K ranges over all maximal ideals of R. For the case $A = R/M$, we have $A_M = R_M/MR_M$ and $A_K = 0$ for all other K, from which we conclude that $\text{pd}_R(R/M) = \text{rank}(M)$.

(c) Since R is noetherian, n is the supremum of the numbers gl dim R_M over all maximal ideals M, hence (c) follows immediately from (b).

We conclude this section by deriving the lower bound $u + \text{rank}(M) \leqslant$ r gl dim $R[\theta_1, \ldots, \theta_u]$, where M is any maximal ideal of R such that R/M has positive characteristic. We must also derive lower bounds for r gl dim $R[\theta_1, \ldots, \theta_u]$ related to maximal ideals M such that R/M has characteristic zero, but this depends on the differential dimension of M, which we develop in the next section.

PROPOSITION 10. *Let R be a commutative noetherian u-differential ring with* gl dim $R < \infty$, *and let M be a maximal ideal of R. If R/M has characteristic $p > 0$, then*

$$\text{r gl dim } R[\theta_1, \ldots, \theta_u] \geqslant u + \text{rank}(M).$$

PROOF. According to Proposition 9, the simple module R/M satisfies the property $\text{pd}_R(R/M) = \text{rank}(M) < \infty$. If A is any nonzero R-module with a composition series such that all the composition factors are isomorphic to R/M, then it follows from the long exact sequence for Ext (by induction on length) that $\text{pd}_R(A) = \text{rank}(M)$.

Now let J be the ideal of R generated by pR and $\{x^p \mid x \in M\}$, and note that $\delta_i(J) \subseteq J$ for all $i = 1, \ldots, u$. Since $\text{char}(R/M) = p$, we see that $J \subseteq M$, whence $R/J \neq 0$. Inasmuch as M/J is a nil ideal in the noetherian ring R/J, Levitzki's Theorem says that M/J must be nilpotent, from which we infer that R/J has a composition series with all composition factors isomorphic to R/M. Now $\text{pd}_R(R/J) = \text{rank}(M)$, hence the desired inequality follows from Corollary 8.

COROLLARY 11. *Let R be a commutative noetherian u-differential ring with* gl dim $R = n < \infty$. *If R has positive characteristic, then* r gl dim $R[\theta_1, \ldots, \theta_u] = n + u$.

PROOF. In view of Proposition 9, we must have $\text{rank}(M) = n$ for some maximal ideal M, whence Proposition 10 yields r gl dim $R[\theta_1, \ldots, \theta_u] \geqslant n + u$. According to Proposition 1, we also have r gl dim $R[\theta_1, \ldots, \theta_u] \leqslant n + u$.

3. **Differential dimension.** The purpose of this section is to introduce a concept of differential dimension for prime ideals P of R, and to obtain the lower bounds

$$\operatorname{rank}(P) + \operatorname{diff}\dim(P) \leqslant \operatorname{r\,gl\,dim} R[\theta_1, \ldots, \theta_u].$$

This differential dimension of P is meant to measure the "differentiability" of P in the sense that it indicates how large a collection of R-linear combinations of the derivations $\delta_1, \ldots, \delta_u$ can map P into itself. In particular, the differential dimension of P will be u if and only if P is closed under all the δ_i. The details follow.

Given any commutative u-differential ring R, make $\operatorname{Hom}_Z(R, R)$ into a left R-module by defining $(rf)(x) = r(fx)$ for all $r, x \in R$, $f \in \operatorname{Hom}_Z(R, R)$, and let Δ denote the left R-submodule of $\operatorname{Hom}_Z(R, R)$ generated by $\delta_1, \ldots, \delta_u$. For any prime ideal P of R, the set $D(P) = \{f \in \Delta \mid f(P) \subseteq P\}$ is a left R-submodule of Δ, and it is clear that $\Delta/D(P)$ is a torsion-free left (R/P)-module. We define the *differential codimension* of P, abbreviated diff codim(P), to be the rank of this torsion-free (R/P)-module $\Delta/D(P)$, i.e., the vector space dimension $[Q[\Delta/D(P)] : Q]$, where Q stands for the quotient field of R/P. [Alternately, diff codim(P) may be defined as the Goldie dimension of the left R-module $\Delta/D(P)$.] Finally, we define the *differential dimension* of P, denoted diff dim(P), to be u − diff codim(P).

PROPOSITION 12. *Let R be a commutative u-differential ring. Let P be any prime ideal of R, and set $S = R_P$, $M = PR_P$. Then each δ_i induces a linear transformation δ_i^* in the dual space $V = \operatorname{Hom}_{S/M}(M/M^2, S/M)$, and the subspace W of V spanned by $\delta_1^*, \ldots, \delta_u^*$ has dimension exactly* diff codim(P).

PROOF. Each δ_i induces a derivation on S as in Proposition 3, and this gives us additive maps δ_i: $M \longrightarrow S$. Observing that $\delta_i(M^2) \subseteq M$, we see that δ_i induces an additive map δ_i^*: $M/M^2 \longrightarrow S/M$, and an easy check confirms that δ_i^* is an (S/M)-homomorphism.

There is a left R-homomorphism ϕ: $\Delta \longrightarrow W$ such that $\phi(\delta_i) = \delta_i^*$ for each i, and an easy computation shows that $\ker \phi = D(P)$. Now $\phi\Delta$ is a left module over the domain $T = (R + M)/M \cong R/P$, from which we infer that ${}_T(\phi\Delta)$ and ${}_{R/P}[\Delta/D(P)]$ have the same rank, i.e., ${}_T(\phi\Delta)$ has rank diff codim(P). Inasmuch as ${}_T(\phi\Delta)$ is torsion-free and S/M is the quotient field of T, the rank of ${}_T(\phi\Delta)$ is just $[S(\phi\Delta): S/M]$. Observing that $S(\phi\Delta) = W$, we conclude that $[W: S/M]$ = diff codim(P).

COROLLARY 13. *Let R be a commutative u-differential ring. If $P \subseteq Q$ are prime ideals of R, then* diff codim(PR_Q) = diff codim(P).

PROOF. Inasmuch as the localization of R_Q at the prime ideal PR_Q is just R_P, this follows immediately from Proposition 12. □

In particular, Corollary 13 shows that diff codim(P) = diff codim(PR_P) for any prime ideal P, which makes it possible to carry out some computations using the maximal ideal PR_P in the local ring R_P. Before proving the inequality rank(P) + diff dim(P) $\leqslant$ r gl dim $R[\theta_1, \ldots, \theta_u]$, we introduce the following easy lemma, which will also be useful later.

LEMMA 14. (a) *Let R be any ring such that* r gl dim $R = n < \infty$. *If $A \subseteq B$ are right R-modules with* $\mathrm{pd}_R(A) = n$, *then* $\mathrm{pd}_R(B) = n$.

(b) *Let R be any ring such that* GWD(R) $= n < \infty$. *If $A \subseteq B$ are R-modules with* $\mathrm{wd}_R(A) = n$, *then* $\mathrm{wd}_R(B) = n$.

PROOF. (a) If $\mathrm{pd}_R(B) < n$, then it follows from the long exact sequence for Ext that $\mathrm{pd}_R(B/A) = n + 1$, which is impossible. (b) is proved similarly.

PROPOSITION 15. *Let R be a commutative noetherian u-differential ring with* gl dim $R < \infty$. *If P is any prime ideal of R, then*

$$\text{r gl dim } R[\theta_1, \ldots, \theta_u] \geqslant \text{rank}(P) + \text{diff dim}(P).$$

PROOF. The local ring R_P is a commutative noetherian u-differential ring with gl dim $R_P < \infty$ and certainly rank(PR_P) = rank(P). Inasmuch as diff dim(PR_P) = diff dim(P) by Corollary 13 and r gl dim $R_P[\theta_1, \ldots, \theta_u] \leqslant$ r gl dim $R[\theta_1, \ldots, \theta_u]$ by Proposition 3, it suffices to consider the case when R is local and P is its maximal ideal. According to Proposition 9, we have gl dim R = rank(P) = $\mathrm{pd}_R(R/P)$; let n denote this common value.

If s = diff codim(P), then Proposition 12 shows that the subspace W of $\mathrm{Hom}_{R/P}(P/P^2, R/P)$ spanned by the induced linear transformations $\delta_1^*, \ldots, \delta_u^*$ has dimension s. Thus W must have a basis consisting of s of the δ_i^*, hence we may arrange the indices $1, \ldots, u$ so that $\delta_1^*, \ldots, \delta_s^*$ is a basis for W.

Since R is semiprime by Proposition 9, the ring $Q = R[\theta_1, \ldots, \theta_s]$ must be a semiprime ring, as well as right and left noetherian, and of course $R[\theta_1, \ldots, \theta_u] = Q[\theta_{s+1}, \ldots, \theta_u]$. Now PQ is a right ideal of Q and $Q/PQ \cong (R/P) \otimes_R Q$, whence $\mathrm{pd}_Q(Q/PQ) \leqslant \mathrm{pd}_R(R/P) = n$. On the other hand, since Q/PQ contains an R-submodule isomorphic to R/P, we obtain $\mathrm{pd}_R(Q/PQ) = n$ from Lemma 14, and then Proposition 1 says that $\mathrm{pd}_Q(Q/PQ) \geqslant n$. Therefore $\mathrm{pd}_Q(Q/PQ) = n$.

Given any $j \in \{s + 1, \ldots, u\}$, we must have $\delta_j^* = r_{j1}\delta_1^* + \ldots + r_{js}\delta_s^*$ for suitable $r_{ji} \in R$, whence $(\delta_j - r_{j1}\delta_1 - \ldots - r_{js}\delta_s)(P) \subseteq P$. Setting

$q_j = r_{j1}\theta_1 + \ldots + r_{js}\theta_s \in Q$, we compute that $(\delta_j - q_j)(PQ) \subseteq PQ$. Given any $i, j \in \{s+1, \ldots, u\}$, we have

$$\left(\delta_i - \sum_{k=1}^{s} r_{ik}\delta_k\right)(P) \subseteq P \quad \text{and} \quad \left(\delta_j - \sum_{t=1}^{s} r_{jt}\delta_t\right)(P) \subseteq P,$$

from which it follows that

$$\left[\left(\delta_j - \sum_{t=1}^{s} r_{jt}\delta_t\right)\left(\delta_i - \sum_{k=1}^{s} r_{ik}\delta_k\right) - \left(\delta_i - \sum_{k=1}^{s} r_{ik}\delta_k\right)\left(\delta_j - \sum_{t=1}^{s} r_{jt}\delta_t\right)\right](P) \subseteq P.$$

We compute that

$$\left(\delta_j - \sum_{t=1}^{s} r_{jt}\delta_t\right)\left(\delta_i - \sum_{k=1}^{s} r_{ik}\delta_k\right) - \left(\delta_i - \sum_{k=1}^{s} r_{ik}\delta_k\right)\left(\delta_j - \sum_{t=1}^{s} r_{jt}\delta_t\right)$$

$$= \sum_{k=1}^{s}\left[\left(\delta_i - \sum_{t=1}^{s} r_{it}\delta_t\right)(r_{jk}) - \left(\delta_j - \sum_{t=1}^{s} r_{jt}\delta_t\right)(r_{ik})\right]\delta_k;$$

hence we obtain

$$\sum_{k=1}^{s}\left[\left(\delta_i - \sum_{t=1}^{s} r_{it}\delta_t\right)(r_{jk}) - \left(\delta_j - \sum_{t=1}^{s} r_{jt}\delta_t\right)(r_{ik})\right]\delta_k^* = 0.$$

Inasmuch as $\delta_1^*, \ldots, \delta_s^*$ are linearly independent over R/P, we see that

$$\left(\delta_i - \sum_{t=1}^{s} r_{it}\delta_t\right)(r_{jk}) - \left(\delta_j - \sum_{t=1}^{s} r_{jt}\delta_t\right)(r_{ik}) \in P \quad \text{for } k = 1, \ldots, s,$$

from which we compute that $(\delta_i - q_i)(q_j) - (\delta_j - q_j)(q_i) \in PQ$. According to Corollary 8, we obtain r gl dim $Q[\theta_{s+1}, \ldots, \theta_u] \geqslant u - s + n$. Inasmuch as $u - s =$ diff dim(P) and $n =$ rank(P), we are done.

4. **Upper bounds.** The purpose of this section is to introduce two kinds of upper bounds which are needed in the computation of the global dimension of $R[\theta_1, \ldots, \theta_u]$. First, we prove a theorem which shows that the global dimension of $R[\theta_1, \ldots, \theta_u]$ is the supremum of the projective dimensions of its simple modules. The second upper bound, which is needed only in the case

that R is an algebra over the rationals, shows that, for any maximal ideal M of R, all factor modules of $R[\theta_1, \ldots, \theta_u]/MR[\theta_1, \ldots, \theta_u]$ have projective dimension at most rank(M) + diff dim(M).

For the first theorem, we need the concepts of Krull dimension (for noncommutative rings) and critical modules, as defined in [7].

THEOREM 16. *Let R be any nonzero right noetherian, left coherent ring. If* r gl dim $R = n < \infty$, *then* $n = \sup\{\mathrm{pd}_R(A) \mid A_R$ *is simple*$\}$.

PROOF. Since this is clear for $n = 0$, we may assume that $n > 0$. Inasmuch as R is right noetherian, we have GWD$(R) = n$ and $\mathrm{pd}_R(A) = \mathrm{wd}_R(A)$ for all simple modules A_R, hence it suffices to show that R has a simple right module with weak dimension n. According to [3, Theorem 2.1], all direct products of flat right R-modules are flat, from which we infer that the weak dimension of any direct product of right R-modules equals the supremum of the weak dimensions of the factors.

In view of [7, Proposition 1.3], all finitely generated right R-modules have Krull dimension, and there certainly exist finitely generated right R-modules with weak dimension n. Now let α be minimal among the Krull dimensions of those finitely generated right R-modules which have weak dimension n, and choose some finitely generated right R-module B such that K dim$(B) = \alpha$ and $\mathrm{wd}_R(B) = n$. Since $n > 0$, we have $B \neq 0$. All factor modules of B are finitely generated and hence have Krull dimension, whence [7, Theorem 2.1] says that every nonzero factor module of B contains a critical submodule. Thus B must have a chain of submodules $B_0 = 0 < B_1 < \ldots < B_k = B$ such that each B_i/B_{i-1} is critical. Inasmuch as $\mathrm{wd}_R(B) \leqslant \sup\{\mathrm{wd}_R(B_i/B_{i-1})\}$, we must have $\mathrm{wd}_R(B_i/B_{i-1}) = n$ for some i. Setting $A = B_i/B_{i-1}$, we see by [7, Lemma 1.1] that K dim$(A) \leqslant \alpha$, hence it follows from the minimality of α that K dim$(A) = \alpha$.

We now have a finitely generated α-critical right R-module A such that $\mathrm{wd}_R(A) = n$. We claim that $\alpha = 0$, i.e., that A is simple.

Assume on the contrary that $\alpha > 0$. Then every nonzero submodule of A is α-critical too [7, Proposition 2.3], and thus is not simple; so A has no simple submodules. Thus the intersection of all nonzero submodules of A is zero, hence we obtain an embedding $A \longrightarrow P$, where P is the direct product of all proper factors of A. Since A is α-critical, each proper factor of A is a finitely generated module with Krull dimension strictly less than α, so by the minimality of α we see that each proper factor of A has weak dimension at most $n - 1$. However, this implies that $\mathrm{wd}_R(P) \leqslant n - 1$, which contradicts Lemma 14. Therefore $\alpha = 0$ and A is simple.

We now turn to considering factors of $R[\theta_1, \ldots, \theta_u]/MR[\theta_1, \ldots, \theta_u]$, where M is a maximal ideal of R, and R is an algebra over the rationals. For conciseness, we here use the term *u-differential Ritt algebra* to stand for a commutative u-differential ring which is an algebra over the rationals. In such a case, the rings $R[\theta_1, \ldots, \theta_j]$ will also be algebras over the rationals, but we do not refer to them as Ritt algebras since they are usually not commutative.

LEMMA 17. *Let R be any differential ring which is an algebra over the rationals, and let M be any maximal right ideal of R. If $(\delta + a)(M) \not\subseteq M$ for all $a \in R$, then $MR[\theta]$ is a maximal right ideal of $R[\theta]$.*

PROOF. Suppose on the contrary that $R[\theta]$ has a right ideal J such that $MR[\theta] < J < R[\theta]$, and pick an element $x \in J - MR[\theta]$ of minimal degree. Observing that $J \cap R = M$, we see that x must have degree $n > 0$, and we write $x = x_0 + \ldots + x_n\theta^n$ with $x_0, \ldots, x_n \in R$ and $x_n \neq 0$. In view of the minimality of n, we infer that $x_n \notin M$, whence $x_nr + y = 1$ for some $r \in R$, $y \in M$. Then $xr + y\theta^n$ has leading term θ^n, hence $xr + y\theta^n$ is an element of $J - MR[\theta]$ with degree n. Thus, replacing x by $xr + y\theta^n$, we may assume that $x_n = 1$.

Given any $m \in M$, it is clear that $xm - m\theta^n \in J$. Observing that $xm - m\theta^n$ has degree at most $n - 1$, we obtain $xm - m\theta^n \in MR[\theta]$, by the minimality of n. Since the coefficient of θ^{n-1} in $xm - m\theta^n$ is $x_{n-1}m + n(\delta m)$, we thus get $x_{n-1}m + n(\delta m) \in M$. But now $(\delta + x_{n-1}/n)(M) \subseteq M$, which is impossible.

LEMMA 18. *Let R be a u-differential Ritt algebra, and let M be a maximal ideal of R. Assume that s is a nonnegative integer such that the induced maps $\delta_1^*, \ldots, \delta_s^* \in \mathrm{Hom}_{R/M}(M/M^2, R/M)$ are linearly independent over R/M. Then $MR[\theta_1, \ldots, \theta_s]$ is a maximal right ideal of $R[\theta_1, \ldots, \theta_s]$.*

PROOF. We first prove the following series of statements $P_0, \ldots, P_{s-1}$.

P_j: If $a \in R[\theta_1, \ldots, \theta_j]$ and $r_{j+1}, \ldots, r_s \in R$ such that

$$(a + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_j],$$

then $a \in R + MR[\theta_1, \ldots, \theta_j]$ and $r_{j+1}, \ldots, r_s \in M$.

To prove P_0, assume that we have $a, r_1, \ldots, r_s \in R$ such that $(a + r_1\delta_1 + \ldots + r_s\delta_s)(M) \subseteq M$. Since $aM \subseteq M$ as well, we obtain $(r_1\delta_1 + \ldots + r_s\delta_s)(M) \subseteq M$, for which it follows that $r_1\delta_1^* + \ldots + r_s\delta_s^* = 0$. In view of the linear independence of $\delta_1^*, \ldots, \delta_s^*$ over R/M, this implies that $r_1, \ldots, r_s \in M$. Therefore P_0 holds.

Now let $0 < j \leqslant s - 1$ and assume that P_{j-1} holds. If P_j fails, then there exist elements $a \in R[\theta_1, \ldots, \theta_j]$ and $r_{j+1}, \ldots, r_s \in R$ such that

$$(a + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_j],$$

but either $a \notin R + MR[\theta_1, \ldots, \theta_j]$ or else some $r_i \notin M$. In case $a \in R + MR[\theta_1, \ldots, \theta_j]$, then $aM \subseteq MR[\theta_1, \ldots, \theta_j]$ and hence $(r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_j]$, from which we obtain $(r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq M$. In this situation, however, P_0 says that $r_{j+1}, \ldots, r_s \in M$, which is impossible. Thus we must have $a \notin R + MR[\theta_1, \ldots, \theta_j]$, and in particular $a \neq 0$. We may also assume that a has the lowest degree in θ_j of those elements of $R[\theta_1, \ldots, \theta_j]$ for which there exist $r_{j+1}, \ldots, r_s \in R$ with

$$(a + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_j].$$

Now write $a = a_0 + a_1\theta_j + \ldots + a_k\theta_j^k$, where $a_0, \ldots, a_k \in R[\theta_1, \ldots, \theta_{j-1}]$ and $a_k \neq 0$. In view of P_{j-1}, we must have $k > 0$, and then it follows from the minimality of k that $a_k \notin MR[\theta_1, \ldots, \theta_{j-1}]$.

If $k \geqslant 2$, then for any $m \in M$ we compute that

$$(a + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(m)$$

leads off with the terms $a_k m\theta_j^k + [a_{k-1}m + ka_k(\delta_j m)]\theta_j^{k-1}$, from which we obtain

$$a_k m,\ a_{k-1}m + ka_k(\delta_j m) \in MR[\theta_1, \ldots, \theta_{j-1}].$$

First, we have $a_k M \subseteq MR[\theta_1, \ldots, \theta_{j-1}]$, hence P_{j-1} says that $a_k = r + b$ for some $r \in R$, $b \in MR[\theta_1, \ldots, \theta_{j-1}]$. Inasmuch as $a_k \notin MR[\theta_1, \ldots, \theta_{j-1}]$, we see that $r \notin M$. Second, we have $(a_{k-1} + ka_k\delta_j)(M) \subseteq MR[\theta_1, \ldots, \theta_{j-1}]$, and clearly $(kb\delta_j)(M) \subseteq MR[\theta_1, \ldots, \theta_{j-1}]$ as well, whence

$$(a_{k-1} + kr\delta_j)(M) \subseteq MR[\theta_1, \ldots, \theta_{j-1}].$$

According to P_{j-1}, we obtain $kr \in M$, and then $r \in M$ (because R is a Ritt algebra). This is a contradiction.

Therefore $k < 2$, so the only possibility left is $k = 1$. Now $a = a_0 + a_1\theta_j$, hence for any $m \in M$ we have

$$(a + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(m)$$
$$= a_1 m\theta_j + [a_0 m + a_1(\delta_j m) + r_{j+1}(\delta_{j+1}m) + \ldots + r_s(\delta_s m)].$$

Thus $a_1 M \subseteq MR[\theta_1, \ldots, \theta_{j-1}]$ and also

$$(a_0 + a_1\delta_j + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_{j-1}].$$

As above, it follows from the first inclusion that $a_1 = r + b$ for some $r \in R - M$, $b \in MR[\theta_1, \ldots, \theta_{j-1}]$, and then we infer from the second inclusion that

$$(a_0 + r\delta_j + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_{j-1}].$$

But now P_{j-1} gives us $r \in M$, which is impossible.

Therefore P_j must hold, and the induction works. We now return to the proof of the lemma and show that for $j = 0, \ldots, s$, $MR[\theta_1, \ldots, \theta_j]$ is a maximal right ideal of $R[\theta_1, \ldots, \theta_j]$. For $j = 0$, this is part of our hypotheses. Now let $0 < j \leqslant s$ and assume that $MR[\theta_1, \ldots, \theta_{j-1}]$ is a maximal right ideal of $R[\theta_1, \ldots, \theta_{j-1}]$. In view of P_{j-1}, we must have

$$(\delta_j + a)(MR[\theta_1, \ldots, \theta_{j-1}]) \not\subseteq MR[\theta_1, \ldots, \theta_{j-1}]$$

for all $a \in R[\theta_1, \ldots, \theta_{j-1}]$, whence Lemma 17 shows that $MR[\theta_1, \ldots, \theta_j]$ is a maximal right ideal of $R[\theta_1, \ldots, \theta_j]$.

PROPOSITION 19. *Let R be a noetherian u-differential Ritt algebra, and set* $T = R[\theta_1, \ldots, \theta_u]$. *Let M be any maximal ideal of R. If J is any right ideal of T which contains M, then* $\mathrm{pd}_T(T/J) \leqslant \mathrm{rank}(M) + \mathrm{diff\ dim}(M)$.

PROOF. If $s = \mathrm{diff\ codim}(M)$, then according to Proposition 12 the subspace W of $\mathrm{Hom}_{R/M}(M/M^2, R/M)$ spanned by $\delta_1^*, \ldots, \delta_u^*$ has dimension s; hence we may arrange the indices $1, \ldots, u$ so that $\delta_1^*, \ldots, \delta_s^*$ is a basis for W. Setting $Q = R[\theta_1, \ldots, \theta_s]$, we now see from Lemma 18 that MQ is a maximal right ideal of Q.

Given any $j \in \{s + 1, \ldots, u\}$, we must have $\delta_j^* = r_{j1}\delta_1^* + \ldots + r_{js}\delta_s^*$ for suitable $r_{ji} \in R$, whence $(\delta_j - r_{j1}\delta_1 - \ldots - r_{js}\delta_s)(M) \subseteq M$. Setting $q_j = r_{j1}\theta_1 + \ldots + r_{js}\theta_s \in Q$, we compute that $(\theta_j - q_j)M \subseteq MT$. If now X denotes the set of all products of nonnegative powers of $\theta_{s+1} - q_{s+1}, \ldots, \theta_u - q_u$, then we obtain $XMT \subseteq MT$.

In particular, $XMQ \subseteq MT \subseteq J$. Observing that T is generated as a right Q-module by X, we infer that $(T/J)_Q$ is a sum of homomorphic images of Q/MQ. Inasmuch as Q/MQ is a simple right Q-module, it follows that $(T/J)_Q$ is isomorphic to a direct sum of copies of Q/MQ, whence $\mathrm{pd}_Q(T/J) \leqslant \mathrm{pd}_Q(Q/MQ)$. Since $Q/MQ \cong (R/M) \otimes_R Q$, we also have $\mathrm{pd}_Q(Q/MQ) \leqslant \mathrm{pd}_R(R/M)$. In addition, $\mathrm{pd}_R(R/M) = \mathrm{rank}(M)$ by Proposition 9, and thus $\mathrm{pd}_Q(T/J) \leqslant \mathrm{rank}(M)$. According to Proposition 1, $\mathrm{pd}_T(T/J) \leqslant u - s + \mathrm{rank}(M)$. Inasmuch as $u - s = \mathrm{diff\ dim}(M)$, this gives us the required inequality.

5. Global dimension formulas.

THEOREM 20. *Let R be a noetherian u-differential Ritt algebra with* $\operatorname{gl}\dim R < \infty$. *Then*

$$\operatorname{r\,gl}\dim R[\theta_1, \ldots, \theta_u] = \sup\{\operatorname{rank}(P) + \operatorname{diff}\dim(P) \mid P \text{ is a prime ideal of } R\}.$$

PROOF. If $S = R[\theta_1, \ldots, \theta_u]$, $n = \operatorname{r\,gl}\dim S$, and

$$k = \sup\{\operatorname{rank}(P) + \operatorname{diff}\dim(P) \mid P \text{ is a prime ideal of } R\},$$

then $n \geqslant k$ by Proposition 15. According to Proposition 2, $n \leqslant u + \operatorname{gl}\dim R < \infty$. Inasmuch as S is right and left noetherian, Theorem 16 says that there exists a simple right S-module A with $\operatorname{pd}_S(A) = n$, and we note that $\operatorname{wd}_S(A) = n$ also.

Choose a nonzero element $x \in A$ whose R-annihilator $P = \{r \in R \mid xr = 0\}$ is maximal among the R-annihilators of all nonzero elements of A. According to [10, Theorem 6], P is a prime ideal of R. If $T = R_P[\theta_1, \ldots, \theta_u]$, then the right R_P-module A_P can be made into a right T-module by defining $(a/s)\theta_i = [a\theta_i s + a(\delta_i s)]/s^2$ for all i and all $a/s \in A_P$. Since the R-annihilator of x is P, the natural map $A \longrightarrow A_P$ is not zero. However, this map is an S-homomorphism and A is a simple S-module, hence $A \longrightarrow A_P$ must be a monomorphism. In view of Lemma 14, we thus obtain $\operatorname{wd}_S(A_P) = n$.

Now $A = xS$ and thus $A_P = (x/1)T$, from which we infer that $A_P \cong T/J$ for some right ideal J of T which contains PR_P. According to Proposition 19, $\operatorname{pd}_T(A_P) \leqslant \operatorname{rank}(PR_P) + \operatorname{diff}\dim(PR_P)$. In view of Corollary 13, we now obtain $\operatorname{wd}_T(A_P) \leqslant \operatorname{rank}(P) + \operatorname{diff}\dim(P) \leqslant k$. Inasmuch as T_S is flat by Proposition 3, $\operatorname{wd}_S(A_P) \leqslant \operatorname{wd}_T(A_P)$, and therefore $n \leqslant k$.

THEOREM 21. *Let R be any commutative noetherian u-differential ring such that* $\operatorname{gl}\dim R < \infty$. *Set*

$$k = \sup\{\operatorname{rank}(P) + \operatorname{diff}\dim(P) \mid P \text{ is a prime ideal of } R \text{ and } \operatorname{char}(R/P) = 0\},$$

$$q = \sup\{\operatorname{rank}(M) \mid M \text{ is a maximal ideal of } R \text{ and } \operatorname{char}(R/M) > 0\}.$$

[*In either case, if there are no ideals of the type required, the supremum is considered to be* $-\infty$.] *Then*

$$\operatorname{r\,gl}\dim R[\theta_1, \ldots, \theta_u] = \max\{k, q + u\}.$$

PROOF. In view of Propositions 10 and 15, we have $\operatorname{r\,gl}\dim R[\theta_1, \ldots, \theta_u] \geqslant \max\{k, q + u\}$. According to Proposition 4, the reverse inequality will hold provided r gl dim $R_M[\theta_1, \ldots, \theta_u] \leqslant \max\{n, q + u\}$ for each maximal ideal M of R.

First consider the case when $\mathrm{char}(R/M) > 0$. According to Proposition 9, gl dim $R_M = \mathrm{rank}(M) \leqslant q$, hence Proposition 2 shows that r gl dim $R_M[\theta_1, \ldots, \theta_u] \leqslant q + u$.

Now assume that $\mathrm{char}(R/M) = 0$. Here $nR_M \not\subseteq MR_M$ for all nonzero integers n, hence all nonzero integers are invertible in R_M. Thus R_M is a Ritt algebra, and so Theorem 20 is applicable. According to [10, Theorem 34], any prime ideal of R_M must have the form PR_M for some prime ideal P of R which is contained in M, and since $\mathrm{char}(R/M) = 0$ we see that $\mathrm{char}(R/P) = 0$, too. In view of Corollary 13, we obtain

$$\mathrm{rank}(PR_M) + \mathrm{diff\ dim}(PR_M) = \mathrm{rank}(P) + \mathrm{diff\ dim}(P) \leqslant k,$$

and therefore Theorem 20 shows that r gl dim $R_M[\theta_1, \ldots, \theta_u] \leqslant k$. □

In particular, Theorem 21 gives a formula for the global dimension of $R[\theta]$ when R is only a 1-differential ring. For this case, the formula can be improved somewhat as follows, since the differential dimension of any prime ideal P depends only on whether or not P is a differential ideal. Also, for this case it is possible to restrict attention to just the maximal ideals of R.

THEOREM 22. *Let R be any commutative noetherian differential ring with* gl dim $R = n < \infty$. *Let $\mathfrak{M}$ denote the collection of all differential maximal ideals of R, together with all maximal ideals M such that* $\mathrm{char}(R/M) > 0$, *and set* $k = \sup\{\mathrm{rank}(M) \mid M \in \mathfrak{M}\}$. *[If $\mathfrak{M}$ is empty, then k is considered to be $-\infty$.] Then* r gl dim $R[\theta] = \max\{n, k + 1\}$.

PROOF. According to Proposition 2, r gl dim $R[\theta] \geqslant n$. Inasmuch as diff dim$(M) = 1$ for any differential maximal ideal M of R, Theorem 21 shows that r gl dim $R[\theta] \geqslant k + 1$.

Suppose that P is any prime ideal of R with $\mathrm{char}(R/P) = 0$. If P is not maximal, then it is clear from Proposition 9 that $\mathrm{rank}(P) < n$. Since diff dim$(P) \leqslant 1$, we get $\mathrm{rank}(P) + \mathrm{diff\ dim}(P) \leqslant n$ in this case. Now assume that P is a maximal ideal. If P is not a differential ideal, then diff dim$(P) = 0$ and $\mathrm{rank}(P) + \mathrm{diff\ dim}(P) \leqslant n$, using Proposition 9 again. On the other hand, if P is a differential ideal, then $\mathrm{rank}(P) + \mathrm{diff\ dim}(P) = 1 + \mathrm{rank}(P) \leqslant k + 1$, by definition of k.

Thus we have $\mathrm{rank}(P) + \mathrm{diff\ dim}(P) \leqslant \max\{n, k + 1\}$ for all prime ideals P of R such that $\mathrm{char}(R/P) = 0$. In view of Theorem 21, we conclude that r gl dim $R[\theta] \leqslant \max\{n, k + 1\}$.

We conclude this section by using Theorem 22 to derive a formula for the global dimension of $R[\theta]$ which involves only differential ideals of R. We recall that a proper ideal J in a commutative ring R is said to be *primary* provided all zero-divisors in the ring R/J are nilpotent. .

THEOREM 23. *Let R be any commutative noetherian differential ring with* gl dim $R = n < \infty$, *and set* $k = \sup\{\mathrm{pd}_R(R/J) \mid J$ *is a primary differential ideal of* $R\}$. [*If R has no primary differential ideals, then k is considered to be* $-\infty$.] *Then* r gl dim $R[\theta] = \max\{n, k+1\}$.

PROOF. According to Proposition 2, r gl dim $R[\theta] \geqslant n$. Inasmuch as R is semiprime by Proposition 9, Corollary 8 shows that r gl dim $R[\theta] \geqslant k+1$.

Now consider any maximal ideal M of R such that $\mathrm{char}(R/M) = p > 0$. If J is the ideal of R generated by pR and $\{x^p \mid x \in M\}$, then as in Proposition 10 we see that M/J is nilpotent and that $\mathrm{pd}_R(R/J) = \mathrm{pd}_R(R/M)$. Inasmuch as M/J is nilpotent, R/J must be local, from which we infer that J is a primary ideal of R. Also, J is clearly a differential ideal, whence $\mathrm{pd}_R(R/J) \leqslant k$. Since $\mathrm{pd}_R(R/M) = \mathrm{rank}(M)$ by Proposition 9, we thus obtain $\mathrm{rank}(M) \leqslant k$.

Thus we have $\mathrm{rank}(M) \leqslant k$ for all maximal ideals M of R such that $\mathrm{char}(R/M) > 0$. Since any differential maximal ideal M of R is a primary differential ideal, we also have $\mathrm{rank}(M) \leqslant k$ for all differential maximal ideals M. According to Theorem 22, we thus obtain r gl dim $R[\theta] \leqslant \max\{n, k+1\}$.

6. **Applications.** For any ring S and any positive integer u, the *Weyl algebra* of degree u over S is the ring $A_u(S) = S[x_1, \ldots, x_u][\theta_1, \ldots, \theta_u]$, where the x_i are ordinary polynomial indeterminates, and we use the derivations $\delta_i = \partial/\partial x_i$ on $S[x_1, \ldots, x_u]$. J.-E. Roos has shown that for a field F of characteristic 0, r gl dim $A_u(F) = u$ [13, Théorème 1], while G. S. Rinehart has shown that, for a field F of positive characteristic, r gl dim $A_u(F) = 2u$ [11, Theorem, p. 345]. We generalize these results in the following theorem, which has also been proved (using entirely different methods) in [12, Theorem 2.6].

THEOREM 24. *Let S be any commutative noetherian ring with* gl dim $S = n < \infty$, *and set* $k = \sup\{\mathrm{rank}(M) \mid M$ *is a maximal ideal of S and* $\mathrm{char}(S/M) > 0\}$. [*If S has no such maximal ideals, then k is considered to be* $-\infty$.] *Then for any positive integer u,* r gl dim $A_u(S) = \max\{n+u, k+2u\}$.

PROOF. Set $R = S[x_1, \ldots, x_u]$ and $\delta_i = \partial/\partial x_i$ for $i = 1, \ldots, u$. Since gl dim $R = n+u$, Proposition 2 shows that r gl dim $A_u(S) \geqslant n+u$.

If S has any maximal ideals M such that $\mathrm{char}(S/M) > 0$, then we may choose such an M with $\mathrm{rank}(M) = k$. Inasmuch as S/M is a field, the ring $R/MR \cong (S/M)[x_1, \ldots, x_u]$ has Krull dimension u, whence R/MR must have a maximal ideal K/MR of rank u. Then K is a maximal ideal of R such that $\mathrm{char}(R/K) > 0$, and clearly $\mathrm{rank}(K) \geqslant k+u$, hence Theorem 21 says that r gl dim $A_u(S) \geqslant k+2u$.

Therefore r gl dim $A_u(S) \geq \max\{n + u, k + 2u\}$. According to Theorem 21, to prove the reverse inequality it is enough to show that $\operatorname{rank}(M) \leq k + u$ for any maximal ideal M of R with $\operatorname{char}(R/M) > 0$, and that $\operatorname{rank}(P) + \operatorname{diff\,dim}(P) \leq n + u$ for any prime ideal P of R such that $\operatorname{char}(R/P) = 0$.

First consider any maximal ideal M of R for which $\operatorname{char}(R/M) > 0$. Choosing a maximal ideal K of S which contains $S \cap M$, we have $\operatorname{char}(S/K) > 0$ and so $\operatorname{rank}(S \cap M) \leq \operatorname{rank}(K) \leq k$. By induction on [10, Theorem 149], we find that $\operatorname{rank}(M) \leq k + u$.

Now consider any prime ideal P of R with $\operatorname{char}(R/P) = 0$, and set $s =$ diff dim(P). If $T = R_P$, $M = PR_P$, and W is the subspace of $\operatorname{Hom}_{T/M}(M/M^2, T/M)$ spanned by $\delta_1^*, \ldots, \delta_u^*$, then by Proposition 12 W has dimension $u - s$. Thus we may arrange the indices $1, \ldots, u$ so that $\delta_{s+1}^*, \ldots, \delta_u^*$ is a basis for W. Set $Q = P \cap (S[x_1, \ldots, x_s])$ and note that $S[x_1, \ldots, x_s]/Q$ has characteristic 0. We claim that $\delta_i(Q) \subseteq Q$ for $i = 1, \ldots, s$. Given $1 \leq i \leq s$, we must have $\delta_i^* = t_{s+1}\delta_{s+1}^* + \ldots + t_u\delta_u^*$ for suitable $t_j \in T$. Multiplying out the denominators in this equation, we obtain

$$a\delta_i^* = r_{s+1}\delta_{s+1}^* + \ldots + r_u\delta_u^* \quad \text{for some } a \in R - P, r_{s+1}, \ldots, r_u \in R.$$

Thus $(a\delta_i - r_{s+1}\delta_{s+1} - \ldots - r_u\delta_u)(M) \subseteq M$, from which we infer that

$$(a\delta_i - r_{s+1}\delta_{s+1} - \ldots - r_u\delta_u)(P) \subseteq P.$$

Since $Q \subseteq P$ and $\delta_{s+1}, \ldots, \delta_u$ all vanish on Q, we thus obtain $a\delta_i(Q) \subseteq P$. Now P is a prime ideal of R and $a \in R - P$, hence it follows that $\delta_i(Q) \subseteq P$, from which we conclude that $\delta_i(Q) \subseteq Q$, as claimed.

All of the rings $S[x_1, \ldots, x_i]/(Q \cap S[x_1, \ldots, x_i])$ $(i = 1, \ldots, s)$ have characteristic 0, hence with the help of the relations $\delta_i(Q) \subseteq Q$ an easy induction shows that $Q \cap (S[x_1, \ldots, x_i]) = (Q \cap S)[x_1, \ldots, x_i]$ for each $i = 1, \ldots, s$. Consequently $Q = (Q \cap S)[x_1, \ldots, x_s]$, whence [10, Theorem 149] shows that $\operatorname{rank}(Q) = \operatorname{rank}(Q \cap S)$. That same theorem also shows that $\operatorname{rank}(P) \leq u - s + \operatorname{rank}(Q)$, and it is clear from Proposition 12 that $\operatorname{rank}(Q \cap S) \leq n$, hence we obtain $\operatorname{rank}(P) \leq n + u - s$. Therefore $\operatorname{rank}(P) + \operatorname{diff\,dim}(P) \leq n + u$.

COROLLARY 25. *Let S be any commutative noetherian ring with* gl dim $S = n < \infty$, *and let u be any positive integer. If S is an algebra over the rationals, then* r gl dim $A_u(S) = n + u$.

Corollary 25 has also been obtained in [1, Corollary 2.6].

Given any ring S and any positive integer u, then following [2] we can define a ring $F_u(S) = S[[x_1, \ldots, x_u]][\theta_1, \ldots, \theta_u]$ analogous to the Weyl algebra $A_u(S)$. If S is a commutative noetherian ring with gl dim $S = n < \infty$, and if S is an algebra over the rationals, then J.-E. Björk has shown in [2, Theorem 4.2] that r gl dim $F_u(S) = n + u$. We shall generalize this result, but first some facts about power series rings must be developed. [We note that our proofs do not depend on Björk's result, and our methods are completely different from his.]

LEMMA 26. *Let S be a commutative noetherian ring with* gl dim $S = n < \infty$. *If u is any positive integer, then* gl dim $S[[x_1, \ldots, x_u]] = n + u$.

PROOF. It obviously suffices to prove the case $n = 1$. The indeterminate x lies in the Jacobson radical of $S[[x]]$, and x is not a zero-divisor in $S[[x]]$. Since $S[[x]]/xS[[x]] \cong S$, [9, Part III, Theorem 10] shows that r gl dim $S[[x]] = n + 1$.

LEMMA 27. *If F is a field and u any positive integer, then $F[[x_1, \ldots, x_u]]$ has Krull dimension u.*

PROOF. This is immediate from Lemma 26 and Proposition 9. □

For use in the next lemma, we recall that if S is a commutative local ring with maximal ideal M, then $S[[x]]$ is a local ring with maximal ideal generated by M and x. Clearly, the ideal J of $S[[x]]$ generated by M and x is a maximal ideal. Also, if p is any element of $S[[x]]$ which does not belong to J, then the constant term of p is not in M and so is invertible in S, whence p is invertible in $S[[x]]$.

LEMMA 28. *Let S be a commutative noetherian ring, and let Q be any prime ideal of S, u any positive integer. Then $Q[[x_1, \ldots, x_u]]$ is a prime ideal of $S[[x_1, \ldots, x_u]]$ with rank equal to* rank(Q). *Also, if P is any prime ideal of $S[[x_1, \ldots, x_u]]$ such that $P \cap S = Q$, then* rank$(P) \leqslant u +$ rank(Q).

PROOF. We may obviously assume that rank$(Q) < \infty$. Also, we clearly need only prove the case $u = 1$. Finally, since P is disjoint from $S - Q$, all the ranks we are interested in remain the same after localizing at Q, hence we may assume, without loss of generality, that S is local with maximal ideal Q. As remarked above, it follows that $S[[x]]$ is local with maximal ideal M generated by Q and x.

Now M is a prime ideal in the noetherian ring $S[[x]]$, and x is an element of M which is not a zero-divisor in $S[[x]]$, hence [10, Theorem 155] says that the rank of $M/xS[[x]]$ in $S[[x]]/xS[[x]]$ equals rank$(M) - 1$. Inasmuch as $S[[x]]/xS[[x]] \cong S$, we infer that rank$(M/xS[[x]])$ = rank(Q), and thus rank$(M) = 1 +$ rank(Q). Observing that $P \subseteq M$, we obtain rank$(P) \leqslant 1 +$ rank(Q). Finally, since rank$(M) = 1 +$ rank$(Q) < \infty$ and $Q[[x]]$ is properly contained in M, we must have rank$(Q[[x]]) \leqslant$ rank(Q), from which we conclude that rank$(Q[[x]])$ = rank(Q). □

With the help of these three lemmas, we may use the proof of Theorem 24, mutatis mutandis, to prove the following generalization of Björk's theorem:

THEOREM 29. *Let S be any commutative noetherian ring with* gl dim $S = n < \infty$, *and set* $k = \sup\{\operatorname{rank}(M) \mid M$ *is a maximal ideal of* S *and* $\operatorname{char}(S/M) > 0\}$. [*If S has no such maximal ideals, then k is considered to be* $-\infty$.] *Then, for any positive integer u,*

$$\text{r gl dim } F_u(S) = \max\{n + u, k + 2u\}.$$

J. Cozzens and J. Johnson have shown that for any u-differential field F, r gl dim $F[\theta_1, \ldots, \theta_u] = u$ [4, Theorem 1(b)]. In view of Corollary 8 and Proposition 2, this result generalizes to semisimple artinian rings:

THEOREM 30. *If R is any semisimple artinian u-differential ring, then* r gl dim $R[\theta_1, \ldots, \theta_u] = u$.

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