

HOLOMORPHIC FUNCTIONS ON NUCLEAR SPACES

BY

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ABSTRACT. The space $H(E)$ of holomorphic functions on a quasi-complete nuclear space is investigated. If $H(E)$ is endowed with the compact open topology, it is shown that $H(E)$ is nuclear if and only if E' (continuous dual of E) is nuclear. If E is a \mathcal{DFN} (dual of a Fréchet nuclear space) and F is a closed subspace of E , then the restriction mapping $H(E) \rightarrow H(F)$ is a surjective strict morphism.

1. **Topologies on $H(E)$.** In this article, E will denote a quasi-complete nuclear locally convex space over the complex numbers \mathbb{C} . E' will denote the continuous dual of E endowed with the strong topology (topology of uniform convergence on bounded subsets of E). In a quasi-complete nuclear space E , bounded subsets are relatively compact. Therefore the strong topology on E' coincides with the Mackey topology (topology of uniform convergence on convex compact subsets of E). If U is a convex balanced neighborhood of 0 in E , p_U will denote its Minkowski functional.

If K is a convex balanced compact subset of E , $[K]$ will denote the linear span of K in E . If $x \in [K]$, then $p_K(x) = \inf\{\lambda: x \in \lambda K \text{ and } \lambda > 0\}$. The polar K° of K is the set $\{\varphi: \varphi \in E' \text{ where } |\varphi|_K = \sup_{x \in K} |\varphi(x)| \leq 1\}$. As K varies through the convex balanced compact subsets of E , the sets K° form a base of neighborhoods for 0 in E' . p_{K° is the seminorm defined on E' by $p_{K^\circ}(\varphi) = |\varphi|_K$, and we let $E'(K^\circ) = E'/p_{K^\circ}^{-1}(0)$. p_{K° naturally induces a norm on $E'(K^\circ)$ which will also be denoted p_{K° .

E' is a nuclear space if given any convex balanced compact subset K of E , there exists another convex balanced compact subset K_1 of E containing K such that the mapping $E'(K_1^\circ) \rightarrow E'(K^\circ)$ is a nuclear mapping between normed spaces. $E'(K_1^\circ) \rightarrow E'(K^\circ)$ is nuclear if there exist $(a_n)_n \subseteq [K_1]$ and $(\varphi_n)_n \subseteq E'$ such that for each $\varphi \in E'$, $\varphi = \sum_N \langle a_n, \varphi \rangle \varphi_n$ (convergence in $E'(K^\circ)$) and $\sum_N p_{K_1}(a_n) p_{K^\circ}(\varphi_n) = M < +\infty$. By replacing K_1 by a sufficiently large multiple of itself, we may assume that $K \subseteq K_1$ and that $M < C$ where $C = 1/4 \sup_{r \geq 1} (r'/r!)^{1/r}$.

DEFINITION 1.1. $L(mE)$, $P(mE)$ will denote, respectively, the continuous m -linear forms and the continuous m -homogeneous polynomials on E . For each

compact set $K \subseteq E$ and $A \in L({}^m E)$ (respectively $p \in P({}^m E)$), let $\epsilon_K(A) = \sup_{x_i \in K} |A(x_1, \dots, x_m)|$ (respectively $\epsilon_K(p) = \sup_{x \in K} |p(x)|$). ϵ will be used to denote the topologies on $L({}^m E)$ and $P({}^m E)$ generated by all seminorms of the type ϵ_K as K varies through the compact subsets of E . $L_s({}^m E)$ will denote the continuous symmetric m -linear forms on E , and $P_f({}^m E)$ will denote the continuous m -homogeneous polynomials of finite type on E ($p \in P({}^m E)$ is of finite type if p can be represented in the form $p = \sum_{i=1}^n \varphi_i^m$ where $\varphi_i \in E'$, $i = 1, \dots, n$).

PROPOSITION 1.1. *Let E be quasi-complete and nuclear, and E' nuclear. Then $L({}^m E), \epsilon$ is a nuclear space.*

PROOF. Let K be a convex balanced compact subset of E . Since E' is nuclear there exists a convex balanced compact subset K_1 of E containing K such that the mapping $E'(K_1^\circ) \rightarrow E'(K^\circ)$ is nuclear with nuclear norm $< C$. Hence let $(a_n)_n \subseteq [K_1]$ and $(\varphi_n) \subseteq E'$ be such that for each $\varphi \in E'$, $\varphi = \sum_N \langle a_n, \varphi \rangle \varphi_n$ (convergence in $E'(K^\circ)$) and $\sum_N p_{K_1}(a_n) p_{K^\circ}(\varphi_n) = M < C$. The claim is that the identity mapping $I_{K_1, K}^m: L({}^m E), \epsilon_{K_1} \rightarrow L({}^m E), \epsilon_K$ is nuclear and that its nuclear norm is $\leq M^m$.

Let $A \in L({}^m E)$. From Boland [2], we know that A may be represented in the form $A(x_1, \dots, x_m) = \sum_{i=1}^\infty \langle x_1, \psi_{i1} \rangle \cdots \langle x_m, \psi_{im} \rangle$ where each $\psi_{ij} \in E'$ and for some convex balanced neighborhood U of 0 in E the series $\sum_{i=1}^\infty p_{U^\circ}(\psi_{i1}) \cdots p_{U^\circ}(\psi_{im})$ converges.

Therefore

$$\begin{aligned} A &= \sum_{i=1}^\infty \langle \cdot, \psi_{i1} \rangle \cdots \langle \cdot, \psi_{im} \rangle \\ &= \sum_{i=1}^\infty \left\langle \cdot, \sum_N \langle a_n, \psi_{i1} \rangle \varphi_n \right\rangle \cdots \left\langle \cdot, \sum_N \langle a_n, \psi_{im} \rangle \varphi_n \right\rangle \\ &= \sum_{i=1}^\infty \sum_{n_1, \dots, n_m} \langle a_{n_1}, \psi_{i1} \rangle \cdots \langle a_{n_m}, \psi_{im} \rangle \varphi_{n_1} \cdots \varphi_{n_m} \\ &= \sum_{n_1, \dots, n_m} \left[\sum_{i=1}^\infty \langle a_{n_1}, \psi_{i1} \rangle \cdots \langle a_{n_m}, \psi_{im} \rangle \right] \varphi_{n_1} \cdots \varphi_{n_m} \\ &= \sum_{n_1, \dots, n_m} \langle A, a_{n_1 \dots n_m} \rangle \varphi_{n_1} \cdots \varphi_{n_m} \end{aligned}$$

where $\langle A, a_{n_1 \dots n_m} \rangle = \sum_{i=1}^\infty \langle a_{n_1}, \psi_{i1} \rangle \cdots \langle a_{n_m}, \psi_{im} \rangle$.

Furthermore if U_1 is the unit ball in $L({}^m E), \epsilon_{K_1}$ and $p_{U_1^\circ}(a_{n_1 \dots n_m}) = \sup_{A \in U_1} |\langle A, a_{n_1 \dots n_m} \rangle|$, then

$$\begin{aligned} & \sum_{n_1, \dots, n_m} p_{U_1^0}(a_{n_1 \dots n_m}) \epsilon_K(\varphi_{n_1} \cdots \varphi_{n_m}) \\ & \leq \sum_{n_1, \dots, n_m} p_{K_1}(a_{n_1}) \cdots p_{K_1}(a_{n_m}) \epsilon_K(\varphi_{n_1}) \cdots \epsilon_K(\varphi_{n_m}) \leq M^m. \end{aligned}$$

This shows that $I_{K_1, K}^m$ is nuclear with nuclear norm

$$\gamma(I_{K_1, K}^m) \leq M^m = \left(\sum_N p_{K_1}(a_n) p_{K^0}(\varphi_n) \right)^m.$$

COROLLARY 1.1. $L_s(^mE), \epsilon$ is nuclear.

COROLLARY 1.2. $P(^mE), \epsilon$ is nuclear.

PROOF. $P(^mE), \epsilon$ and $L_s(^mE), \epsilon$ are homeomorphic via the mapping $A \in L_s(^mE) \rightarrow p_A \in P(^mE)$ where $p_A(x) = A(x, \dots, x)$. Hence Corollary 1.2 follows from Corollary 1.1.

As in the proof of Proposition 1.1, it may be shown that the mapping $I_{K_1, K}^m: P(^mE), \epsilon_{K_1} \rightarrow P(^mE), \epsilon_K$ is nuclear and its nuclear norm $\gamma(I_{K_1, K}^m) \leq m^m M^m / m!$. This follows from elementary properties of compositions of nuclear and continuous mappings, and the following sequence of continuous mappings:

$$\begin{aligned} P(^mE), \epsilon_{K_1} & \rightarrow L_s(^mE), \epsilon_{K_1} \rightarrow L(^mE), \epsilon_{K_1} \xrightarrow{I_{K_1, K}^m} L(^mE), \epsilon_K \\ & \rightarrow L_s(^mE), \epsilon_K \rightarrow P(^mE), \epsilon_K. \end{aligned}$$

(See Nachbin [5].)

DEFINITION 1.2. $H(E)$ will denote the space of all C-valued holomorphic functions on E . For each compact subset K of E , ϵ_K will denote the seminorm defined on $H(E)$ by $\epsilon_K(f) = \sup_{x \in K} |f(x)|$. ϵ will denote the topology generated by all seminorms of the form ϵ_K . ϵ is the compact open topology.

THEOREM 1.1. *Let E be quasi-complete and nuclear, and E' nuclear. Then $H(E), \epsilon$ is nuclear.*

PROOF. Let K be a given compact set, and choose K_1 as in Corollary 1.2. We know that for each $m \geq 1$,

$$P(^mE), \epsilon_{K_1} \xrightarrow{I_{K_1, K}^m} P(^mE), \epsilon_K$$

is nuclear with nuclear norm

$$\gamma(I_{K_1, K}^m) \leq \frac{m^m}{m!} M^m \leq \frac{m^m}{m!} \left(1/4 \sup_{r \geq 1} \left(\frac{r^r}{r!} \right)^{1/r} \right)^m < \frac{1}{2^m}.$$

Hence for each $m \exists (a_{mn})_n \subseteq (P(^mE), \epsilon_{K_1})'$ and $(y_{mn})_n \subseteq P(^mE), \epsilon_K$ such that for $p \in P(^mE), \epsilon_{K_1}$,

$$p = \sum_n \langle p, a_{mn} \rangle y_{mn} \quad (\text{convergence in } P({}^m E), \epsilon_K)$$

and $\sum_n p_{U_1^\circ}(a_{mn})\epsilon_K(y_{mn}) < 1/2^m$ (where U_1 is the unit ball in $P({}^m E)$, ϵ_{K_1}). Now each a_{mn} may be extended to \tilde{a}_{mn} which is continuous on $H(E)$, ϵ , by defining

$$\tilde{a}_{mn}(f) = a_{mn}(\hat{d}^m f(0)/m!).$$

Hence, given $f \in H(E)$, ϵ_{K_1} ,

$$\begin{aligned} f &= \sum_{m=0}^\infty \frac{\hat{d}^m f(0)}{m!} \\ &= \sum_{m=0}^\infty \sum_{n=0}^\infty \left\langle \frac{\hat{d}^m f(0)}{m!}, a_{mn} \right\rangle y_{mn} \\ &= \sum_{m=0}^\infty \sum_{n=0}^\infty \langle f, \tilde{a}_{mn} \rangle y_{mn} = \sum_{m,n \geq 0} \langle f, \tilde{a}_{mn} \rangle y_{mn}. \end{aligned}$$

Moreover, if U_1 is the unit ball in $H(E)$, ϵ_{K_1} , then

$$\sum_{m,n \geq 0} p_{U_1^\circ}(\tilde{a}_{mn})\epsilon_K(y_{mn}) \leq \sum_{m=0}^\infty \frac{1}{2^m} = 2.$$

Therefore $H(E)$, $\epsilon_{K_1} \rightarrow H(E)$, ϵ_K is nuclear.

COROLLARY 1.3. *Let E be a nuclear quasi-complete space. Then $H(E)$, ϵ is nuclear $\Leftrightarrow E'$ is nuclear.*

COROLLARY 1.4. *Let E be a DFN space (the strong dual of a Fréchet nuclear space). Then $H(E)$, ϵ is a Fréchet nuclear space. In particular it is reflexive and separable.*

EXAMPLES. $H(\mathcal{D})$, $H(\mathcal{D}')$, $H(E)$, $H(S)$, $H(\Sigma_N \mathbb{C})$ are all nuclear spaces when endowed with the compact open topology. $H(E')$, $H(S')$, $H(H'(C))$ are Fréchet nuclear spaces when endowed with the compact open topology.

Note. In Barroso, Matos and Nachbin [1] it has been shown that when E is a Silva space, all of the standard topologies $\tau_\delta, \tau_\omega, \tau_0$ ($= \epsilon$) coincide on $H(E)$. Hence in particular we know that if E is a DFN space, then all of the standard topologies $\tau_\delta, \tau_\omega, \tau_0$ on $H(E)$ are equivalent Fréchet nuclear topologies.

2. Polynomials on E .

PROPOSITION 2.1. *Suppose E is a DFN space. Then given any $p \in P({}^m E)$,*

p may be represented in the form $p = \sum_{i=1}^{\infty} \varphi_i^m$ ($\varphi_i \in E^i$) where for each compact subset K of E , $\sum_{i=1}^{\infty} |\varphi_i^m|_K < +\infty$.

PROOF. Since E is a \mathcal{DFN} space, there exists an increasing sequence $(K_n)_n$ of convex compact subsets of E such that given any compact K , then for some n , $K \subseteq K_n$. Now we know that $P_f({}^m E)$ is dense in $P({}^m E)$ for the compact open topology. In Boland [2] it was shown that the compact open topology (ϵ topology) and the π topology coincide on $P({}^m E)$. π is the topology generated by all π_K , where π_K is defined on $P_f({}^m E)$ in the following manner:

$$\pi_K(q) = \inf \left\{ \sum_{i=1}^s |\varphi_i^m|_K : q = \sum_{i=1}^s \varphi_i^m \right\}.$$

Now suppose $p \in P({}^m E)$. Then by these remarks there exists a sequence $(p_r)_r \subseteq P_f({}^m E)$ which converges to p (in both the π and ϵ topologies). In fact (by appropriately choosing a subsequence) we may assume that for $r \geq n$, $\pi_{K_n}(p_r - p_{r-1}) < 1/2^r$ for each n . Let $q_1 = p_1$ and $q_r = p_r - p_{r-1}$ for $r > 1$. Then $\sum_{s=1}^r q_s = p_r \rightarrow p$ as $r \rightarrow \infty$. For each $r > 1$ choose $\varphi_{r1}, \dots, \varphi_{rj_r}$ such that $q_r = \sum_{i=1}^{j_r} \varphi_{ri}^m$ and $\sum_{i=1}^{j_r} |\varphi_{ri}^m|_{K_r} < 1/2^r$. Now for each $x \in E$, $\sum_{r=1}^{\infty} \sum_{i=1}^{j_r} \varphi_{ri}^m(x) = p(x)$, and furthermore given a compact subset K of E , $K \subseteq K_s$ for some s and hence

$$\begin{aligned} \sum_{r=1}^{\infty} \sum_{i=1}^{j_r} |\varphi_{ri}^m|_K &\leq \sum_{r=1}^{\infty} \sum_{i=1}^{j_r} |\varphi_{ri}^m|_{K_s} \\ &\leq \sum_{r=1}^s \sum_{i=1}^{j_r} |\varphi_{ri}^m|_{K_s} + \sum_{r=s+1}^{\infty} \sum_{i=1}^{j_r} |\varphi_{ri}^m|_{K_r} \\ &\leq \sum_{r=1}^s \sum_{i=1}^{j_r} |\varphi_{ri}^m|_{K_s} + \sum_{r=s+1}^{\infty} \frac{1}{2^r} < \infty. \end{aligned}$$

Hence $p = \sum_{r=1}^{\infty} \sum_{i=1}^{j_r} \varphi_{ri}^m$ is a representation of the desired form.

3. Extending holomorphic functions. In this section E will be assumed to be a \mathcal{DFN} space.

LEMMA 3.1. For each compact $K \subseteq E$, $\tilde{\pi}_K(f) = \sum_{n=0}^{\infty} \pi_K(\hat{d}^n f(0)/n!)$ defines a seminorm on $H(E)$. Furthermore the π topology on $H(E)$ generated by all such $\tilde{\pi}_K$ is equivalent to the ϵ (compact open) topology on $H(E)$.

PROOF. $H(E)$, ϵ is a Fréchet nuclear space. Hence by Dineen [3, Proposition 1.3], the topology ϵ on $H(E)$ is that generated by all seminorms $\tilde{\epsilon}_K$ where $\tilde{\epsilon}_K(f) = \sum_{n=0}^{\infty} \epsilon_K(\hat{d}^n f(0)/n!)$. By Boland [2, Corollary 1.5], given a compact

$K \subseteq E$, there exists a compact K_1 (such that for some $\alpha > 0$, $K \subseteq \alpha K_1$) and a constant $C_K > 0$ such that $\pi_K(p) \leq C_K^m \epsilon_{K_1}(p)$ for each $p \in P(mE)$, $m = 0, 1, \dots$. By appropriately choosing K_1 , we may assume $C_K = 1$. Hence

$$\tilde{\pi}_K(f) = \sum_{n=0}^{\infty} \pi_K \left(\frac{\hat{d}^n f(0)}{n!} \right) \leq \sum_{n=0}^{\infty} \epsilon_{K_1} \left(\frac{\hat{d}^n f(0)}{n!} \right) = \tilde{\epsilon}_{K_1}(f).$$

Hence $\tilde{\pi}_K$ is a seminorm on $H(E)$, and π is at least as fine as ϵ . Since $\tilde{\epsilon}_K(f) \leq \tilde{\pi}_K(f)$, it follows that ϵ and π coincide on $H(E)$.

LEMMA 3.2. *Let F be a closed subspace of the \mathcal{DFN} space E . Then the restriction mapping $R: H(E) \rightarrow H(F)$ is almost open (given any neighborhood U of 0 in $H(E)$, there exists a neighborhood V of 0 in $H(F)$ such that $V \subseteq \overline{R(U)}$).*

PROOF. Since F is closed, F is also a \mathcal{DFN} space. Hence $P(mE)$, $P(mF)$ ($m = 0, 1, 2, \dots$), and $H(E)$, $H(F)$ are Fréchet nuclear spaces.

Let $U = \{ \tilde{f}: \tilde{f} \in H(E): \tilde{\pi}_K(\tilde{f}) < \alpha \}$ where $\alpha < 1$. We will show that $\overline{R(U)}$ is a neighborhood of 0 in $H(F)$.

The restriction mapping $R: E' \rightarrow F'$ is open, continuous and onto. Hence there exists a compact K' in F such that $\{ \varphi: \varphi \in F', |\varphi|_{K'} < 1 \} \subseteq R \{ \tilde{\varphi}: \tilde{\varphi} \in E', |\tilde{\varphi}|_K < \alpha \}$. In particular given $\varphi \in F'$ and $\beta > 0$ such that $|\varphi|_{K'} < \beta$, there exists an extension $\tilde{\varphi} \in E'$ of φ such that $|\tilde{\varphi}|_K < \alpha\beta$.

Let m be any integer ≥ 2 . Suppose $p \in P_f(mF)$ such that $\pi_{K'}(p) < \beta$. Then p may be written in the form $p = \sum_{i=1}^r \varphi_i^m$ where $\varphi_i \in F' \forall i$, and $\sum_{i=1}^r |\varphi_i^m|_{K'} < \beta$. For each i , let $\tilde{\varphi}_i$ be an extension of φ_i to E such that $\tilde{\varphi}_i \in E'$ and $\sum_{i=1}^r |\tilde{\varphi}_i^m|_K < \alpha^m \beta$. Hence by letting $\tilde{p} = \sum_{i=1}^r \tilde{\varphi}_i^m$ we have $\tilde{p} \in P_f(mE)$ and $\pi_K(\tilde{p}) < \alpha^m \beta$. Therefore if $w^m = \{ \tilde{p}: \tilde{p} \in P(mE), \pi_K(\tilde{p}) < \alpha^m \}$, then $\{ p: p \in P_f(mF), \pi_{K'}(p) < \beta \} \subseteq R(w^m)$. Hence $\{ p: p \in P(mF), \pi_{K'}(p) < 1 \} \subseteq \overline{R(w^m)^{P(mF)}}$. We have shown that the restriction of R to $P(mE)$ is an almost open mapping onto a dense subset of $P(mF)$. Hence it is an open mapping onto $P(mF)$ (Horvath [4, Theorem 3.17.2, Proposition 3.17.12]) and we may assume

$$\{ p: p \in P(mF), \pi_{K'}(p) < 1 \} \subseteq R \{ \tilde{p}: \tilde{p} \in P(mE), \pi_K(\tilde{p}) < \alpha^m \}.$$

In particular given $p \in P(mF)$ and $\beta > 0$ such that $\pi_{K'}(p) < \beta$, there exists an extension $\tilde{p} \in P(mE)$ of p such that $\pi_K(\tilde{p}) < \alpha^m \beta$. Now let

$$V = \left\{ f: f \in H(F), \sum_{m=0}^{\infty} \pi_{K'} \left(\frac{\hat{d}^m f(0)}{m!} \right) < \beta = \frac{\alpha(1-\alpha)}{2} \right\}.$$

V is a neighborhood of 0 in $H(F)$, and we will show that $V \subseteq \overline{R(U)}$. Suppose $f \in V$. Since $\sum_{m=0}^r \hat{d}^m f(0)/m! \rightarrow f$ as $r \rightarrow \infty$, to show that $f \in \overline{R(U)}$ it suffices to show that $\sum_{n=0}^r \hat{d}^n f(0)/n! \in R(U)$ for each r . For each $m = 1, 2, \dots$, let $\tilde{p}_m \in P(mE)$ be an extension of $\hat{d}^m f(0)/m!$ such that $\pi_K(\tilde{p}_m) \leq \alpha^m \beta$. Then

$$\sum_{m=0}^r \pi_K(\tilde{p}_m) \leq \sum_{m=0}^r \alpha^m \beta < \beta/(1-\alpha) < \alpha.$$

Hence $\sum_{m=0}^r \tilde{p}_m \in U$ and $\sum_{m=0}^r \hat{d}^m f(0)/m! \in R(U)$.

THEOREM 3.1. *Let F be a closed subspace of the DFN space E . Then if $H(E)$, $H(F)$ are endowed with the compact open topologies, the restriction mapping $R: H(E) \rightarrow H(F)$ is a strict morphism onto (i.e. continuous, open, linear and onto).*

PROOF. R is clearly a continuous linear mapping, and by Lemma 3.2 it is an almost open mapping with dense image. Since $H(E)$, $H(F)$ are Fréchet spaces, it follows that R is open and onto (see Horvath [4, Theorem 3.17.2, Proposition 3.17.12]).

COROLLARY 3.1. *Let F be a closed subspace of the DFN space E . Then given $f \in H(F)$, there exists an extension $\tilde{f} \in H(E)$ of f .*

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