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MINIMAL COVERS AND HYPERDEGREES[1]

BY

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ABSTRACT. Every hyperdegree at or above that of Kleene's O is the hyperjump and the supremum of two minimal hyperdegrees (Theorem 3.1). There is a nonempty Σ_1^1 class of number-theoretic predicates each of which has minimal hyperdegree (Theorem 4.7). If $V = L$ or a generic extension of L, then there are arbitrarily large hyperdegrees which are not minimal over any hyperdegree (Theorems 5.1, 5.2). If $O^{\#}$ exists, then there is a hyperdegree such that every larger hyperdegree is minimal over some hyperdegree (Theorem 5.4). Several other theorems on hyperdegrees and degrees of nonconstructibility are presented.

1. **Introduction.** In this paper are proved several results concerning hyperdegrees. The methods of proof in §§3 and 4 are based on the methods of Gandy and Sacks [9]. In §§5 and 7 some ideas of modern set theory are applied. The significance of the results is discussed in §6.

In this introductory section, the contents of the paper are summarized and compared to some of the recent literature on Turing degrees. A Turing degree $\mathbf{m}$ is said to be minimal if $\mathbf{m} > \mathbf{0}$ and there is no Turing degree strictly between $\mathbf{m}$ and $\mathbf{0}$. S. B. Cooper [3], [4] has investigated minimal Turing degrees and has proved the following two theorems. Let $\mathbf{0}'$ be the complete r.e. Turing degree. (1) $\mathbf{0}'$ is the supremum of two minimal Turing degrees. (2) Every Turing degree $\geqslant \mathbf{0}'$ is the jump of a minimal Turing degree. The proofs of (1) and (2) involve delicate applications of the priority method. In §3 below is proved a theorem for hyperdegrees which has as corollaries the hyperdegree analogs of (1) and (2). It is found that, once the basic techniques from Gandy-Sacks [9] have been mastered, the proof of the hyperdegree analogs is much easier than the proofs of (1) and (2). In particular, the priority method is not used for the hyperdegree analogs. Pursuant to Cooper's theorem (2), L. P. Sasso [28][2] has

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(1) The preparation of this paper was partially supported by NSF Contract GP–24352. The main theorems of the paper were announced in [31] and [32].

(2) Subsequently Cooper, R. Epstein, and Sasso [5] sharpened Sasso's [28] result by constructing a minimal Turing degree $\mathbf{m}$ such that $\mathbf{m} < \mathbf{0}' < \mathbf{m}'$. See also Yates [43].

shown that not every minimal Turing degree $\mathbf{m}$ has $\mathbf{m}' = \mathbf{m} \cup \mathbf{0}'$ where $'$ denotes Turing jump. (Sasso's proof does not use the priority method.) In §2 it is observed that the analogous statement for hyperdegrees is false.

The results mentioned in the previous paragraph illustrate the general principle that the behavior of the hyperdegrees is often less complicated than that of the Turing degrees (with respect to both simplified methods of proof, and increased regularity of structure). Another manifestation of the same phenomenon is Spector's result [34] that the hyperdegree analog of the Friedberg-Muchnik theorem [24, p. 163] is false. A possible explanation for this phenomenon is discussed in §6.

It is shown in §4 that there exists a nonempty Σ_1^1 class M of number-theoretic predicates such that every element of M is of minimal hyperdegree. (There does not seem to be anything analogous to M for Turing degrees. See also the discussion in §6.) The elements of M are made to have the following strong independence property: If $x_1, \cdots, x_m, y_1, \cdots, y_n$ are distinct elements of M and if Kleene's O is not hyperarithmetic in either of the sequences $\langle x_1, \cdots, x_m\rangle$ and $\langle y_1, \cdots, y_n\rangle$, then the hyperdegrees of these two sequences have greatest lower bound zero. In particular, any two distinct elements of M are hyperarithmetically incomparable. As a by-product, the following apparently new theorem of point-set topology is obtained. Let n be a positive integer, and let B be a Borel set in n-dimensional Euclidean space. Then there exist perfect, closed sets $P_1, \cdots, P_n$ in 1-dimensional Euclidean space such that $P_1 \times \cdots \times P_n$ is either a subset of B or disjoint from B.(3)

A Turing degree $\mathbf{b}$ is said to be a minimal cover if there is a Turing degree $\mathbf{a} < \mathbf{b}$ such that there is no Turing degree strictly between $\mathbf{a}$ and $\mathbf{b}$. C. G. Jockusch [12] proved that there exists a Turing degree $\mathbf{b}_0$ such that every Turing degree above $\mathbf{b}_0$ is a minimal cover. Jockusch's proof [12] had a feature which set it apart from the rest of current Turing degree theory; namely, it used the power set axiom of *ZFC* (via the result of Paris [23] concerning Gale-Stewart games). This raised two questions (see also [15]): (i) whether the hyperdegree statement analogous to Jockusch's theorem is provable in *ZFC* (it is obviously provable from the hypothesis of projective determinacy); (ii) whether Jockusch's theorem is provable without the power set axiom. In §5 below it is shown that the hyperdegree analog, though probably consistent with *ZFC*, is false in a wide class of models of *ZFC* and hence not provable in *ZFC*. This answers question (i) in the negative. Subsequently, Harrington and Kechris [39]

(3) By a different proof, the theorem can be sharpened as follows. Let B be an arbitrary subset of n-dimensional Euclidean space. Suppose that B is nonmeager and has the property of Baire. Then the conclusion holds with $P_1 \times \cdots \times P_n \subseteq B$.

answered question (ii) by giving a new proof of Jockusch's theorem which does not use the power set axiom and so is formalizable in analysis.

In §7, some results on L-degrees (degrees of nonconstructibility) are stated without proof. The paper ends with some open questions.

The theorems of this paper were mostly proved in February 1971, while the author was a graduate student at the Massachusetts Institute of Technology under the supervision of Professor Sacks. We thank Professors Kreisel and Sacks for helpful discussions. However, in fairness to Sacks, it should be noted that Sacks attempted to dissuade the author from thinking about the matters discussed here.

2. **Notation and preliminaries.** All set-theoretical and logical notation is standard. Lower case Greek letters $\alpha, \beta, \cdots$ are variables ranging over ordinals, except ω which always denotes the least infinite ordinal. Thus $\omega = \{0, 1, 2, \cdots\}$ = the natural numbers. Letters $i, j, k, m, n, \cdots$ are variables ranging over ω. A *real* is a subset of ω. Letters $x, y, z, \cdots$ are variables ranging over reals.

The basic paper on relative hyperarithmeticity is Spector [34]. See also Chapter 16 of Rogers [24]. Write $x \leqslant_h y$ if x is *hyperarithmetic* in y. Also $x <_h y$ if $x \leqslant_h y \wedge y \nleqslant_h x$. Also $hj(x)$ = the complete $\Pi_1^{1,x}$ subset of ω = the *hyperjump* of x. The least nonrecursive ordinal is denoted ω_1. The least ordinal not recursive in x is denoted ω_1^x. The following lemmas are well known (see Rogers [24, pp. 415 and 421]).

LEMMA 2.1. (i) $x \leqslant_h y \longrightarrow \omega_1^x \leqslant \omega_1^y$.
(ii) $x \leqslant_h y \longrightarrow (\omega_1^x < \omega_1^y \longleftrightarrow hj(x) \leqslant_h y)$.

LEMMA 2.2. $\forall x \exists y(x <_h y <_h hj(x))$.

We write $hd(x) = \{y | x \leqslant_h y \wedge y \leqslant_h x\}$ = the *hyperdegree* of x. Lower case boldface letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \cdots$ are variables ranging over hyperdegrees (except in the Introduction where they range over Turing degrees). If $\mathbf{a} = hd(y)$ and $\mathbf{b} = hd(y)$ then write $\mathbf{a} \leqslant \mathbf{b}$ for $x \leqslant_h y$, $\mathbf{a} < \mathbf{b}$ for $x <_h y$, $\mathbf{a} \cup \mathbf{b}$ for $hd(x \oplus y)$ where

$$x \oplus y = \{2i | i \in x\} \cup \{2i + 1 | i \in y\},$$

and $\mathbf{a}' = hd(hj(x))$. In particular $\mathbf{0} = hd(0)$ = the hyperdegree of hyperarithmetic reals, and $\mathbf{0}' = hd(hj(0))$ = the hyperdegree of Kleene's O.

Say $\mathbf{b}$ is *minimal over* $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and there is no hyperdegree $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$. Also $\mathbf{b}$ is a *minimal cover* if it is minimal over some hyperdegree. Also $\mathbf{b}$ is *minimal* if it is minimal over $\mathbf{0}$.

THEOREM 2.3. *Let* $\mathbf{a}$ *be a hyperdegree.*
(i) $\mathbf{a}'$ *is not minimal over* $\mathbf{a}$.

(ii) *If* $\mathbf{b}$ *is minimal over* $\mathbf{a}$ *then* $\mathbf{b}' = \mathbf{b} \cup \mathbf{a}'$.

PROOF. Part (i) is just a restatement of Lemma 2.2. For part (ii), let $\mathbf{a} = hd(x)$ and $\mathbf{b} = hd(y)$ and suppose $\mathbf{b}$ is minimal over $\mathbf{a}$. Then $x \leqslant_h y$ and, by 2.2, $hj(x) \nleqslant_h y$. Hence, by 2.1, $\omega_1^x = \omega_1^y$. Put $z = y \oplus hj(x)$. Then, by 2.1, $\omega_1^x < \omega_1^z$. So, by 2.1 again, $hj(y) \leqslant_h z$. But $z \leqslant_h hj(y)$ trivially. So $\mathbf{b}' = hd(hj(y)) = hd(z) = \mathbf{b} \cup \mathbf{a}'$. □

The end of a proof is indicated by □.

The *constructible hierarchy* is defined by recursion on the ordinals as follows: $L_0 = \{\varnothing\}$; $L_{\alpha+1} = \{X \subseteq L_\alpha | X$ is first-order definable over $\langle L_\alpha, \in\rangle$ allowing parameters from $L_\alpha\}$; $L_\lambda = \bigcup\{L_\alpha | \alpha < \lambda\}$ for limit ordinals λ; $L = \bigcup\{L_\alpha | \alpha$ an ordinal$\}$. This definition is *relativized* to any class U by modifying the successor steps as follows: $L_{\alpha+1}(U) = \{X \subseteq L_\alpha(U) | X$ is first-order definable over $\langle L_\alpha(U), \in, L_\alpha(U) \cap U\rangle$ allowing parameters from $L_\alpha(U)\}$. The following characterization of relative hyperarithmeticity is well known (cf. Kleene [18], Boolos-Putnam [2], and Barwise-Gandy-Moschovakis [1]):

$$x \leqslant_h y \longleftrightarrow x \in L_{\omega_1^y}(y).$$

This characterization will be used throughout the paper but especially in §5. Note that $L_{\omega_1^y}(y)$ is the smallest admissible set containing ω and y as elements. In §4 it is also useful to keep in mind that a set $D \subseteq 2^\omega \cap L_{\omega_1}$ is Π_1^1 if and only if it is Σ_1 over L_{ω_1}; see Barwise-Gandy-Moschovakis [1]. We sometimes write $L_\alpha(x_1, \cdots, x_m)$ for $L_\alpha(\{\langle i, j\rangle | 1 \leqslant i \leqslant m \wedge j \in x_i\})$.

3. **Pairs of minimal hyperdegrees.** The space of all reals $2^\omega = \{x | x \subseteq \omega\}$ is given the product topology where $2 = \{0, 1\}$ gets the discrete topology. If T is a closed subset of 2^ω, then the *code* of T is the real $\{\bar{x}(n) | x \in T \wedge n < \omega\}$ where $\bar{x}(n) = 2^n \cdot (\Sigma\{2^{i+1} | i \in x \cap n\} + 1)$. Thus closed sets are coded by reals, and a closed set is uniquely determined by its code.

As in Gandy-Sacks [9] and §2 of Sacks [26], a *condition* is a perfect, closed subset of 2^ω whose code is hyperarithmetic. Letters $P, Q, \cdots$ are variables ranging over conditions. Sets of conditions are classified as to definability by means of the corresponding sets of codes. For instance, if D is a set of conditions, then D is said to be Π_1^1 if and only if $\{\text{code}(P) | P \in D\}$ is Π_1^1. The set of all conditions is denoted $\mathbf{P}$. Let D be a subset of $\mathbf{P}$. D is said to be *dense* in $\mathbf{P}$ if for every $P \in \mathbf{P}$ there exists $Q \in D$ such that $Q \subseteq P$. D is said to be *definable* if it is first-order definable over the structure $\langle L_{\omega_1}, \in\rangle$. As in Lemma 2.8 of Sacks [26], it can be shown that D is definable if and only if it is arithmetical in $hj(0)$. A real x is said to *meet* D if $\exists P(x \in P \in D)$, i.e., $x \in \bigcup D$. A real x is said to be $\mathbf{P}$*-generic over* L_{ω_1} if x meets every dense, definable subset of $\mathbf{P}$. The main theorem of Gandy-Sacks [9] is that if x is $\mathbf{P}$-generic over L_{ω_1} then x is of minimal hyperdegree.

Let E be a subset of $\mathbf{P} \times \mathbf{P}$. E is said to be *dense* in $\mathbf{P} \times \mathbf{P}$ if for every $\langle P, Q\rangle \in \mathbf{P} \times \mathbf{P}$ there exists $\langle P', Q'\rangle \in E$ such that $P' \subseteq P$ and $Q' \subseteq Q$. The notion of a pair of reals $\langle x, y\rangle$ being $\mathbf{P} \times \mathbf{P}$-generic over L_{ω_1} is defined in the obvious way. It can be shown that if $\langle x, y\rangle$ is $\mathbf{P} \times \mathbf{P}$-generic over L_{ω_1} then x and y are of different minimal hyperdegrees and $\{hd(0), hd(x), hd(y), hd(x) \cup hd(y)\}$ is an initial segment of the hyperdegrees.(4) In particular $\mathbf{0}' \not\leqslant hd(x) \cup hd(y)$. So the question arises whether there exist minimal hyperdegrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{0}' \leqslant \mathbf{a} \cup \mathbf{b}$. Theorem 3.1 below answers this question affirmatively. In the proof of Theorem 3.1, reals x, y are constructed such that x and y are each $\mathbf{P}$-generic over L_{ω_1} and are of different hyperdegrees, but the pair $\langle x, y\rangle$ is not $\mathbf{P} \times \mathbf{P}$-generic over L_{ω_1}.

THEOREM 3.1. *Let* $\mathbf{c}$ *be a hyperdegree* $\geqslant \mathbf{0}'$. *Then there exist minimal hyperdegrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{a}' = \mathbf{b}' = \mathbf{c}$.

PROOF. We first introduce some special notation. For each condition P put $(P)_0 = \{x \mid n \in x \in P\}$ and $(P)_1 = \{x \mid n \notin x \in P\}$ where n is the least natural number such that $\exists x \exists y (n \in x \in P \wedge n \notin y \in P)$. Note that $(P)_0$ and $(P)_1$ are conditions, $(P)_0 \cup (P)_1 = P$, and $(P)_0 \cap (P)_1 = \varnothing$. For each condition P and natural number m put $P[m] = ((\cdots(P)_0)_0\cdots{}_0)_1$ where there are m 0's. The point of this definition is to insure the following facts:

(i) $P[m]$ is a condition, and $P[m] \subseteq P$;

(ii) m can be computed from the code of P and any element of $P[m]$;

(iii) the code of $P[m]$ can be computed from m and the code of P.

Adopt a metarecursive [19] indexing of conditions by natural numbers. Then the set of indices is Π^1_1, and the relation $\{\langle e, P\rangle \mid e$ is an index of $P\}$ is Π^1_1. Let $\langle D_n \mid n < \omega\rangle$ be a natural enumeration of the dense, definable sets of conditions. Such an enumeration has the property that the relation $\{\langle e, n\rangle \mid e$ is an index of an element of $D_n\}$ is hyperarithmetic in $hj(0)$.

Let $\mathbf{c}$ be a hyperdegree $\geqslant \mathbf{0}'$, and let z be such that $hd(z) = \mathbf{c}$. Descending sequences of conditions $\langle P_n \mid n < \omega\rangle$ and $\langle Q_n \mid n < \omega\rangle$ will be defined. Put $P_0 = Q_0 = 2^\omega =$ the trivial condition. Suppose P_{2n} and Q_{2n} have been defined. Let i be the least e such that e is an index of a condition $P \in D_n$ such that $P \subseteq P_{2n}$. Put $P_{2n+1} = P$ and $Q_{2n+1} = Q_{2n}[2i]$ if $n \in z$, $Q_{2n}[2i + 1]$ otherwise. Let j be the least e such that e is an index of a condition $Q \in D_n$ such that $Q \subseteq Q_{2n+1}$. Put $Q_{2n+2} = Q$ and $P_{2n+2} =$

(4)See also Thomason [37] where finite, distributive, initial segments of the hyperdegrees are discussed.

$P_{2n+1}[2j]$ if $n \in z$, $P_{2n+1}[2j+1]$ otherwise.

Let x, y respectively be the unique element of $\bigcap\{P_n \mid n < \omega\}$, $\bigcap\{Q_n \mid n < \omega\}$ respectively. Put $\mathbf{a} = hd(x)$ and $\mathbf{b} = hd(y)$. The construction insures that x and y are each $\mathbf{P}$-generic over L_{ω_1}, so $\mathbf{a}$ and $\mathbf{b}$ are minimal by Gandy-Sacks. Let f, g respectively be the sequence (of indices corresponding to) $\langle P_n \mid n < \omega\rangle$, $\langle Q_n \mid n < \omega\rangle$ respectively. Since $hj(0)$ is hyperarithmetic in z, f and g are hyperarithmetic in z. On the other hand, z is hyperarithmetic in f by (ii), and by (ii) and (iii) the pair $\langle f, g\rangle$ is hyperarithmetic in each of the pairs $\langle x, y\rangle$, $\langle x, hj(0)\rangle$, $\langle y, hj(0)\rangle$. So $\mathbf{a} \cup \mathbf{b} = \mathbf{a} \cup \mathbf{0}' = \mathbf{b} \cup \mathbf{0}' = \mathbf{c}$. Moreover $\mathbf{a} \cup \mathbf{0}' = \mathbf{a}'$ and $\mathbf{b} \cup \mathbf{0}' = \mathbf{b}'$ by 2.3. So Theorem 3.1 is proved. $\square$

REMARKS. 1. Theorem 3.1 and its proof were inspired by §1 of Spector [35].

2. A corollary of Theorem 3.1 is that every hyperdegree $\geqslant \mathbf{0}'$ is the hyperdegree of a hyperjump. Earlier proofs of this corollary had been given by Thomason [36], Harrison [10], and Gandy (unpublished).

3. It is probable that Cooper's methods in [3], [4] will yield the theorem for Turing degrees analogous to 3.1.

4. A Σ_1^1 class of minimal hyperdegrees. As in §3, the set of all conditions is denoted $\mathbf{P}$. The goal of this section is to prove Theorem 4.7.

LEMMA 4.1. *There is a sequence* $\langle D_\alpha \mid \alpha < \omega_1\rangle$ *such that*

(i) $\forall \alpha < \omega_1$ (D_α *is a dense subset of* $\mathbf{P}$);

(ii) *the* D_α*'s are uniformly* Π_1^1, *i.e., the relation* $\{\langle x, \alpha\rangle \mid x$ *is the code of an element of* $D_\alpha\}$ *is* Σ_1 *over* L_{ω_1};

(iii) *if* x *meets* D_α *for each* $\alpha < \omega_1$ *then for all* $y \in L_{\omega_1}(x)$ *either* $y \in L_{\omega_1}$ *or* $x \in L_{\omega_1^y}(y)$.

PROOF. As in Gandy-Sacks [9] let L be a ramified language appropriate for describing $L_{\omega_1}(x)$ where x is a real variable. Let $\langle t_\alpha(x) \mid \alpha < \omega_1\rangle$ be a natural enumeration of the ranked L-terms which denote reals. Thus $2^\omega \cap L_{\omega_1}(x) = \{t_\alpha(x) \mid \alpha < \omega_1\}$. For each $\alpha < \omega_1$ the relation $\{\langle x, y\rangle \mid t_\alpha(x) = y\}$ is hyperarithmetic, so the functions $\lambda x t_\alpha(x)$ are just the "effective Borel" functions from 2^ω into 2^ω.

For each $\alpha < \omega_1$, let D_α be the set of all conditions P such that the restriction of $\lambda x t_\alpha(x)$ to P is either a constant function or one-one. The proof of Lemma 8 of Gandy-Sacks [9] says that D_α is dense. Moreover $P \in D_\alpha$ iff

$$\forall x\, \forall y (x \in P \wedge y \in P \longrightarrow t_\alpha(x) = t_\alpha(y))$$

or

$$\forall x\ \forall y(x \in P \wedge y \in P \wedge x \neq y \longrightarrow t_\alpha(x) \neq t_\alpha(y))$$

so by inspection the D_α's are uniformly Π_1^1. So 4.1(i) and 4.1(ii) are proved.

Now suppose x meets D_α for each $\alpha < \omega_1$, and $y \in L_{\omega_1}(x)$. Then $y = t_\alpha(x)$ for some $\alpha < \omega_1$. Let $P \in D_\alpha$ be such that $x \in P$. If $\lambda x t_\alpha(x)$ is constant on P then we have

$$\forall n(n \in y \longleftrightarrow \forall z(z \in P \longrightarrow n \in t_\alpha(z)))$$

and

$$\forall n(n \in y \longleftrightarrow \exists z(z \in P \wedge n \in t_\alpha(z)))$$

so y is Δ_1^1; hence, by the Kleene-Souslin theorem [29, p. 185], y is hyperarithmetic; hence $y \in L_{\omega_1}$. If $\lambda x t_\alpha(x)$ is one-one on P then we have

$$\forall n(n \in x \longleftrightarrow \forall z(z \in P \wedge t_\alpha(z) = y \longrightarrow n \in z))$$

and

$$\forall n(n \in x \longleftrightarrow \exists z(z \in P \wedge t_\alpha(z) = y \wedge n \in z))$$

so x is Δ_1^1 in y; hence $x \in L_{\omega_1^y}(y)$ by the Kleene-Souslin theorem relativized to y. This proves 4.1(iii). □

Technical Note. By a slight additional argument, the conclusion $x \in L_{\omega_1^y}(y)$ in 4.1(iii) can be strengthened to say that $x \in L_{\omega_1}(y)$.

Write $\mathbf{P}^m$ for the Cartesian product $\mathbf{P} \times \cdots \times \mathbf{P}$ where there are m copies of $\mathbf{P}$. If $E \subseteq \mathbf{P}^m$ then E is said to be *dense* in $\mathbf{P}^m$ if for each $\langle P_1, \cdots, P_m\rangle \in \mathbf{P}^m$ there exists $\langle P_1', \cdots, P_m'\rangle \in E$ such that $P_1' \times \cdots \times P_m' \subseteq P_1 \times \cdots \times P_m$.

Let m be a positive integer, and B be a Δ_1^1 (equivalently, hyperarithmetic or "effective Borel") subset of $(2^\omega)^m$. Let E_B be the set of all $\langle P_1, \cdots, P_m\rangle \in \mathbf{P}^m$ such that either $P_1 \times \cdots \times P_m \subseteq B$ or $P_1 \times \cdots \times P_m \cap B = \varnothing$.

SUBLEMMA 4.2. *E_B is dense in* $\mathbf{P}^m$.

PROOF. Let L^m be the ramified language appropriate for describing $L_{\omega_1}(x_1, \cdots, x_m)$ where $x_1, \cdots, x_m$ are real variables. Let F be a ranked sentence of L^m such that

$$B = \{\langle x_1, \cdots, x_m\rangle \mid L_{\omega_1}(x_1, \cdots, x_m) \models F\}.$$

Then 4.2 is proved by induction on the rank of F. The details are omitted. See the proof of Lemma 3 of Gandy-Sacks [9]. □

If $x_1, \cdots, x_m$ are reals and $E \subseteq \mathbf{P}^m$, then $\langle x_1, \cdots, x_m\rangle$ is said to *meet* E if there exists $\langle P_1, \cdots, P_m\rangle \in E$ such that $\langle x_1, \cdots, x_m\rangle \in P_1 \times \cdots \times P_m$.

LEMMA 4.3. *There exist* $E_\alpha^{m,n}$ $(1 \leqslant m < \omega, 1 \leqslant n < \omega, \alpha < \omega_1)$ *such that*

(i) $E_\alpha^{m,n}$ *is a dense subset of* $\mathbf{P}^{m+n}$;

(ii) *the* $E_\alpha^{m,n}$*'s are uniformly* Π_1^1;

(iii) *if* $\langle x_1, \cdots, x_m, y_1, \cdots, y_n\rangle$ *meets* $E_\alpha^{m,n}$ *for each* $\alpha < \omega_1$, *then* $L_{\omega_1}(x_1, \cdots, x_m) \cap L_{\omega_1}(y_1, \cdots, y_n) = L_{\omega_1}$.

PROOF. Let $\langle\langle t_\alpha^m(x_1, \cdots, x_m), t_\alpha^n(y_1, \cdots, y_n)\rangle \mid \alpha < \omega_1\rangle$ be a natural enumeration of the pairs $\langle t^m, t^n\rangle$ where t^m, t^n respectively is a ranked term of L^m, L^n respectively which denotes a real. Thus $2^\omega \cap L_{\omega_1}(x_1, \cdots, x_m) = \{t_\alpha^m(x_1, \cdots, x_m) \mid \alpha < \omega_1\}$ and similarly for n. Define $E_\alpha^{m,n}$ to be the set of all $\langle P_1, \cdots, P_m, Q_1, \cdots, Q_n\rangle \in \mathbf{P}^{m+n}$ such that (a) or (b) or (c) holds:

(a) the restriction of $\lambda x_1 \cdots x_m t_\alpha^m(x_1, \cdots, x_m)$ to $P_1 \times \cdots \times P_m$ is a constant function;

(b) the restriction of $\lambda y_1 \cdots y_n t_\alpha^n(y_1, \cdots, y_n)$ to $Q_1 \times \cdots \times Q_n$ is a constant function;

(c) the (restricted) functions mentioned in (a) and (b) have disjoint ranges.

The density of $E_\alpha^{m,n}$ in $\mathbf{P}^{m+n}$ will now be proved by means of a splitting argument like the proofs of Lemmas 2 and 8 in Gandy-Sacks [9]. Let $\langle P_1, \cdots, P_m, Q_1, \cdots, Q_n\rangle$ be given. Suppose there is no $\langle P_1', \cdots, P_m', Q_1', \cdots, Q_n'\rangle$ having property (c) and such that $P_1' \times \cdots \times Q_n' \subseteq P_1 \times \cdots \times Q_n$. Define conditions P_k^{ij} $(1 \leqslant k \leqslant m, i < \omega, j < 2^i)$ so that $P_k^{00} = P_k$ and $P_k^{i+1,2j} \cup P_k^{i+1,2j+1} \subseteq P_k^{ij}$ and $P_k^{i+1,2j} \cap P_k^{i+1,2j+1} = \varnothing$. By 4.2 this can be done so that for each i and all $j_1, \cdots, j_m < 2^i, P_1^{ij_1} \times \cdots \times P_m^{ij_m}$ is either contained in or disjoint from $\{\langle x_1, \cdots, x_m\rangle \mid i \in t^m(x_1, \cdots, x_m)\}$. But, for each i, the containment or disjointness is necessarily the same for all $j_1, \cdots, j_m < 2^i$, since otherwise property (c) could be obtained by an application of 4.2 to $\langle Q_1, \cdots, Q_n\rangle$ and $\{\langle y_1, \cdots, y_n\rangle \mid i \in t_\alpha^n(y_1, \cdots, y_n)\}$. Put $P_k' = \bigcap\{\bigcup\{P_k^{ij} \mid j < 2^i\} \mid i < \omega\}$. This P_k' is a (hyperarithmetically coded) condition because of the uniformities inherent in the definition of $\langle P_k^{ij} \mid i < \omega, j < 2^i\rangle$. Moreover $\langle P_1', \cdots, P_m'\rangle$ has property (a).

The proof of 4.3(i) has just been given. Also 4.3(ii) and 4.3(iii) are immediate from the definition of the $E_\alpha^{m,n}$'s. So Lemma 4.3 is proved. □

A set of conditions I is said to be *metafinite* if the corresponding set of codes is hyperarithmetic, i.e. an element of L_{ω_1}. The terminology is meant to suggest the use of inductive, metarecursive constructions involving metafinite sets of conditions (cf. Kreisel-Sacks [19]). This idea will bear fruit in the proof of Lemma 4.6.

SUBLEMMA 4.4. *Let* I *be a metafinite set of conditions. Let* $\langle E_m \mid 1 \leqslant m < \omega\rangle$ *be such that* E_m *is a dense subset of* $\mathbf{P}^m$, *and the* E_m*'s are uniformly*

Π_1^1. *Then there exists a metafinite set of conditions* I^* *such that*

(i) $(\forall P \in I)(\exists Q \in I^*)\, Q \subseteq P$,

(ii) $(\forall Q \in I^*)(\exists P \in I)\, Q \subseteq P$,

(iii) *if* $x_1, \cdots, x_m$ *are distinct elements of* $\bigcup I^*$ *then* $\langle x_1, \cdots, x_m\rangle$ *meets* E_m *for each* m, $1 \leqslant m < \omega$.

N.B. $\bigcup I^* = \{x \mid \exists P(x \in P \in I^*)\}$. Sublemma 4.4 can be proved by means of a splitting argument which is omitted here.

SUBLEMMA 4.5 (i) *Let* S *be a nonempty* Σ_1^1 *class of reals. Then* $\exists x(x \in S \wedge \omega_1^x = \omega_1)$.

(ii) $\{x \mid \omega_1^x = \omega_1\}$ *is* Σ_1^1.

PROOF. Part (i) follows from Gandy's basis theorem (Rogers [24, p. 420]) and Lemma 2.1. Part (ii) is in Sacks [27, p. 393]. □

LEMMA 4.6. *Let* $E_{\alpha,m}$ $(1 \leqslant m < \omega, \alpha < \omega_1)$ *be such that* $E_{\alpha,m}$ *is a dense subset of* $\mathbf{P}^m$, *and the* $E_{\alpha,m}$*'s are uniformly* Π_1^1. *Then there is a* Σ_1^1 *class of reals* K *such that*

(i) $\exists x(x \in K)$;

(ii) $\forall x(x \in K \longrightarrow \omega_1^x = \omega_1)$;

(iii) $\forall x(x \in K \longrightarrow x$ *is not hyperarithmetic*);

(iv) *if* $x_1, \cdots, x_m$ *are distinct elements of* K *then* $\langle x_1, \cdots, x_m\rangle$ *meets* $E_{\alpha,m}$ *for each* $\alpha < \omega_1$.

PROOF. A metarecursive (i.e., $\Delta_1(L_{\omega_1})$, cf. Barwise-Gandy-Moschovakis [1] and Kreisel-Sacks [19]) sequence of metafinite sets of conditions $\langle I_\alpha \mid \alpha < \omega_1\rangle$ will be defined so as to have the following properties:

(a) If $\alpha < \beta < \omega_1$ and $P \in I_\alpha$ then there exist $Q_0, Q_1 \in I_\beta$ such that $Q_0 \subseteq (P)_0$ and $Q_1 \subseteq (P)_1$.(5)

(b) If $\alpha < \beta < \omega_1$ and $x \in Q \in I_\beta$ then $x \in P$ for some $P \in I_\alpha$.

(c) If $\alpha < \omega_1$ and $x_1, \cdots, x_m$ are distinct elements of $\bigcup I_{\alpha+1}$ then $\langle x_1, \cdots, x_m\rangle$ meets $E_{\alpha,m}$.

The definition is begun by putting $I_0 = \{2^\omega\}$. Thus the trivial condition is the unique element of I_0. If I_α has been defined then $I_{\alpha+1}$ is gotten as follows. First put

$$I_\alpha^+ = \{(P)_0 \mid P \in I_\alpha\} \cup \{(P)_1 \mid P \in I_\alpha\}.$$

Then put $I_{\alpha+1} = (I_\alpha^+)^*$ where the * operation is as in Sublemma 4.4 taking

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(5)The notation $(P)_0$, $(P)_1$ was defined at the beginning of the proof of Theorem 3.1.

$I = I_\alpha^+$ and $E_m = E_{\alpha,m}$. Thus property (c) for α is assured, and properties (a) and (b) hold up to $\alpha + 1$ if they held up to α.

Now let λ be a limit ordinal less than ω_1, and suppose that I_α has been defined for each $\alpha < \lambda$. It is safe to assume that properties (a) and (b) hold for $\alpha < \beta < \lambda$. Let $<_L$ be the canonical, $\Delta_1(L_{\omega_1})$ well-ordering of L_{ω_1}. Let $\langle \alpha_n \mid n < \omega\rangle$ be the $<_L$-least ω-sequence of ordinals such that $\alpha_n < \alpha_{n+1}$ and $\lambda = \bigcup_n \alpha_n$. For each condition $P \in \bigcup\{I_\alpha \mid \alpha < \lambda\}$ a condition $P' \subseteq P$ will be constructed by means of a splitting argument. Then I_λ will be defined by $I_\lambda = \{P' \mid P \in \bigcup\{I_\alpha \mid \alpha < \lambda\}\}$. Given $P \in \bigcup\{I_\alpha \mid \alpha < \lambda\}$, let $k = k_P$ be the least n such that $P \in I_\alpha$ for some $\alpha < \alpha_n$. Define P^{ij} $(i < \omega, j < 2^i)$ as follows. Put $P^{00} = P$. If P^{ij} has been defined, then by (a) there exist Q_0 and Q_1 in $I_{\alpha_{k+i}}$ such that $Q_0 \subseteq (P^{ij})_0$ and $Q_1 \subseteq (P^{ij})_1$. Choose such Q_0 and Q_1 in a metarecursive fashion, and put $P^{i+1,2j} = Q_0$ and $P^{i+1,2j+1} = Q_1$. Finally put

$$P' = \bigcap\{\bigcup\{P^{ij} \mid j < 2^i\} \mid i < \omega\}$$

and define I_λ as above. Clearly $P' \subseteq P$, and P' is a (hyperarithmetically coded) condition because of the metarecursive uniformity in the construction of $\langle P^{ij} \mid i < \omega, j < 2^i\rangle$. Also $P' \subseteq \bigcap\{\bigcup I_{\alpha_{k+i}} \mid i < \omega\}$ so properties (a) and (b) continue to hold up to λ. So the definition of the I_α's and the proof of (a), (b), and (c) are complete.

Define $J = \bigcap\{\bigcup I_\alpha \mid \alpha < \omega_1\}$. To see that J is Σ_1^1, note that it is defined explicitly by

$$J(x) \leftrightarrow (\forall \alpha < \omega_1) B_\alpha(x)$$

where the I_α's are uniformly metafinite so the predicates

$$B_\alpha(x) \leftrightarrow (\exists P \in I_\alpha) x \in P$$

are Σ_1^1 (in fact Δ_1^1) uniformly in $\alpha < \omega_1$. To see that J is nonempty consider an ω-sequence of ordinals $\langle \beta_n \mid n < \omega\rangle$ such that $\beta_n < \beta_{n+1}$ and $\omega_1 = \bigcup_n \beta_n$. Use (a) to pick a descending sequence of conditions $P_0 \supseteq P_1 \supseteq \cdots \supseteq P_n \supseteq \cdots$ such that $P_n \in I_{\beta_n}$. Then $\bigcap_n P_n$ is an intersection of nonempty compact sets, hence nonempty. But $\bigcap_n P_n \subseteq J$ in view of (b).

It is harmless to assume that no element of J is hyperarithmetic. This is because, for each hyperarithmetic x, the set $D_x = \{P \mid x \notin P\}$ is Π_1^1 and dense in $\mathbf{P}$. Thus the D_x's could have been included among the $E_{\alpha,1}$'s.

Now define $K = J \cap \{x \mid \omega_1^x = \omega_1\}$. Then 4.6(ii), 4.6(iii), and 4.6(iv) are obvious. Also 4.6(i) follows from 4.5(i). That K is Σ_1^1 follows from 4.5(ii). □

Technical Note. Sublemma 4.5 is well known. However, its application here (in the passage from J to K in the proof of 4.6) is essential, for the following reason. Let $\langle B_\alpha \mid \alpha < \omega_1\rangle$ be any metarecursive sequence of (indices for) Δ_1^1

sets of reals. A compactness theorem of Kreisel (later generalized by Barwise) says that if, for each $\beta < \omega_1$, $\bigcap\{B_\alpha | \alpha < \beta\}$ has nonhyperarithmetic elements, then so does $\bigcap\{B_\alpha | \alpha < \omega_1\}$. (This theorem could have been used to give an alternative proof that J is nonempty.) A theorem of Friedman and Harrington (unpublished, but see [6]) says that if $\bigcap\{B_\alpha | \alpha < \omega_1\}$ has nonhyperarithmetic elements, then it has an element x such that $hd(x) = hd(hj(0))$; hence $\omega_1^x > \omega_1$. In particular J contains such an element. So the use of 4.5 is needed in order to get K to have property 4.6(ii).

THEOREM 4.7. *There exists a* Σ_1^1 *class of reals* M *such that*
(i) $\exists x(x \in M)$;
(ii) $\forall x(x \in M \longrightarrow hd(x)$ *is minimal*);
(iii) *if* $x_1, \cdots, x_m, y_1, \cdots, y_n$ *are distinct elements of* M *then* $L_{\omega_1}(x_1, \cdots, x_m) \cap L_{\omega_1}(y_1, \cdots, y_n) = L_{\omega_1}$. *In particular, any two distinct elements of* M *are hyperarithmetically incomparable.*

PROOF. Immediate from Lemmas 4.1, 4.3, 4.6, and the following characterization of minimal hyperdegrees: $hd(x)$ is minimal iff $\omega_1^x = \omega_1$ and $x \notin L_{\omega_1}$ and for each $y \in L_{\omega_1}(x)$ either $y \in L_{\omega_1}$ or $x \in L_{\omega_1}(y)$. □

Technical Notes. 1. Feferman and Harrison [10] have shown that there exists a nonempty, Σ_1^1 class of reals Q such that no element of Q is hyperarithmetic, and any two distinct elements of Q are hyperarithmetically incomparable. Their class Q does not have the stronger independence property 4.7(iii). In fact, if x and y are elements of Q with $\omega_1^{x \oplus y} = \omega_1$ then $hd(x)$ and $hd(y)$ have no greatest lower bound. Furthermore, no element of Q has minimal hyperdegree.

2. It seems to be an open question whether there exists a Σ_1^1 class S such that $\exists x(x \in S \wedge x$ not hyperarithmetic) and $\forall x \forall y(x \in S \wedge y \in S \longrightarrow \omega_1^{x \oplus y} = \omega_1)$. This question has been raised by H. Friedman and the author, independently.

5. Upper segments of minimal covers. A hyperdegree $\mathbf{b}$ is said to be a *minimal cover* if there is a hyperdegree $\mathbf{a} < \mathbf{b}$ such that there is no hyperdegree strictly between $\mathbf{a}$ and $\mathbf{b}$. An *upper segment* of the hyperdegrees is a set of hyperdegrees of the form $\{\mathbf{b} | \mathbf{b} \geqslant \mathbf{b}_0\}$ where $\mathbf{b}_0$ is a fixed hyperdegree.

ZFC is Zermelo-Fraenkel set theory plus the axiom of choice. In general, the results of this paper are stated and proved in *ZFC*. Let V be the universe of set theory, and let L be the inner model of constructible sets (defined in §2). Gödel has shown that $ZFC + V = L$ is consistent (provided ZFC is, but this does not seem problematic).

THEOREM 5.1. *Assume* $V = L$. *Then there is no upper segment of minimal covers in the hyperdegrees.*

PROOF. Let $C = \{x \mid x \in L_{\omega_1^x}\}$, i.e., C is the class of all reals x such that x is constructible by an ordinal recursive in x. Clearly the E_α's of Boolos and Putnam [2] are elements of C. Hence $\forall x \in L\ \exists y \in L\,(x \leqslant_h y \wedge y \in C)$ by the Main Technical Lemma of [2]. So it will suffice to show that $\forall y \in C\,(hd(y)$ is not a minimal cover). So, suppose $x <_h y \in C$. Then $y \in L_{\omega_1^y}$; hence $L_{\omega_1^y} = L_{\omega_1^y}(y)$ since the latter is an admissible set. Hence $x \in L_{\omega_1^y} \wedge y \notin L_{\omega_1^x}(x)$ so $\omega_1^x < \omega_1^y$. Hence by 2.1 $hj(x) \leqslant_h y$, so by 2.2 $hd(y)$ is not minimal over $hd(x)$. $\square$

REMARK. 1. The class C used above is of considerable interest to effective descriptive set theorists. It can be characterized as the maximum Π_1^1 class of reals having no perfect subclass. More folklore on C can be found in Kechris [17].

2. Theorem 5.1 extends an observation of Jockusch and Soare [15, p. 858].

V is said to be a *generic extension* of L if there exist a partially ordered set P and a subset $G \subseteq P$ such that $P \in L$, G is P-generic over L, and $V = L(G)$. For background on forcing and genericity the reader may consult Shoenfield [30]. Theorem 5.1 will now be generalized to a wide class of models of ZFC.

THEOREM 5.2. *Assume that* V *is a generic extension of* L. *Then there is no upper segment of minimal covers in the hyperdegrees.*

PROOF. Define a class of reals S by

$$y \in S \longleftrightarrow \forall x(x <_h y \longrightarrow \omega_1^x < \omega_1^y).$$

As in the proof of Theorem 5.1, it can be shown that $y \in S \longrightarrow hd(y)$ is not a minimal cover. So Theorem 5.2 is reduced to the following lemma.

LEMMA 5.3. *Suppose* x *is a real in some generic extension of* L. *Then* $\exists y(x \leqslant_h y \in S)$.

PROOF. Suppose $x \in L(G)$ where G is P-generic over L. Let κ be the next cardinal after the cardinality of the powerset of P. Let V^* be a generic extension of V in which κ is countable. The main theorem of Sacks [25](6) reads as follows: Let α be a countable admissible ordinal greater than ω; then $\exists z(\omega_1^z = \alpha \wedge z \in S)$. Applying Sacks' theorem in V^*, find a real z such that $\kappa < \omega_1^z$ and $z \in S$.

(6) This theorem was announced in [26].

Now suppose the conclusion of Lemma 5.3 fails for x, i.e., there is no y (in V) such that $x \leqslant_h y \in S$. Let $\underset{\sim}{x}$ be a forcing term denoting x in $L(G)$. Let $p \in G$ force that $\underset{\sim}{x}$ is a real and there is no real y such that $\underset{\sim}{x} \leqslant_h y \in S$. Since $\kappa < \omega_1^z$, there exists $G^* \in L_{\omega_1^z}(z)$ such that $p \in G^*$ and G^* is P-generic over L. Let x^* be the real denoted by $\underset{\sim}{x}$ in $L(G^*)$. Then $x^* \leqslant_h z$ so V^* satisfies $\exists y(x^* \leqslant_h y \in S)$. But the predicate $y \in S$ is easily seen to be Σ_2^1. So by Shoenfield's absoluteness theorem [29, p. 319], $L(G^*)$ satisfies $\exists y(x^* \leqslant_h y \in S)$. This is a contradiction. □

Write $x \leqslant_T y$ to mean that x is Turing reducible to y, i.e., x is recursive in y. Also write $x \equiv_T y$ to mean $x \leqslant_T y \land y \leqslant_T x$. Consider the following hypothetical statement.

Let A be a Σ_1^1 class of reals such that

$$(STD) \qquad \forall x\ \forall y(x \equiv_T y \in A \rightarrow x \in A) \quad \text{and} \quad \forall x\ \exists y(x \leqslant_T y \in A).$$

Then $\exists x\ \forall y(x \leqslant_T y \rightarrow y \in A)$.

This hypothetical statement STD is known in the literature as Σ_1^1 Turing degree determinateness. It is known that STD can be proved from the hypothetical statement that Ramsey cardinals exist (Martin [20], [21]). It can be shown that STD follows from the hypothetical statement that $0^\#$ exists. (See Solovay [33] and Martin-Solovay [22] for background material on $0^\#$.) It is an open question whether the existence of $0^\#$ can be proved from STD. It is well known that STD is false assuming $V = L$. (See the proof of Theorem 5.1 and Remark 1 following it.) So the situation regarding STD may be summarized as follows: While STD is not provable from the currently accepted axioms of set theory, it is provable from certain currently studied set-theoretical hypotheses.(7)

THEOREM 5.4. *Assume STD. Then there is an upper segment of minimal covers in the hyperdegrees.*

PROOF. The proof of Theorem 4.7 can be uniformly relativized. This yields a Σ_1^1 relation $M \subseteq 2^\omega \times 2^\omega$ such that $\forall x \exists y M(x, y)$ and $\forall x \forall y(M(x,y) \rightarrow hd(y)$ is minimal over $hd(x))$. So let $A = \{z \mid \exists x \exists y(M(x, y) \land z \equiv_T x \oplus y)\}$, and apply STD. □

COROLLARY 5.5. *STD is inconsistent with V being a generic extension of L.*

PROOF. By 5.2 and 5.4. □

(7) Indeed, there exists a large body of (useless) statistical evidence that $ZFC + STD$ may be consistent. This evidence consists in the fact that a large number of people have tried and failed to deduce a contradiction from (systems stronger than) $ZFC + STD$.

REMARK. The proof of Lemma 5.3 is adapted from H. Friedman's proof of Theorem 1 of [7]. Recently, Corollary 5.5 has been proved independently by Friedman [8]. The Σ_1^1 class used by Friedman has the virtue that its definition is short and explicit, in contrast to the proof of Lemma 4.6.

6. **Discussion.** In this section we attempt to analyze the results of §§2, 3, 4, and 5 by putting them into perspective. A superficial analysis was already attempted in §1 where hyperdegrees were compared with Turing degrees. Precisely the structures $\mathcal{H} = \langle H, \leqslant_H, j_H\rangle$ and $\mathcal{D} = \langle D, \leqslant_D, j_D\rangle$ were considered and their first-order theories compared, D (H) being the set of all Turing degrees (hyperdegrees) and $\leqslant_D, j_D$ ($\leqslant_H, j_H$) respectively the usual partial ordering and (hyper) jump operation. Striking first-order differences between $\mathcal{D}$ and $\mathcal{H}$ have been known for a long time (more such differences were pointed out in §1), so the analogy between $\mathcal{D}$ and $\mathcal{H}$ is far from perfect.

Recent work has established a much more powerful analogy between $\mathcal{D} = \mathcal{D}_\omega$ and the structures $\mathcal{D}_\alpha = \langle D_\alpha, \leqslant_\alpha, j_\alpha\rangle$ (or $\mathcal{D}_{c\alpha} = \langle D_{c\alpha}, \leqslant_{c\alpha}, j_{c\alpha}\rangle$) for each admissible ordinal α, where D_α ($D_{c\alpha}$) is the set of all α-degrees (α-calculability degrees), $\leqslant_\alpha$ ($\leqslant_{c\alpha}$) is the partial ordering induced by relative α-recursiveness (α-calculability), and j_α ($j_{c\alpha}$) is the α-jump (α-calculability jump) operation [41], [42]. (One may also want to enrich all these structures by adding relations corresponding to "recursively enumerable in.") The hyperdegrees are embedded here as a substructure since, for $\alpha = \omega_1$ and $x, y \subseteq \omega$, $x \leqslant_h y$ if and only if x is α-calculable from y, and the α-calculability jump of x is virtually the same thing as the hyperjump of x [19]. On the other hand, ω is an α-finite set, and not every α-calculability degree contains a subset of ω. From this point of view it is entirely reasonable to expect the theory of $\mathcal{H}$ to be independent of and simpler than that of $\mathcal{D}_{c\alpha}$ since only subsets of ω are considered, just as the theory of Turing degrees of subsets of a fixed 6-element set is much simpler than the theory of the full structure $\mathcal{D}$. In support one has the vague feeling that the proofs of 2.3, 3.1, and 4.7 depend heavily on the fact that ω is α-finite. (A precise explication of this vague feeling would be welcome.) All this should not be interpreted as saying that $\mathcal{H}$ is less interesting than $\mathcal{D}_{c\alpha}$. It is true that close analogies between $\mathcal{H}$ and $\mathcal{D}$ are not to be expected, but we regard $\mathcal{H}$ as having independent interest.

There are some worthwhile open questions regarding Theorem 4.7 and its analysis in terms of metarecursion theory (i.e., α-recursion theory with $\alpha = \omega_1$ [19]). Jockusch and Soare [13] attempt an analogy between Π_1^0 classes and Turing degrees on the one hand, and Σ_1^1 classes and hyperdegrees on the other. If one takes this analogy seriously, then 4.7 comes as an unpleasant surprise since by [13] every nonempty Π_1^0 class of reals contains an r.e. Turing degree, and no r.e. Turing degree is minimal. (However, Cooper's methods [4] can be used to construct a perfect tree T recursive in the Turing degree $\mathbf{0}'$ such that every path through T is of minimal Turing degree $\mathbf{m}$ with $\mathbf{m}' = \mathbf{m} \cup \mathbf{0}'$. Also, it seems at least plausible that the methods of Jockusch and Soare [14], [16] will produce a Π_1^0 class of at least two reals such that if $\mathbf{a}$ and $\mathbf{b}$ are the Turing degrees of two distinct elements of the class then $\mathbf{a} > \mathbf{0}$ and $\mathbf{b} > \mathbf{0}$ and $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.) In contrast, from the point of view of metarecursion theory, 4.7 is not at all surprising since only subsets of ω are being considered. So the question is, do the elements of M in 4.7 have minimal metadegree or metacalculability degree (among *all* subsets of ω_1)? If not, what is the correct metarecursive analog of the theorem [13] that every nonempty Π_1^0 class of reals has an element of nonminimal Turing degree? We thank G. Kreisel for raising these questions.

It is also useful to look at 4.7 in the context of subsystems of analysis. The proof of 4.7 is obviously rather complicated and in particular constitutes a refinement of Gandy-Sacks [9]. What is not so obvious is how to analyze the nature of the refinement, i.e., how to formulate a precise sense in which 4.7 is a refinement. To this end, consider the formal system BI discussed by H. Friedman [38]. Clearly BI is strong enough to formalize the elements of hyperdegree theory, e.g., to prove $\Delta_1^1 = HYP$ and that nonhyperarithmetic reals exist. But BI is not strong enough to prove that $hj(0)$ exists. Hence the contents of [9] cannot be formalized in BI. Nevertheless, it turns out that the existence of a minimal hyperdegree is provable in BI, and the most direct way to see this is to formalize the proof of 4.7 within BI. The point here is that, in BI, one has to go through the complications of 4.7 in order merely to prove the existence of a minimal hyperdegree. (Minor technical point: in the proof of 4.6 in BI, one must replace the use of $\langle \beta_n | n < \omega \rangle$ by an application of the Kreisel-Barwise compactness theorem; see the Technical Note following the proof of 4.6.) So 4.7 is seen to be a genuine refinement, inasmuch as the construction of a minimal hyperdegree via 4.7 can, but the Gandy-Sacks [9] construction cannot, be formalized in BI.

There is one corollary of §5 which seems to deserve mention. This is the existence of a first-order sentence which is true in the structure $\langle H, \leqslant_H \rangle$ if $V = L$ or a generic extension of L, and false in $\langle H, \leqslant_H \rangle$ if STD holds. In particular this implies that the first-order theory of $\mathcal{H}$ is probably not absolute. We have to say probably here because STD is not known to be consistent with ZFC. It leaves open whether the word "probably" can be removed. It also leaves open whether the first-order theory of $\mathcal{D}$ is absolute. These questions will be answered elsewhere. Some other questions which remain open are listed in §7.

7. Appendix: L-degrees and open questions. There has not been a systematic effort to extend the theorems of this paper to notions of degree other than Turing degree and hyperdegree. Nevertheless, in this section, we indicate briefly how some of the theorems extend to degrees of nonconstructibility, or L-degrees as they will be called here.

Write $x \leqslant_L y$ if x is constructible from y, i.e., $x \in L(y)$. L-degrees were introduced by Sacks [26] who proved that if $\aleph_2^L$ is countable then there exists a minimal L-degree. This result was improved by Jensen [11] who at the same time obtained the following analog of 4.7:

THEOREM 7.1. *There is a* Π_2^1 *relation* $M \subseteq 2^\omega \times 2^\omega$ *such that*

(i) $\forall x(\aleph_1^{L(x)}$ *countable* $\longrightarrow \exists y M(x, y))$;

(ii) $\forall x \, \forall y(M(x, y) \longrightarrow$ *the* L*-degree of* y *is minimal over the* L*-degree of* $x)$;

(iii) *if* $y_1, \cdots, y_m, z_1, \cdots, z_n$ *are distinct elements of* $\{y \mid M(x, y)\}$ *then* $L(y_1, \cdots, y_m) \cap L(z_1, \cdots, z_n) = L(x)$.

Jensen's relation M has the further interesting property that if $m > 0$ and $y_1, \cdots, y_m$ are distinct elements of $\{y \mid M(x, y)\}$ then $L(y_1, \cdots, y_m)$ is a generic extension of $L(x)$. This is in contrast to Technical Note 2 following the proof of Theorem 4.7.

We now consider the problem of how to extend the auxiliary notions of hyperdegree theory (such as $hj(x)$ and ω_1^x) to L-degree theory. This should of course be done in such a way as to preserve the basic properties, e.g., those expressed in Lemmas 2.1 and 2.2. A number of people including the author seem to have come to the independent conclusion that Solovay's sharp operation [33] should play the role for L-degrees that the hyperjump operation plays for hyperdegrees. So for the rest of this section we shall assume that $\forall x(x^\#$ exists). Then we have the following analogs of 2.2 and 3.1.

THEOREM 7.2. $\forall x \exists y(x <_L y <_L x^\#)$.

THEOREM 7.3. *Suppose* $x^\# \leqslant_L y$. *Then there exist* z_1 *and* z_2 *whose* L*-degrees are minimal over the* L*-degree of* x, *and such that* $z_1^\# \equiv_L z_2^\# \equiv_L z_1 \oplus z_2 \equiv_L y$.

Note that the relation $\{\langle x, y\rangle \mid y = x^\#\}$ is Π_2^1. The following analog of the Gandy basis theorem is due to Harrington and Kechris [40]:

THEOREM 7.4. *Let* $A \subseteq 2^\omega$ *be* Π_2^1 *in* x, *and suppose* $\exists y(y \in A \wedge x^\# \nleqslant_L y)$. *Then* $\exists y(y \in A \wedge y^\# \leqslant_L x^\#)$.

In Theorems 7.2, 7.3, and 7.4, the sharp operation was taken as the correct analog of the hyperjump operation. It is not so clear how to assign ordinals to L-degrees in analogy with the assignment ω_1^x for hyperdegrees. Minimal requirements on such an assignment are

$$(*)\qquad \begin{cases} x \leqslant_L y \longrightarrow \lambda^x \leqslant \lambda^y; \\ x \leqslant_L y \longrightarrow (\lambda^x < \lambda^y \leftrightarrow x^\# \leqslant_L y). \end{cases}$$

Recently Kunen has proved the following theorem (of ZFC): Suppose there exists a weakly compact[8] cardinal κ such that $(\kappa^+)^{L(x)} < \kappa^+$; then $x^\#$ exists. (Here κ^+ denotes the next cardinal after κ.) From this theorem of Kunen, it follows that the assignment $\lambda^x = (\aleph_1^+)^{L(x)}$ satisfies $(*)$. (Here $\aleph_1$ always denotes the $\aleph_1$ of V.) But it is not clear that this assignment of ordinals is the most natural one satisfying $(*)$. These last two observations are due to A. S. Kechris.

At any rate, the mere existence of an assignment satisfying $(*)$ yields the following result (cf. the proof of Theorem 2.3):

THEOREM 7.5. $x \leqslant_L y \longrightarrow (x^\# \leqslant_L y \vee y^\# \equiv_L y \oplus x^\#)$.

We end the paper with some open questions.

Question 1. Do there exist two transitive models of ZFC such that M_2 is a generic extension of M_1 and there is an elementary difference between the partial orderings of hyperdegrees in M_1 and M_2?

Question 2. Do there exist two transitive models of ZFC such that M_1 and M_2 have the same ordinals and there is an elementary difference between the partial orderings of Turing degrees in M_1 and M_2?

(We conjecture that the answers to Question 1 and Question 2 are affirmative. In fact, we conjecture that (provably in ZFC) the elementary theory of the partial ordering of Turing degrees or hyperdegrees is recursively isomorphic to the truth set of second-order arithmetic.)

Question 3. Is there a Σ_1^1 class of reals S such that S has a nonhyperarithmetic element, and $\omega_1^{x \oplus y} = \omega_1$ for all $x, y \in S$?

Question 4. Is Σ_1^1 hyperdegree determinateness consistent with V being a generic extension of L?

Question 5 (Sacks [26]). Does every countable set of hyperdegrees have a minimal upper bound?

Question 6 (Friedman). Does there exist a real x such that $hd(x)$ is mini-

(8) Weak compactness is taken to imply strong inaccessibility.

mal and $\{x\} = A \cap L_{\omega_1}(x)$ for some arithmetical $A \subseteq 2^\omega$? If we had such a real x then $L_{\omega_1}(x)$ would be an ω-model for the independence of the Kleene-Souslin theorem from hyperarithmetic analysis.

ADDED IN PROOF (April 10, 1975). Recently we have established the conjecture stated after Question 2, thus answering Questions 1 and 2 affirmatively. Friedman and Harrington have answered Question 3 negatively. We strongly suspect that Question 4 has a negative answer. Questions 5 and 6 remain open.

BIBLIOGRAPHY

1. K. J. Barwise, R. O. Gandy and Y. N. Moschovakis, *The next admissible set*, J. Symbolic Logic 36 (1971), 108–120. MR 46 #36.

2. G. Boolos and H. Putnam, *Degrees of unsolvability of constructible sets of integers*, J. Symbolic Logic 33 (1968), 497–513. MR 39 #1331.

3. S. B. Cooper, *Degrees of unsolvability complementary between recursively enumerable degrees*. I, Ann. Math. Logic 4 (1972), 31–73. MR 45 #3199.

4. ———, *Minimal degrees and the jump operator*, J. Symbolic Logic 38 (1973), 249–271.

5. S. B. Cooper, R. Epstein, L. P. Sasso, *A minimal degree below* $\mathbf{0}'$ *not realizing least possible jump* (abstract), Recursive Function Theory Newsletter, No. 6 (July 1973), p. 7.

6. H. Friedman, *Borel sets and hyperdegrees*, J. Symbolic Logic 38 (1973), 405–409.

7. ———, *Determinateness in the low projective hierarchy*, Fund. Math. 72 (1971), 79–95. MR 45 #8518.

8. ———, Σ_1^1 *degree determinateness fails in all Boolean extensions*, Mimeographed Notes, SUNY at Buffalo, 1973.

9. R. O. Gandy and G. E. Sacks, *A minimal hyperdegree*, Fund. Math. 61 (1967), 215–223. MR 37 #1246.

10. J. Harrison, *Recursive pseudo-well-orderings*, Trans. Amer. Math. Soc. 131 (1968), 526–543. MR 39 #5366.

11. R. B. Jensen, *Definable sets of minimal degree*, Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968), edited by Y. Bar-Hillel, North-Holland, Amsterdam, 1970, pp. 122–128. MR 46 #5130.

12. C. G. Jockusch, Jr., *An application of* Σ_4^0 *determinacy to the degrees of unsolvability*, J. Symbolic Logic 38 (1973), 293–294.

13. C. G. Jockusch, Jr. and R. I. Soare, *Degrees of members of* Π_1^0 *classes*, Pacific J. Math. 40 (1972), 605–616. MR 46 #8827.

14. ———, *A minimal pair of* Π_1^0 *classes*, J. Symbolic. Logic 36 (1971), 66–78. MR 44 #68.

15. ———, *Minimal covers and arithmetical sets*, Proc. Amer. Math. Soc. 25 (1970), 856–859. MR 42 #67.

16. ———, Π_1^0 *classes and degrees of theories*, Trans. Amer. Math. Soc. 173 (1972), 33–56. MR 47 #4775.

17. A. S. Kechris, *The theory of countable analytical sets*, Trans. Amer. Math. Soc. 202 (1975), 259–297.

18. S. C. Kleene, *Quantification of number-theoretic functions*, Compositio Math. 14 (1959), 23–40. MR 21 #2586.

19. G. Kreisel and G. E. Sacks, *Metarecursive sets*, J. Symbolic Logic 30 (1965), 318–338. MR 35 #4097.

20. D. A. Martin, *The axiom of determinateness and reduction principles in the analytic hierarchy*, Bull. Amer. Math. Soc. 74 (1968), 687–689. MR 37 #2607.

21. ———, *Measurable cardinals and analytic games*, Fund. Math. 66 (1969/70), 287–291. MR 41 #3283.

22. D. A. Martin and R. M. Solovay, *A basic theorem for* Σ_3^1 *sets of reals*, Ann. of Math. (2) 89 (1969), 138–159. MR 41 #53.

23. J. B. Paris, ZF $\vdash \Sigma_4^0$ *determinateness*, J. Symbolic Logic 37 (1972), 661–667.

24. H. Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967. MR 37 #61.

25. G. E. Sacks, *Countable admissible ordinals and hyperdegrees*, Advances in Math. (to appear).

26. ———, *Forcing with perfect closed sets*, Proc. Sympos. Pure Math., vol. 13, part I, Amer. Math. Soc., Providence, R. I., 1971, pp. 331–355. MR 43 #1827.

27. ———, *Measure-theoretic uniformity in recursion theory and set theory*, Trans. Amer. Math. Soc. 142 (1969), 381–420. MR 40 #7108.

28. L. P. Sasso, *A minimal degree not realizing least possible jump*, J. Symbolic Logic 39 (1974), 571–574.

29. J. R. Shoenfield, *Mathematical logic*, Addison-Wesley, Reading, Mass., 1967. MR 37 #1224.

30. ———, *Unramified forcing*, Proc. Sympos. Pure Math., vol. 13, part I, Amer. Math. Soc., Providence, R. I., 1971, pp. 357–381. MR 43 #6079.

31. S. G. Simpson, *Some results on hyperdegrees*, Notices Amer. Math. Soc. 18 (1971), 565. Abstract #71T-E36.

32. ———, *A remark on the elementary theory of the hyperdegrees*, Notices Amer. Math. Soc. 18 (1971), 661. Abstract #71T-E40.

33. R. M. Solovay, *A nonconstructible* Δ_3^1 *set of integers*, Trans. Amer. Math. Soc. 127 (1967), 50–75. MR 35 #2748.

34. C. Spector, *Recursive well-orderings*, J. Symbolic Logic 20 (1955), 151–163. MR 17, 570; 1437.

35. ———, *On degrees of recursive unsolvability*, Ann. of Math. (2) 64 (1956), 581–592. MR 18, 552; 1118.

36. S. K. Thomason, *The forcing method and the upper semilattice of hyperdegrees*, Trans. Amer. Math. Soc. 129 (1967) 38–57. MR 36 #2503.

37. ———, *On initial segments of hyperdegrees*, J. Symbolic Logic 35 (1970), 189–197. MR 44 #5225.

38. H. M. Friedman, *Bar induction and* Π_1^1*-CA*, J. Symbolic Logic 34 (1969), 353–362. MR 40 #4109.

39. L. Harrington and A. S. Kechris, *A basis result for* Σ_3^0 *sets of reals with an application to minimal covers*, Proc. Amer. Math. Soc. (to appear).

40. ———, Π_2^1 *singletons and* $O^{\#}$, Fund. Math. (to appear).

41. G. E. Sacks and S. G. Simpson, *The* α*-finite injury method*, Ann. Math. Logic 4 (1972), 343–367.

42. S. G. Simpson, *Degree theory on admissible ordinals*, Generalized Recursion Theory, edited by J. E. Fenstad and P. G. Hinman, North-Holland, Amsterdam, 1974, 165–193.

43. C. E. M. Yates, *Prioric games and minimal degrees below* $\mathbf{0}'$, Fund. Math. 82 (1974), 217–237.

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