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TOPOLOGICAL DYNAMICS AND C^*-ALGEBRAS[1]

BY

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ABSTRACT. If G is a group of automorphisms of a C^*-algebra A with identity, then G acts in a natural way as a transformation group on the state space $S(A)$ of A. Moreover, this action is uniformly almost periodic if and only if G has compact pointwise closure in the space of all maps of A into A. Consideration of the enveloping semigroup of $(S(A), G)$ shows that, in this case, this pointwise closure $\overline{G}$ is a compact topological group consisting of automorphisms of A. The Haar measure on $\overline{G}$ is used to define an analogue of the canonical center-valued trace on a finite von Neumann algebra. If A possesses a sufficiently large group G_0 of inner automorphisms such that $(S(A), G_0)$ is uniformly almost periodic, then A is a central C^*-algebra. The notion of a uniquely ergodic system is applied to give necessary and sufficient conditions that an approximately finite dimensional C^*-algebra possess exactly one finite trace.

Introduction. The purpose of this paper is to apply some ideas from topological dynamics to the study of C^*-algebras. If X is a compact Hausdorff space and (X, Γ) is a topological transformation group, then Γ has a natural representation as a group of automorphisms of the commutative C^*-algebra $C(X)$: for $t \in \Gamma$ and $f \in C(X)$ put

$$(tf)(x) = f(xt), \qquad x \in X.$$

It is often possible to express properties of (X, Γ) in terms of the system $(\Gamma, C(X))$; for example, (X, Γ) is uniformly almost periodic iff for each $f \in C(X)$, the set $\{tf\colon t \in \Gamma\}$ is relatively compact in $C(X)$. If A is an arbitrary C^*-algebra with identity and G is a group of automorphisms of A, we may view the pair (G, A) as a noncommutative version of $(\Gamma, C(X))$. We shall see that some of the relationships between (X, Γ) and $(\Gamma, C(X))$ have noncommutative analogues, and that these analogues can be used to obtain information about the structure of certain C^*-algebras.

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1. **Preliminaries.** We shall generally follow the terminology of [6] for topological dynamics, that of [5] for C^*-algebras, and that of [12] for uniform spaces and topologies on function spaces. We shall however translate Dixmier's "morphisme" by "$*$-homomorphism", and we define "trace" below.

DEFINITION 1.1. Let G be a semigroup with identity e, and let X be a set. A *right action* of G on X is a mapping

$$\pi: X \times G \longrightarrow X: (x, \alpha) \longrightarrow x\alpha = \pi(x, \alpha)$$

such that

(1) $xe = x$ for all $x \in X$, and

(2) $(x\alpha)\beta = x(\alpha\beta)$ for all $x \in X$ and all $\alpha, \beta \in G$.

When there is no danger of confusion, we shall write $x\alpha$ for $\pi(x, \alpha)$.

A *left action* of G on X is a mapping $(\alpha, x) \longrightarrow \alpha x$ of $G \times X$ into X such that $ex = x$ and $\alpha(\beta x) = (\alpha\beta)x$ for all $x \in X$ and all $\alpha, \beta \in G$. We make the convention that the term "action" will mean "right action" unless we specify otherwise. An action is *continuous* if it is continuous from the product topology on $X \times G$.

REMARK 1.2. If X and Y are sets, we shall write Y^X for the set of all mappings of X into Y. When $Y = X$, we may use composition of mappings to provide X^X with two natural semigroup structures: Let (p, q) be an ordered pair of elements of X^X. If we write our mappings on the right, we shall define pq by

$$x(pq) = (xp)q, \qquad x \in X.$$

If we write our mappings on the left, we shall compose in the opposite order:

$$(pq)x = p(qx), \qquad x \in X.$$

These definitions give actions of X^X on X on the right and left respectively. We shall find it convenient to write maps of C^*-algebras on the left and maps of their state spaces on the right.

REMARK 1.3. Let A be a C^*-algebra with identity. We write $S(A)$ for its state space, and we give $S(A)$ the weak* topology. Then the set $S(A)$ is convex, and the topological space $S(A)$ is compact and Hausdorff, hence is a uniform space in a unique way. We note that the uniformity on $S(A)$ is determined by the family of all pseudo-norms of the form $(p, q) \longrightarrow |p(a) - q(a)|$, where a is a positive element of A. It follows that a net ρ_γ in $S(A)^{S(A)}$ converges to $\rho \in S(A)^{S(A)}$ in the topology of uniform convergence iff for each positive $a \in A$ we have

$$\sup_{p \in S(A)} |(p\rho_\gamma)(a) - (p\rho)(a)| \longrightarrow 0$$

[12, pp. 226–227]. We note also that as $S(A)$ is compact, a family of maps in $S(A)^{S(A)}$ is equicontinuous iff it is uniformly equicontinuous.

We write $ES(A)$ for the set of pure states of A with the weak* topology.

REMARK 1.4. Let (X, G) be a transformation group with compact Hausdorff phase space X. For each $t \in G$, let π^t denote the map $x \longrightarrow xt$, $x \in X$. The pointwise closure in X^X of the set $\{\pi^t: t \in G\}$ is a semigroup, called the enveloping semigroup of (X, G) [6, 3.2]. The following are equivalent:

(1) (X, G) is uniformly almost periodic.

(2) $\{\pi^t: t \in G\}$ is an equicontinuous family.

(3) The enveloping semigroup of (X, G) is a group of continuous maps.

(4) If $f \in C(X)$, then f is almost periodic, i.e. $\{f \circ \pi^t: t \in G\}$ has compact closure in $C(X)$ [6, 4.4 and 4.15].

(The proof given in [6, 4.15] for real functions applies equally well to $C(X)$.)

Let A be a C^*-algebra with identity. If $p \in S(A)$, we write L^p for the representation of A obtained by applying the Gelfand-Naimark-Segal construction to p, and we say that L^p is *associated* to p. The *left kernel* of p is the left ideal $\{a \in A: p(a^*a) = 0\}$. A state τ of A is a *trace* on A if τ is invariant under the inner automorphisms of A, i.e. $\tau(a) = \tau(uau^*)$ for all $a \in A$ and all unitary $u \in A$. Since every element of A is a linear combination of unitaries, a state τ of A is a trace iff $\tau(ab) = \tau(ba)$ for all $a, b \in A$. We denote the set of all traces on A by $T(A)$, and we write $ET(A)$ for the set of extremal traces of A, i.e. extreme points of $T(A)$. A trace τ is extremal iff L^τ is a factor representation [5, 6.7.3 and 6.8.5].

A *face* of a compact convex set K is a convex subset F of K such that if $p, q \in K$ and $\frac{1}{2}p + \frac{1}{2}q \in F$, then p and q are in F. An extreme point of a face of K is also an extreme point of K, and the inverse image of an extreme point under an affine map is a face.

If A is a C^*-algebra, we denote by $\mathrm{Max}(A)$ the space of all maximal ideals of A equipped with the relative topology from the Jacobson topology on $\mathrm{Prim}(A)$ [5, 3.1.1]. We write ZA for the center of A. Suppose A has an identity. Then there is a mapping ρ of $\mathrm{Prim}(A)$ onto $\mathrm{Max}(ZA)$ given by $\rho: P \longrightarrow P \cap ZA$. This mapping is continuous, and since $\mathrm{Prim}(A)$ is compact and $\mathrm{Max}(ZA)$ is Hausdorff, it is also closed. If ρ is one-to-one (i.e. a homeomorphism), then A is said to be a *central* C^*-algebra [1].

REMARK 1.5. Let A be a C^*-algebra with identity I, and let p be a state of A such that L^p is a factor representation (e.g. a pure state or an extremal trace). We identify the center of $L^p(A)'$ with $\mathbf{C}$. Then p coincides with L^p on ZA, hence is multiplicative on ZA. Suppose moreover that $\ker L^p$ is a primitive ideal. Then the character of ZA which corresponds to $\ker L^p \cap ZA$ is $p|_{ZA}$. For if the value of this character on z is λ, then $z - \lambda I \in \ker L^p \cap ZA \subseteq \ker L^p$, so $p(z - \lambda I) = L^p(z - \lambda I) = 0$, and $p(z) = \lambda$.

2. Uniformly almost periodic groups of automorphisms. In this section A will denote a C^*-algebra with identity I. An *automorphism* of A is an invertible $*$-homomorphism of A onto A, and we write $\mathrm{Aut}(A)$ for the group of all automorphisms of A. We shall characterize those subgroups of $\mathrm{Aut}(A)$ which act uniformly almost periodically on $S(A)$.

Let G be a subgroup of $\mathrm{Aut}(A)$. We say that an element a of A is *G-invariant* if $\alpha(a) = a$ for all $\alpha \in G$. A state p of A is G-invariant if $p \circ \alpha = p$ for all $\alpha \in G$. We denote the algebra of all G-invariant elements of A by $Z_G A$ and the set of all G-invariant states of A by $S_G(A)$. Then $Z_G A$ is a C^*-subalgebra of A, and $S_G(A)$ is a compact convex subspace of $S(A)$.

Let A^A have the pointwise (product) topology. Then the set $\mathrm{Aut}(A)$ is not in general closed in A^A, since a net of automorphisms may converge pointwise to a map which is not onto. It will therefore be convenient for us to consider a slightly larger subset of A: Let $H(A)$ be the set of all $*$-homomorphisms α of A into A such that $\alpha(I) = I$. Then $H(A)$ is pointwise closed, the elements of $H(A)$ are norm-decreasing positive maps, and an element of $H(A)$ is an automorphism iff it is an invertible mapping. Moreover, $H(A)$ is closed under composition of mappings, hence is a subsemigroup of A^A. We note that a net $\{\alpha_\gamma\}$ converges to α in $H(A)$ iff $\alpha_\gamma(a) \longrightarrow \alpha(a)$ for each positive $a \in A$.

LEMMA 2.1. *$H(A)$ is a topological semigroup, and* $\mathrm{Aut}(A)$ *is a topological group.*

PROOF. Suppose $(\alpha_\gamma, \beta_\gamma) \longrightarrow (\alpha, \beta)$ in $H(A) \times H(A)$ and let $a \in A$. Then

$$\|\alpha_\gamma\beta_\gamma(a) - \alpha\beta(a)\| \leqslant \|\alpha_\gamma\|\,\|\beta_\gamma(a) - \beta(a)\| + \|\alpha_\gamma(\beta(a)) - \alpha(\beta(a))\|.$$

As $\alpha_\gamma \longrightarrow \alpha$, $\beta_\gamma \longrightarrow \beta$, and $\|\alpha_\gamma\| \leqslant 1$ for all γ, this tends to zero, so $\alpha_\gamma\beta_\gamma \longrightarrow \alpha\beta$. Thus $H(A)$ is a topological semigroup.

To show that $\mathrm{Aut}(A)$ is a topological group we suppose that $\alpha_\gamma \longrightarrow \alpha$ in $\mathrm{Aut}(A)$. Let $a \in A$. Automorphisms of C^*-algebras are isometric, so

$$\|\alpha_\gamma^{-1}(a) - \alpha^{-1}(a)\| = \|a - \alpha_\gamma\alpha^{-1}(a)\| = \|\alpha(\alpha^{-1}(a)) - \alpha_\gamma(\alpha^{-1}(a))\| \longrightarrow 0.$$

Thus inversion is continuous on $\mathrm{Aut}(A)$, and $\mathrm{Aut}(A)$ is a topological group.

If $\alpha \in H(A)$ and p is a state of A, then $p \circ \alpha$ is again a state of A. Thus there is a natural action of $H(A)$ on $S(A)$ defined by

$$(p, \alpha) \longrightarrow p \circ \alpha = p\alpha, \qquad p \in S(A),\ \alpha \in H(A).$$

This action is continuous: if $(p_\gamma, \alpha_\gamma) \longrightarrow (p, \alpha)$ in $S(A) \times H(A)$ and $a \in A$, then

$$|p_\gamma \circ \alpha_\gamma(a) - p \circ \alpha(a)| \leqslant \|\alpha_\gamma(a) - \alpha(a)\| + |p_\gamma(\alpha(a)) - p(\alpha(a))| \longrightarrow 0..$$

It follows that if G is any subgroup of $\mathrm{Aut}(A)$, then the restriction of this action to $S(A) \times G$ makes $(S(A), G)$ into a transformation group.

Since $H(A)$ is closed in A^A, we have for any subset G of $H(A)$ that the closures of G in $H(A)$ and in A^A coincide. We shall find the following theorem very useful in providing examples of uniformly almost periodic actions on state spaces.

THEOREM 2.2. *Let G be a subset of $H(A)$, and let S be any subset of A such that the linear span of S is dense in A. Then the closure of G in $H(A)$ is compact iff for every $a \in S$, the set $G[a] = \{\alpha(a): \alpha \in G\}$ has compact closure in A.*

PROOF. Let $\overline{G}$ be the closure of G. As $\alpha \longrightarrow \alpha(a)$ is continuous, we have $\overline{G}[a] = \{\alpha(a): \alpha \in \overline{G}\} \subseteq \overline{G[a]}$ for every $a \in A$.

If $\overline{G}$ is compact, then for each $a \in A$ we have $\overline{G}[a]$ compact, whence $\overline{G}[a] = \overline{G[a]}$. In particular, $\overline{G[a]}$ is then compact for every $a \in S$.

Conversely, suppose that for every $a \in S$, the set $\overline{G[a]}$ is compact. The elements of $\overline{G}$ are linear and norm-decreasing, so the restriction mapping r: $\alpha \longrightarrow \alpha|_S$ is a one-to-one map of $\overline{G}$ into A^S. Let A^S have the product topology. Then $\alpha_\gamma \longrightarrow \alpha$ in $\overline{G}$ iff $r(\alpha_\gamma) \longrightarrow r(\alpha)$ in A^S. For if $r(\alpha_\gamma) \longrightarrow r(\alpha)$, then $\alpha_\gamma \longrightarrow \alpha$ pointwise on S, hence pointwise on the linear span of S. As this span is dense, and as $\overline{G}$ consists of maps uniformly bounded in norm, an $\epsilon/3$-argument shows that $\alpha_\gamma(a) \longrightarrow \alpha(a)$ for all $a \in A$.

Thus r is a homeomorphism of $\overline{G}$ onto its image in A^S. To see that this image is closed in A^S, suppose $r(\alpha_\gamma)$ is a net in the image such that $r(\alpha_\gamma) \longrightarrow \theta$ in A^S. It is enough to show that θ has an extension to a map α in A^A such that $\alpha_\gamma \longrightarrow \alpha$ in A^A. We extend θ to the linear span of S by $\theta(\Sigma_{i=1}^n \lambda_i a_i) = \Sigma_{i=1}^n \lambda_i \theta(a_i)$. We have

$$\sum_{i=1}^n \lambda_i \theta(a_i) = \sum_{i=1}^n \lambda_i \lim_\gamma \alpha_\gamma(a_i) = \lim_\gamma \alpha_\gamma\left(\sum_{i=1}^n \lambda_i a_i\right).$$

It follows that θ is well defined and linear and that α_γ converges pointwise to θ on the span of S. Since the α_γ are norm-decreasing, θ is also norm-decreasing on this span. Thus θ has an extension to a linear, norm-decreasing map α of A into A. By the same $\epsilon/3$-argument as above, $\alpha_\gamma \longrightarrow \alpha$ in A^A.

To complete the proof we observe that the image of $\overline{G}$ is contained in $\times_{a\in S} \overline{G}[a] \subseteq \times_{a \in S} \overline{G[a]}$ and apply the Tychonoff Theorem.

We shall need to consider more closely the maps of $S(A)$ into itself which are induced by the elements of $H(A)$. For each $\alpha \in H(A)$, let $i(\alpha)$ be the mapping $p \longrightarrow p\alpha$. Then i is an injection of $H(A)$ into $S(A)^{S(A)}$, for if $p(\alpha(a)) = p(\beta(a))$ for all $p \in S(A)$ and all $a \in A$, then $\alpha(a) = \beta(a)$ for all $a \in A$. Moreover,

i has the following additional properties:

(1) i takes the identity of the semigroup $H(A)$ onto that of the semigroup $S(A)^{S(A)}$, and i is a homomorphism of semigroups: $i(\alpha\beta) = i(\alpha)i(\beta)$.

(2) For each $\alpha \in H(A)$, $i(\alpha)$ is weak*-continuous and affine.

(3) If $\alpha \in H(A)$ and α is invertible, then $i(\alpha)$ is invertible and $i(\alpha)^{-1} = i(\alpha^{-1})$.

LEMMA 2.3. *The map i is bicontinuous from $H(A)$ into $S(A)^{S(A)}$ when $S(A)^{S(A)}$ is given the topology of uniform convergence.*

PROOF. If $a \in A$ is positive and α_γ, $\alpha \in H(A)$, then $\alpha_\gamma(a) - \alpha(a)$ is selfadjoint. Hence $\|\alpha_\gamma(a) - \alpha(a)\| = \sup_{p\in S(A)} |p(\alpha_\gamma(a) - \alpha(a))|$.

LEMMA 2.4. *Let K be a convex subset of the dual of a Banach space B, and let K have the weak* topology. Let F be a family of affine maps of K into K. Then the pointwise closure of F in K^K is again a family of affine maps.*

PROOF. Let $\lambda \in [0, 1]$, let $p, q \in K$, and suppose β_γ converges pointwise in K^K. Then the functionals $\lim_\gamma \beta_\gamma(\lambda p + (1 - \lambda)q)$ and $\lambda \lim_\gamma \beta_\gamma(p) + (1 - \lambda) \lim_\gamma \beta_\gamma(q)$ agree on each element of B.

THEOREM 2.5. *Let G be a subgroup of* Aut(A). *Then the following are equivalent*:

(1) *The transformation group $(S(A), G)$ is uniformly almost periodic.*

(2) *The closure $\bar{G}$ of G in $H(A)$ (or in A^A) is compact.*

Under these conditions $\bar{G}$ is a group, and i is a homeomorphism and a group isomorphism of $\bar{G}$ onto the enveloping semigroup E of $(S(A), G)$.

PROOF. Let T be the topology of uniform convergence. We use below without comment Remark 1.4 and some topological results which can be found in [12, pp. 227, 232–233].

Suppose $\bar{G}$ is compact. Then $i(\bar{G})$ is T-compact by Lemma 2.3. The topology T is jointly continuous on the family of all continuous maps of $S(A)$ into $S(A)$, so $i(\bar{G})$ is equicontinuous. In particular, the subfamily $i(G)$ is equicontinuous, so $(S(A), G)$ is uniformly almost periodic.

Conversely, suppose $(S(A), G)$ is uniformly almost periodic. Let $A_{\mathbf{R}}$ be the selfadjoint part of A. By Theorem 2.2 it suffices to show that for each $a \in A_{\mathbf{R}}$, the orbit $G[a]$ has compact closure (in $A_{\mathbf{R}}$ or in A). For such an a, let $\hat{a}$ be the map $p \longrightarrow p(a)$ of $S(A)$ into the real numbers. Then $a \longrightarrow \hat{a}$ is an isometric linear map of $A_{\mathbf{R}}$ into $C(S(A))$. Thus it suffices to show that for each $a \in A_{\mathbf{R}}$, $\{\widehat{\alpha(a)}: \alpha \in G\}$ has compact closure in $C(S(A))$. Since $\widehat{\alpha(a)}(p) = p(\alpha(a)) = \hat{a}(p\alpha)$, we need only show that each $\hat{a}$ is an almost periodic function, which follows from uniform almost periodicity of $(S(A), G)$.

Now suppose that (1) and (2) are satisfied. Since i is continuous into the topology T, it is also continuous into the pointwise topology. Thus $i(\overline{G})$ is pointwise compact. Since $S(A)^{S(A)}$ is pointwise Hausdorff, i is a homeomorphism of $\overline{G}$ onto $i(\overline{G})$. But then $i(G)$ is pointwise dense in both $i(\overline{G})$ and E, so $i(\overline{G}) = E$. Now E is a group, and i is an isomorphism of the semigroup $\overline{G}$ onto E which takes the identity of $\overline{G}$ to the identity of E. If $\alpha \in \overline{G}$, then there exists $\beta \in \overline{G}$ such that $i(\beta) = i(\alpha)^{-1}$. But then β will be an inverse for α, so $\overline{G}$ is a group.

COROLLARY 2.6. *If $\overline{G}$ is compact, then $\overline{G}$ is a subgroup of* Aut(A). *In particular, $\overline{G}$ is a compact topological group.*

COROLLARY 2.7. *The closure of G in A^A is compact iff the closure of G in* Aut(A) *is compact, and in this case the two closures coincide.*

COROLLARY 2.8. *If $\overline{G}$ is compact, then every element of E maps the set of pure states of A into itself.*

PROOF. As $(S(A), G)$ is uniformly almost periodic, the elements of E are invertible maps. By Lemma 2.4, they are affine. Thus each $\alpha \in E$ must take extreme points to extreme points.

REMARK. The methods of this section can also be used to obtain analogous results for groups of C^*-automorphisms as defined in [11].

3. **Uniformly almost periodic C^*-algebras.** In this section we use uniform almost periodicity of $(S(A), G)$ to obtain information about the traces and the ideal structure of the algebra A. We remark that our discussion of centrality is based on that in [13], in which Mosak obtained most of the results of this section for certain group C^*-algebras.

We continue to assume that A is a C^*-algebra with identity I. Moreover, we assume that G is a group of automorphisms of A such that $(S(A), G)$ is uniformly almost periodic. Let μ be normalized Haar measure on $\overline{G}$, and let $a \in A$. As $\alpha \longrightarrow \alpha(a)$ is continuous on $\overline{G}$, it is weakly μ-measurable. The image of $\overline{G}$ is a compact metric space, hence is separable, so the Bochner integral $\int_{\overline{G}} \alpha(a)\,d\mu(\alpha)$ exists [20, pp. 131–133]. We may thus define a mapping $^\#$ of A into A by $a^\# = \int_{\overline{G}} \alpha(a)\,d\mu(\alpha)$, $a \in A$.

LEMMA 3.1. *The mapping $^\#$ is a positive, linear, idempotent mapping of A onto $Z_G A$. It is norm-decreasing and takes no nonzero positive element of A to zero.*

PROOF. The first statement is proved in [18, Example 1.1]. That $^\#$ is norm-decreasing follows from $\|a^\#\| \leqslant \int_{\overline{G}} \|\alpha(a)\|\,d\mu(\alpha)$. If $(a^*a)^\# = 0$, then for every $p \in S(A)$ we have $\int_{\overline{G}} p(\alpha(a^*a))\,d\mu(\alpha) = 0$. As $\alpha \longrightarrow p(\alpha(a^*a))$ is positive and continuous, it follows that $(a^*a)^\# = 0$ iff $p(\alpha(a^*a)) = 0$ for all $p \in S(A)$ and all $\alpha \in \bar{G}$. Thus $(a^*a)^\# = 0$ implies $a^*a = 0$. (See [13, 3.6].)

Let $r: S_G(A) \longrightarrow S(Z_G A)$ be restriction to $Z_G A$.

LEMMA 3.2. *The mapping r is an affine homeomorphism of $S_G(A)$ onto $S(Z_G A)$.*

PROOF. r is the inverse of the mapping Φ^* in [18, Example 1.1].

If we wish to study the ideals or traces of A, it is natural to consider the group $\mathcal{I}(A)$ of all inner automorphisms of A. This group is generally too large to act uniformly almost periodically on $S(A)$. For suppose A is a *UHF*-algebra (not finite dimensional), and let p be a pure state of A. Then the set of all states of the form $b \longrightarrow p(ubu^*)$, u unitary in A, is weak*-dense in $S(A)$. If the action of $\mathcal{I}(A)$ were uniformly almost periodic, then $S(A)$ would be a minimal set [6, 2.5]. But this contradicts the existence of a trace on A. (I am indebted to Erling Størmer for pointing out this counterexample.)

DEFINITION 3.3. Let A be a C^*-algebra with identity. We say that A is *uniformly almost periodic* if

(1) every state of ZA is the restriction of some trace of A, and

(2) there exists a group G of inner automorphisms of A such that $(S(A), G)$ is uniformly almost periodic and $Z_G A = ZA$.

REMARK 3.4. Let $\mathcal{U}_0$ be a group of unitary elements of A such that the linear span of $\mathcal{U}_0$ is dense in A, and let G_0 be the group of all inner automorphisms of A induced by the elements of $\mathcal{U}_0$. Suppose $(S(A), G_0)$ is uniformly almost periodic. Then A is uniformly almost periodic. For if $a \in A$ commutes with every $u \in \mathcal{U}_0$, then $a \in ZA$, so $Z_{G_0} A = ZA$. By Lemma 3.2, restriction takes $S_{G_0}(A)$ onto $S(ZA)$. If τ is a G_0-invariant state, then for every $a \in A$ and every $u \in \mathcal{U}_0$, we have $\tau(ua - au) = 0$, whence $\tau(ab) = \tau(ba)$ for all $a, b \in A$. Thus $S_{G_0}(A) = T(A)$. It follows that A is uniformly almost periodic.

We give examples of uniformly almost periodic C^*-algebras in the last section. We assume for the remainder of this section that A is uniformly almost periodic and that G is a group of inner automorphisms of A which satisfies condition (2) of Definition 3.3.

THEOREM 3.5. *The sets $S_G(A)$ and $T(A)$ coincide, and $ET(A)$ is a weak* closed subset of $T(A)$. (That is, $T(A)$ is a Bauer simplex.) Moreover, $\tau \in T(A)$ is extremal iff $\tau|_{ZA}$ is a character, and r restricted to $ET(A)$ is a homeomorphism onto $ES(ZA)$.*

PROOF. Since G consists of inner automorphisms, $T(A) \subseteq S_G(A)$. As each $\psi \in S(ZA)$ is the restriction of a trace of A, $r(T(A)) = S(ZA)$. Then $T(A) =$

$S_G(A)$, since r is one-to-one. By Remark 1.5, the restriction of an extremal trace to ZA is a character. If $\tau \in T(A)$ and $r(\tau)$ is pure in $S(ZA)$, then τ must be extremal, since r is affine and one-to-one. Thus r restricts to a bijection of $ET(A)$ and $ES(ZA)$. It follows from Lemma 3.2 that this bijection is a homeomorphism and that $ET(A) = r^{-1}(ES(ZA))$ is weak* closed in $T(A)$.

LEMMA 3.6. *If $\tau \in ET(A)$, then the left kernel of τ is a primitive ideal of A.*

PROOF (AFTER MOSAK). Let $\tau|_{ZA} = \psi$. Then ψ is an irreducible representation of ZA, and we can find an irreducible representation π of A on some Hilbert space H_π such that π is an extension of ψ. Then

$$\tau(a^*a) = \tau((a^*a)^\#) = \psi((a^*a)^\#) = \pi((a^*a)^\#), \qquad a \in A.$$

Thus it suffices to show that if π is a representation of A, then $\pi(a) = 0$ iff $\pi((a^*a)^\#) = 0$, or equivalently that $\pi(a^*a) = 0$ iff $\pi((a^*a)^\#) = 0$.

If $x \in H_\pi$, let $\omega_x(b) = (bx, x)$, $b \in B(H_\pi)$. Then

$$\pi((a^*a)^\#) = 0 \Longleftrightarrow \omega_x \circ \pi((a^*a)^\#) = 0 \quad \forall x \in H_\pi \Longleftrightarrow$$

$$\int_{\overline{G}} \omega_x \circ \pi(\alpha(a^*a))d\mu(\alpha) = 0 \quad \forall x \in H_\pi \Longleftrightarrow$$

$$\omega_x \circ \pi \circ \alpha(a^*a) = 0 \quad \forall x \in H_\pi, \quad \forall \alpha \in \overline{G} \Longleftrightarrow \pi \circ \alpha(a^*a) = 0 \quad \forall \alpha \in \overline{G}.$$

Let K be the kernel of π. Since G consists of inner automorphisms and K is a closed ideal, each α in $\overline{G}$ maps K into K. It follows that $\pi \circ \alpha(a^*a) = 0$ $\forall \alpha \in \overline{G}$ iff $\pi(a^*a) = 0$.

Let θ be the mapping of $ET(A)$ into $\mathrm{Prim}(A)$ defined by sending an extremal trace into its left kernel, and let ρ be the mapping $P \longrightarrow P \cap ZA$ of $\mathrm{Prim}(A)$ onto $\mathrm{Max}(ZA)$. We shall identify a maximal ideal of ZA with the corresponding character. With this identification the mapping r restricted to $ET(A)$ is a homeomorphism of $ET(A)$ and $\mathrm{Max}(ZA)$. By Remark 1.5 this homeomorphism factors into $\rho \circ \theta$, i.e. $\tau|_{ZA} = \ker L^\tau \cap ZA$ when $\tau \in ET(A)$. Since r is one-to-one, θ is also one-to-one from $ET(A)$ into $\mathrm{Prim}(A)$.

LEMMA 3.7. *If $\tau \in ET(A)$, then its left kernel is a maximal ideal and θ is a homeomorphism of $ET(A)$ onto* $\mathrm{Max}(A)$. *Moreover, ρ restricts to a homeomorphism of* $\mathrm{Max}(A)$ *onto* $\mathrm{Max}(ZA)$.

PROOF. Let $M \in \mathrm{Max}(A)$, and let p be a state of A such that L^p has kernel M. Then $p^\#$: $a \longrightarrow p(a^\#)$ is a trace on A. Let $a \in M$. Since the elements of $\overline{G}$ map M into M, p vanishes on each $\alpha(a)$, $\alpha \in \overline{G}$. Thus $p(a^\#) = 0$, and $p^\#$ is a trace which vanishes on M. The set of all traces which vanish on M is a weak* closed face of $T(A)$. Let τ be any extreme point of this face. Then $\ker L^\tau \supseteq M$, and by maximality of M we have $\ker L^\tau = M$. Thus θ maps $ET(A)$ onto a subspace of $\mathrm{Prim}(A)$ which contains $\mathrm{Max}(A)$.

Let $P \in \mathrm{Prim}(A)$, and suppose there exists $\sigma \in ET(A)$ such that $\ker L^\sigma = P$. Choose $M_P \in \mathrm{Max}(A)$ such that $P \subseteq M_P$, and choose $\tau \in ET(A)$ such that $\ker L^\tau = M_P$. By Remark 1.5, the characters corresponding to $P \cap ZA$ and $M_P \cap ZA$ are $\sigma|_{ZA}$ and $\tau|_{ZA}$ respectively. Since $M_P \cap ZA = P \cap ZA$, these characters are equal. As r is one-to-one, we have $\sigma = \tau$, hence $M_P = P$. Thus θ is a bijection of $ET(A)$ and $\mathrm{Max}(A)$.

Now ρ is one-to-one on $\mathrm{Max}(A)$, since $r = \rho \circ \theta$, r is one-to-one, and θ is a bijection. As $\mathrm{Max}(A)$ is compact and ρ is continuous, ρ restricts to a homeomorphism ρ_0 of $\mathrm{Max}(A)$ onto $\mathrm{Max}(ZA)$. It follows that $\theta = \rho_0^{-1} \circ r$ is also a homeomorphism.

LEMMA 3.8. *Every primitive ideal of A is maximal. In particular, θ is a homeomorphism of $ET(A)$ and* $\mathrm{Prim}(A)$.

PROOF (AFTER MOSAK). We define a mapping of $\hat{A}$ into $ET(A)$ as follows. If π is an irreducible representation of A, then $\pi^\#: a \longrightarrow \pi(a^\#)$ is a trace on A. Suppose $\pi^\# = \frac{1}{2}\tau_1 + \frac{1}{2}\tau_2$ with τ_1 and τ_2 in $T(A)$. Then, as $\pi^\#|_{ZA}$ is a pure state of ZA, $\pi^\# = \tau_1 = \tau_2$ on ZA. Since restriction is one-to-one, we have $\tau_1 = \tau_2 = \pi^\#$ on all of A. If π is unitarily equivalent to π_0, let p and p_0 be states associated with π and π_0 respectively. Then there exists a unitary u in A such that $p(uau^*) = p_0(a)$ for all $a \in A$. Hence $\pi^\# = p^\# = p_0^\# = \pi_0^\#$, and $\pi \longrightarrow \pi^\#$ is well defined.

Now $\pi \longrightarrow \pi^\#$ maps onto $ET(A)$. For if $\tau \in ET(A)$, let p be a pure state of A which agrees with τ on ZA. Let $\pi = L^p$, and then $\pi^\# = \pi = p = \tau$ on ZA, so $\pi^\# = \tau$.

If we can show that the mapping $\pi \longrightarrow \ker \pi$ of $\hat{A}$ onto $\mathrm{Prim}(A)$ is the composition of $\pi \longrightarrow \pi^\#$ and θ, then θ must map onto $\mathrm{Prim}(A)$, and hence $\mathrm{Max}(A) = \mathrm{Prim}(A)$. So we must show that if $\pi \in \hat{A}$, then the kernel of π is $\{a: \pi((a^*a)^\#) = 0\}$. But we verified this in the proof of Lemma 3.6.

THEOREM 3.9. *If A is uniformly almost periodic, then A is a central C^*-algebra.*

PROOF. Combine Lemmas 3.7 and 3.8.

4. Uniquely ergodic C^*-algebras. We turn now to uniquely ergodic systems and approximately finite C^*-algebras. If X is a compact metric space and T is a homeomorphism of X onto X, then by [15, 2.1] there exists at least one normalized T-invariant Borel measure on X. The system (X, T) is said to be uniquely ergodic if there exists exactly one such measure, or equivalently if $C(X)$ has exactly one T-invariant state. By analogy we define a C^*-algebra to be *uniquely ergodic* if it possesses exactly one trace.

A C^*-algebra A with identity I is said to be *approximately finite* if there exists an increasing sequence $\{A_n\}$ of finite dimensional C^*-subalgebras of A, each A_n containing I, such that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ [3]. We shall see below that every approximately finite C^*-algebra possesses at least one trace, and we shall characterize those which are uniquely ergodic. We assume in this section that A is approximately finite with $\{A_n\}$ and I as above. We note that A is separable, and hence that $S(A)$ is metrizable.

For each $n \geq 1$, the unitary group U_n of A_n is compact, so there exists a map φ_n of A into A given by

$$\varphi_n: a \longrightarrow \int_{U_n} uau^* \, d\mu_n(u), \qquad a \in A,$$

where μ_n is normalized Haar measure on U_n.

LEMMA 4.1. *For each $n \geq 1$, φ_n is a norm-decreasing, idempotent, positive, linear map of A onto $A_n' = \{a \in A\colon ab = ba \text{ for all } b \in A_n\}$. Each A_n' is a C^*-subalgebra of A and $\bigcap_{n=1}^{\infty} A_n' = ZA$. If $a, b \in A_n$, then $\varphi_n(ab) = \varphi_n(ba)$.*

PROOF. If $a \in \bigcap_{n=1}^{\infty} A_n'$, then a commutes with every element of $\bigcup_{n=1}^{\infty} A_n$, hence with every element of A. It is trivial that φ_n is norm-decreasing, and the rest of the lemma follows from [18, Example 1.1].

LEMMA 4.2. *Let $\{p_n\}$ be a sequence of states of A. Then $\{p_n \circ \varphi_n\}$ is a sequence of states of A and has at least one limit point in $S(A)$. Every limit point is a trace of A.*

PROOF. Clearly $\{p_n \circ \varphi_n\}$ is a sequence of states, and it has a limit point by compactness of $S(A)$. Let $p_{n_i} \circ \varphi_{n_i} \longrightarrow \tau$ in $S(A)$. If $a, b \in \bigcup_{n=1}^{\infty} A_n$, then for all sufficiently large n we have $\varphi_n(ab - ba) = 0$, whence $\tau(ab - ba) = 0$. The map $(a, b) \longrightarrow ab - ba$ is continuous on $A \times A$, so $\tau(ab - ba) = 0$ for all $a, b \in A$.

COROLLARY 4.3. *If ψ is a state of ZA, then there exists a trace τ of A whose restriction to ZA is ψ. If ψ is a character, then τ can be chosen to be extremal.*

PROOF. Let p be a state of A which extends ψ, and let τ be a weak* limit point of $\{p \circ \varphi_n\}$. Then $\tau = \psi$ on ZA. Suppose now ψ is a character. The set $F = \{\tau \in T(A)\colon \tau|_{ZA} = \psi\}$ is a nonempty closed face of $T(A)$. Any extreme point of F is an extremal trace which extends ψ.

REMARK. The following proposition describes the approximately finite C^*-algebras which possess a centering map analogous to the map $^\#$ of the last section. We note however that the map $a \longrightarrow \varphi(a)$ below may annihilate some nonzero positive elements of A.

PROPOSITION 4.4. *The following are equivalent*:

(1) *For each $a \in A$ the sequence $\{\varphi_n(a)\}$ converges in norm to an element $\varphi(a)$ of A.*

(2) *The mapping r: $\tau \longrightarrow \tau|_{ZA}$ of $T(A)$ onto $S(ZA)$ is one-to-one.*

If these conditions are satisfied, then for each $a \in A$ we have $\varphi(a) \in ZA$, and for each $p \in S(A)$ the mapping $a \longrightarrow p(\varphi(a))$ is a trace.

PROOF. (1) $\Rightarrow$ (2): Let $a \in A$. If $\sigma \in T(A)$, then for each n we have $\sigma(\varphi_n(a)) = \sigma(a)$, so $\sigma(a) = \sigma(\varphi(a))$. Suppose σ and τ are in $T(A)$ and $\sigma = \tau$ on ZA. As $\varphi(a) \in \bigcap_{n=1}^\infty A_n' = ZA$, $\sigma(a) = \tau(a)$. Thus r is one-to-one. That $a \longrightarrow p(\varphi(a))$ is a trace follows from the fact that φ is positive and linear and vanishes on $ab - ba$ for all $a, b \in \bigcup_{n=1}^\infty A_n$.

(2) $\Rightarrow$ (1): If r is one-to-one, then it is an affine homeomorphism of $T(A)$ onto $S(ZA)$, and its restriction to $ET(A)$ is a homeomorphism onto $ES(ZA)$. We use this homeomorphism and the Gelfand transform $\hat{}$ to identify ZA with $C(ET(A))$. For each $a \in A$, put $a^\#(\tau) = \tau(a)$, $\tau \in ET(A)$. Then $a^\# \in ZA$, and the mapping $a \longrightarrow a^\#$ is linear, norm-decreasing, positive, and invariant under the inner automorphisms of A. As $z^\#(\tau) = \hat{z}(r(\tau))$, $z^\# = z$ for $z \in ZA$. To show that $\varphi_n(a)$ is convergent for each $a \in A$, it suffices to show that for each positive $a \in A$, $\|\varphi_n(a) - a^\#\| \longrightarrow 0$, i.e.

$$\sup_{p \in S(A)} |p \circ \varphi_n(a) - p(a^\#)| \longrightarrow 0.$$

If this is false, then there exist $a_0 \geqslant 0$ in A, a subsequence $\{\varphi_{n_i}\}$ of $\{\varphi_n\}$, and $p_i \in S(A)$ such that

$$(1) \qquad |p_i \circ \varphi_{n_i}(a_0) - p_i(a_0^\#)| \geqslant \epsilon > 0 \quad \text{for all } i \geqslant 1.$$

By passing to a subsequence we may assume $p_i \circ \varphi_{n_i} \longrightarrow \tau$ and $p_i \longrightarrow p_0$ in $S(A)$. Then τ and $p_0^\#$: $a \longrightarrow p_0(a^\#)$ are traces of A. For $z \in ZA$ we have

$$\tau(z) = \lim_{i\to\infty} p_i \circ \varphi_{n_i}(z) = \lim_{i\to\infty} p_i(z) = p_0(z) = p_0(z^\#),$$

so $r(\tau) = r(p_0^\#)$. But then $\tau = p_0^\#$, which contradicts (1).

If $a \in A$, let $\hat{a}$ be the mapping $p \longrightarrow p(a)$ of $S(A)$ into $\mathbf{C}$. The following is a C^*-algebraic analogue of [15, 5.3].

THEOREM 4.5. *If A is an approximately finite C^*-algebra, then the following are equivalent*:

(1) *A is uniquely ergodic.*

(2) *For each $a \in A$ the sequence $\{\widehat{\varphi_n(A)}\}$ converges uniformly on $S(A)$ to a constant function.*

(3) *For each $a \in A$ there exists a subsequence of $\{\widehat{\varphi_n(a)}\}$ which converges pointwise on $S(A)$ to a constant function.*

If these conditions are satisfied, then the constant function of conditions (2) *and* (3) *has the value $\tau(a)$, where τ is the trace of A.*

PROOF. (1) $\Rightarrow$ (2): Let $a \in A$. By Corollary 4.3, $ZA = \mathbf{C}I$, and by the last proposition $\varphi_n(a)$ converges in norm to $K_a I$ for some complex number K_a. Thus $\sup_{p\in S(A)}|p \circ \varphi_n(a) - K_a| \longrightarrow 0$.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $a \in A$, $\sigma \in T(A)$, and suppose $\{\widehat{\varphi_{n_i}(a)}\}$ converges pointwise to the constant function K_a. Then $\sigma(a) = \sigma(\varphi_{n_i}(a)) \longrightarrow K_a$, so $a \longrightarrow K_a$ is the only trace on A.

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5. Examples.

EXAMPLE 5.1. Let X be a compact Hausdorff space and $A = C(X)$. Let (X, Γ) be a transformation group. As in the introduction we let tf be the function $(tf)(x) = f(xt)$, $x \in X$, where $t \in \Gamma$ and $f \in C(X)$. Let G be the group of all automorphisms of A which have the form $f \longrightarrow tf$, $t \in \Gamma$. It follows from Remark 1.4 and Theorems 2.2 and 2.5 that (X, Γ) is uniformly almost periodic iff $(S(A), G)$ is uniformly almost periodic. It is not difficult to show that if these two transformation groups are uniformly almost periodic, then their enveloping semigroups are homeomorphic and isomorphic.

EXAMPLE 5.2. Let A be a *UHF*-algebra. We may write $A = \overline{\bigcup_{n=1}^\infty A_n}$, where $A_n = M_1 \otimes \ldots \otimes M_n$ and for each $i \geqslant 1$, M_i is a finite dimensional factor. Let $\mathcal{U}$ be the group of all unitaries in $\bigcup_{n=1}^\infty A_n$ which have the form $u_1 \otimes \ldots \otimes u_k$, where $k \geqslant 1$ and u_i is a unitary element of M_i, $i = 1, 2, \ldots, k$. If G is the group of all inner automorphisms of A induced by elements of $\mathcal{U}$, then we claim that $(S(A), G)$ is uniformly almost periodic. By Theorems 2.2 and 2.5, it suffices to show that if $a \in \bigcup_{n=1}^\infty A_n$, then the set $\{uau^*: u \in \mathcal{U}\}$ has compact closure in A. Now $\mathcal{U}$ leaves the generating set $\{a_1 \otimes \ldots \otimes a_n: a_i \in M_i, i = 1, \ldots, n\}$ of A_n invariant, hence leaves A_n invariant, $n \geqslant 1$. It follows that $\{uau^*: u \in \mathcal{U}\}$ lies in the closed ball of radius $\|a\|$ in some A_n, hence has compact closure, since A_n is finite dimensional.

EXAMPLE 5.3. Let Γ be a discrete group, and let A be the group C^*-algebra of Γ [5, 13.9.1]. We identify $L^1(\Gamma)$ with a dense $*$-subalgebra of A, and for each $g \in \Gamma$ we write δ_g for the function which is one at g and zero elsewhere on Γ. Then Γ is isomorphic to the subgroup $\{\delta_g: g \in \Gamma\}$ of the unitary group of A, and we also identify these groups. Then Γ has dense linear span in A.

Let $\mathrm{Aut}(\Gamma)$ be the group of all automorphisms of Γ. Each $\alpha \in \mathrm{Aut}(\Gamma)$ has a unique extension to an automorphism $\widetilde{\alpha}$ of A, and $\alpha \longrightarrow \widetilde{\alpha}$ is a one-to-one group homomorphism from $\mathrm{Aut}(\Gamma)$ into $\mathrm{Aut}(A)$.

A group is said to be *class-finite* if every conjugacy class in the group is a finite set, i.e. every element has a finite orbit under the action of the inner automorphisms. Let G be the group of all inner automorphisms of Γ. By Theorems 2.2 and 2.5, $(S(A), \widetilde{G})$ is uniformly almost periodic iff the orbit of each δ_g has compact closure. Since Γ is a discrete subset of A, $(S(A), \widetilde{G})$ is uniformly almost periodic iff Γ is class-finite.

We remark that class-finite groups are precisely the discrete $[FIA]^-$-groups studied by Mosak in [13]. Thoma studied harmonic analysis on class-finite groups in [19], and Neumann gave a structure theory for such groups in [14].

REMARK 5.4. The algebras given in these three examples are uniformly almost periodic C^*-algebras: In 5.1 put $\mathcal{U}_0$ equal to the unitary group of A, in 5.2 put $\mathcal{U}_0 = \mathcal{U}$, and in 5.3 put $\mathcal{U}_0 = \{\delta_g: g \in \Gamma\}$. In all three cases $\mathcal{U}_0$ has dense linear span and we may apply Remark 3.4.

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