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A GENERALIZATION OF JARNÍK'S THEOREM ON DIOPHANTINE APPROXIMATIONS TO RIDOUT TYPE NUMBERS

BY

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ABSTRACT. Let s be a positive integer, $c > 1$, $\mu_0, \ldots, \mu_s$ reals in $[0, 1]$, $\sigma = \Sigma_{i=0}^{s} \mu_i$, and t the number of nonzero μ_i. Let Π_i $(i = 0, \ldots, s)$ be $s + 1$ disjoint sets of primes and S the set of all $(s + 1)$-tuples of integers $(p_0, \ldots, p_s)$ satisfying $p_0 > 0$, $p_i = p_i^* p_i'$, where the p_i^* are integers satisfying $|p_i^*| \le c|p_i|^{\mu_i}$, and all prime factors of p_i' are in Π_i, $i = 0, \ldots, s$. Let $\lambda > 0$ if $t = 0$, $\lambda > \sigma/\min(s, t)$ otherwise, E_λ the set of all real s-tuples $(\alpha_1, \ldots, \alpha_s)$ satisfying $|\alpha_i - p_i/p_0| < p_0^{-\lambda}$ $(i = 1, \ldots, s)$ for an infinite number of $(p_0, \ldots, p_s) \in S$. The main result is that the Hausdorff dimension of E_λ is σ/λ. Related results are obtained when also lower bounds are placed on the p_i^*. The case $s = 1$ was settled previously (Proc. London Math. Soc. 15 (1965), 458–470). The case $\mu_i = 1$ $(i=0, \ldots, s)$ gives a well-known theorem of Jarník (Math. Z. 33 (1931), 505–543).

1. Introduction. Jarník [3] proved that the Hausdorff dimension of the set E of all real s-tuples $(\alpha_1, \ldots, \alpha_s)$ satisfying $|\alpha_i - p_i q^{-1}| < q^{-\lambda}$, $i = 1, \ldots, s$, for an infinite number of $(s + 1)$-tuples $(q, p_1, \ldots, p_s)$ of integers with $q > 0$, is $(s + 1)\lambda^{-1}$ provided that $\lambda > 1 + s^{-1}$.

In this paper we investigate the case where $q, p_1, \ldots, p_s$ are restricted to certain sets of integers which were considered by Ridout in his extension of Roth's theorem [6]. In [1] it was proved that the set E in this case has Lebesgue measure 0. The Hausdorff dimension for the one-dimensional case of the problem was determined by the authors in [2].

2. Definitions and notation. Let s be a positive integer, $\mu_0, \mu_1, \ldots, \mu_s$ reals in $[0, 1]$ and $\sigma = \Sigma_{i=0}^{s} \mu_i$. Let $\Pi_i = \{P_{i,1}, \ldots, P_{i,n_i}\}$ $(i = 0, \ldots, s)$, be $s + 1$ sets of distinct primes, C_i the set of integers all of whose prime factors belong to Π_i.

We say that condition I is satisfied, if there exists $P_i \in \Pi_i$ for $i = 0, \ldots, s$, such that

(Ia) $P_i \neq P_0$ $(i = 1, \ldots, s)$.

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(Ib) Those among the numbers $(1-\mu_0)/\log P_0, \ldots, (1-\mu_s)/\log P_s$ which are not zero are linearly independent over the field of rational numbers.

In particular, condition (Ib) is satisfied if $\mu_i = 1$, $i = 0, \ldots, s$.

Let $c > 1$. We define $S = S(c; \mu_0, \ldots, \mu_s; C_0, \ldots, C_s)$ to be the set of all $(s+1)$-tuples of integers $(p_0, \ldots, p_s)$, $p_0 > 0$, satisfying

(i) $(p_i, p_0) = 1$, $i = 1, \ldots, s$.

(ii) $p_i = p_i^* p_i'$ with $p_i' \in C_i$ and p_i^* any integer satisfying $|p_i^*| < c|p_i|^{\mu_i}$, $i = 0, \ldots, s$.

Similarly we define $S^T = S^T(c; \mu_0, \ldots, \mu_s; C_0, \ldots, C_s)$ by replacing (ii) by the requirement

(ii)^T $p_i = p_i^* p_i'$ where $p_i' \in C_i$ and p_i^* is any integer satisfying

$$|p_i|^{\mu_i} \le |p_i^*| < c|p_i|^{\mu_i}, \qquad i = 0, \ldots, s.$$

Let $\mu_0', \mu_1', \ldots, \mu_s'$ be reals satisfying (a) $0 \le \mu_i' \le \mu_i$; (b) if $\sigma > 0$, then $0 \le \mu_j' < \mu_j$ for some j. We define a set S' in a similar way to S and S^T, but replacing this time condition (ii) by the requirement

(ii)$'$ $p_i = p_i^* p_i'$ where $p_i' \in C_i$ and p_i^* is any integer satisfying

$$|p_i|^{\mu_i'} \le |p_i^*| < c|p_i|^{\mu_i}, \qquad i = 0, \ldots, s.$$

Let λ, D be positive reals, W an s-dimensional interval with edges parallel to the axes. We define the set $E = E(\lambda, W, S, D)$ to be the set of all s-tuples $(\alpha_1, \ldots, \alpha_s) \in W$ satisfying $|\alpha_i - p_i p_0^{-1}| < D p_0^{-\lambda}$, $i = 1, \ldots, s$, for an infinite number of $(s+1)$-tuples $(p_0, \ldots, p_s)$ from S. Similarly we define $E^T = E^T(\lambda, W, S^T, D)$ and $E' = E'(\lambda, W, S', D)$.

By R^s we denote the Euclidean space of s dimensions, and by $d(x, y)$ the distance between two points x, y of R^s. By $\delta(E)$, $\alpha - m^*E$, $\dim E$ we denote, respectively, the diameter, the Hausdorff measure with respect to the function t^α and the Hausdorff dimension of the set E. By a cube we mean an s-dimensional interval with edges parallel to the axes.

3. Main results. The main results of this paper are

Theorem I. $\dim E^T \le \dim E' \le \dim E \le \sigma/\lambda$.

Theorem II. *Let t be the number of μ_i which are not zero $(i = 0, \ldots, s)$. Let λ satisfy*

$$\lambda > 0 \qquad \text{if } t = 0,$$
$$\lambda > \sigma/\min(s, t) \qquad \text{if } t > 0. \tag{1}$$

If condition I *holds, then*

$$\dim E \geq \dim E' \geq \dim E^T \geq \sigma/\lambda.$$

Theorem III. *If* (1) *and* (Ia) *hold then* $\dim E \geq \dim E' \geq \sigma/\lambda$.

These results imply $\dim E = \dim E' = \sigma/\lambda$ if (1) and (Ia) hold and $\dim E = \dim E' = \dim E^T = \sigma/\lambda$ if (1) holds and condition I is satisfied. The case $\mu_i = 1$, $i = 0, \ldots, s$, gives Jarník's result.

4. **Proof of Theorem I.** Let $b_i > 0$, $i = 1, \ldots, s$. By symmetry, it is enough to prove the theorem when W is defined by

$$W = \{(x_1, \ldots, x_s) | 0 \leq x_i \leq b_i, \; i = 1, \ldots, s\}.$$

We shall prove that, for every $\sigma > 0$, if $\rho = (\sigma + \delta)\lambda^{-1}$ then $\rho - m^*E = 0$. We may also assume that $\delta < 1 - \mu_0$ if $\mu_0 < 1$.

Let $\epsilon > 0$. The set of all cubes whose center is $(p_1/p_0, \ldots, p_s/p_0) \in W$ with $(p_0, \ldots, p_s) \in S$, $p_0 > q_0$, and length of edge $2Dp_0^{-\lambda}$, is obviously a covering for E. If q_0 is large enough, the diameter of each cube is smaller than ϵ. It remains to prove that the series $M = \Sigma (p_0^{-\lambda})^{\rho} = \Sigma\, p_0^{-\sigma-\delta}$ converges, where the summation is over all sets $(p_0, \ldots, p_s) \in S$ such that $(p_1/p_0, \ldots, p_s/p_0) \in W$. Since $p_i = p_i^* p_i'$ for $i = 0, \ldots, s$, the summation can be broken up into a summation over $p_1^*, \ldots, p_s^*$, and over $p_1', \ldots, p_s'$. Therefore,

$$M = \sum_{p_0} M_1, \qquad M_1 \leq \sum\nolimits^{\{2\}} p_0^{-\sigma-\delta} \sum\nolimits^{\{1\}} 1,$$

where $\{1\}$ and $\{2\}$ indicate summations over $p_1^*, \ldots, p_s^*$ and $p_1', \ldots, p_s'$, respectively. Positive constants depending only on c, δ, μ_i, b_i, Π_i $(0 \leq i \leq s)$ are denoted by A below. Since $p_i^* < cp_i^{\mu_i} \leq cb^{\mu_i}p_0^{\mu_i}$ $(1 \leq i \leq s)$, we have $\Sigma^{\{1\}}1 < Ap_0^{\sigma-\mu_0}$. Putting $\eta = \delta/2$, we thus obtain

$$M_1 \leq Ap_0^{-\mu_0-\eta} \sum\nolimits^{\{2\}} p_0^{-\eta} = Ap_0^{-\mu_0-\eta} \prod_{i=1}^{s} \sum\nolimits^{\{3\}} p_0^{-\eta/s},$$

where $\{3\}$ denotes summation over $p_i' \in C_i$. Since $p_i' \leq p_i \leq b_i p_0$ $(1 \leq i \leq s)$, we obtain

$$\sum\nolimits^{\{3\}} p_0^{-\eta/s} \leq A \sum\nolimits^{\{3\}} p_i'^{-\eta/s} \leq A \prod_{j=1}^{n_i} (1 - P_{i,j}^{-\eta/s})^{-1} \leq A.$$

Therefore

$$M_1 \leq Ap_0^{-\mu_0-\eta} \quad \text{and} \quad M \leq A \sum\nolimits^{\{5\}} p_0'^{-\mu_0-\eta} \sum\nolimits^{\{4\}} p_0^{*-\mu_0-\eta},$$

where {4} and {5} denote summations over all $p_0^* \le R = C^{1/(1-\mu_0)} p_0'^{\mu_0/(1-\mu_0)}$ $(\mu_0 < 1)$ and $p_0' \in C_0$, respectively. (If $\mu = 1$, $M < A \sum_1^\infty p_0^{-1-\eta} \le A$.)

$$\sum\nolimits^{\{4\}} p_0^{*\,-\mu_0-\eta} < 1 + \int_1^R x^{-\mu_0-\eta}\, dx \le A p_0'^{\mu_0-\eta\mu_0/(1-\mu_0)}.$$

Therefore $M \le A \sum^{\{5\}} p_0'^{-\eta A} < \infty$, completing the proof.

5. **Proof that Theorem II implies Theorem III.** We may assume that $\sigma > 0$, because otherwise Theorem III is trivially true. Let $P_i \in \Pi_i$, $i = 0, \ldots, s$ and $P_i \ne P_0$, $i = 1, \ldots, s$. If condition I is not satisfied, then

$$(1-\mu_0)/\log P_0, \ldots, (1-\mu_s)/\log P_s$$

are linearly dependent over the rationals.

Let $\epsilon > 0$. There exists j such that $0 \le \mu_j' < \mu_j$. Choose μ_j'' such that $\mu_j' < \mu_j'' < \mu_j$, $\mu_j - \mu_j'' < \epsilon$, and such that the nonzero members among

$$(1-\mu_0)/\log P_0, \ldots, (1-\mu_j'')/\log P_j, \ldots, (1-\mu_s)/\log P_s$$

are linearly independent over the rationals. Let $\mu_i'' = \mu_i$ for $i \ne j$, and let S''^T and S''' be the same as S^T and S' respectively, except that in (ii)^T and (ii)$'$, μ_i is replaced by μ_i'' $(0 \le i \le s)$. Then

$$S''^T \subset S''' \subset S' \subset S, \quad E''^T \subset E''' \subset E' \subset E.$$

By Theorem II,

$$\dim E \ge \dim E' \ge \dim E''' \ge \dim E''^T \ge (\sigma - \epsilon)/\lambda.$$

Since this holds for every $\epsilon > 0$, we have $\dim E \ge \dim E' \ge \sigma/\lambda$, which is Theorem III.

Remark. Condition I is, however, essential in proving $\dim E^T \ge \sigma/\lambda$, as is shown by the following example. Let P_0 and P_1 be two distinct primes, $C_0 = \{P_0^{m_0}\}$, $C_1 = \{P_1^{m_1}\}$, m_0, m_1 nonnegative integers. There exist μ_0 and μ_1 in $[0, 1)$ such that $P_1^{1/(1-\mu_1)} = P_0^{1/(1-\mu_0)} = A > 1$. Let $0 < \epsilon < (A-1)/(A+1)$, and

$$1 < c < \min\left((1+\epsilon)^{1-\mu_1}, (1-\epsilon)^{-(1-\mu_0)}\right).$$

If $(p_0, p_1) \in S^T(c; \mu_0, \mu_1; C_0, C_1)$ and $p_0, p_1 > 0$, then

$$p_i = p_i^* p_i', \quad p_i'^{\mu_i} \le p_i^* < c p_i'^{\mu_i}, \quad p_i' = P_i^{m_i}, \qquad i = 0, 1.$$

This gives

$$P_i^{m_i/(1-\mu_i)} \le p_i < c^{1/(1-\mu_i)} P_i^{m_i/(1-\mu_i)}, \qquad i = 0, 1,$$

and

$$(2) \qquad (1-\epsilon)A^k < c^{-1/(1-\mu_0)}A^k < p_1/p_0 < c^{1/(1-\mu_1)}A^k < (1+\epsilon)A^k,$$

where $k = m_1 - m_0$.

The requirement for ϵ implies that $A(1-\epsilon) > 1+\epsilon$. By (2), the interval $(1+\epsilon, A(1-\epsilon))$ does not contain any p_1/p_0 with $(p_0, p_1) \in S^T$ because, if $k \leq 0$, then $(1+\epsilon)A^k \leq 1+\epsilon$, and if $k > 0$, then $A(1-\epsilon) \leq (1-\epsilon)A^k$.

6. **Lemmas for Theorem II.** It suffices to prove Theorem II for an interval W of the form

$$W = \{(x_1, \dots, x_s) | a_i \leq x_i \leq b_i,\ i = 1, \dots, s\},$$

where the a_i are arbitrary positive reals, $b_i = a_i + L_0$, and L_0 is any sufficiently small real number, to be chosen later in the proof (Lemma 4).

Lemma 1. *It is enough to prove Theorem* II *for the case* $\mu_i \geq \mu_0$, $i = 1, \dots, s$.

Proof. If $\mu_i < \mu_0$ for some $i > 0$, we may assume that $\mu_s = \min(\mu_0, \dots, \mu_s)$. Let $\nu_i = \mu_i$ if $i \neq 0$, s, $\nu_0 = \mu_s$ and $\nu_s = \mu_0$. Let $\psi\colon W \to R^s$ be defined by

$$\psi(x_1, \dots, x_{s-1}, x_s) = (x_1/x_s, \dots, x_{s-1}/x_s, 1/x_s),$$

and let $W_1 = \{(x_1, \dots, x_s) | a_i' \leq x_i \leq b_i',\ 1 \leq i \leq s\}$ be chosen so that $\psi(W_1) \subset W$. It is easily seen that ψ has Jacobian a_s^{-s-1}, which is bounded away from 0 and ∞ on W_1, and therefore preserves Hausdorff dimension. Let S^T, E^T be as defined in §2,

$$S_1^T = S^T(c; \nu_0, \dots, \nu_s; C_s, C_1, \dots, C_{s-1}, C_0), \qquad E_1^T = E^T(\lambda, W_1, S_1^T, D_1),$$

where $D_1 > 0$ is sufficiently small. The conditions of Theorem II hold for E_1^T, and we have, moreover, $\nu_i \geq \nu_0$ $(1 \leq i \leq s)$. Therefore, assuming the validity of the theorem for this case, $\dim E_1^T \geq \sigma/\lambda$. We now prove that for a suitable choice of D_1 we have $\psi(E_1^T) \subset E^T$. Let $(\beta_1, \dots, \beta_s) \in \psi(E_1^T)$. There exists $(\alpha_1, \dots, \alpha_s) \in E_1^T$ such that $(\alpha_1/\alpha_s, \dots, \alpha_{s-1}/\alpha_s, 1/\alpha_s) = (\beta_2, \dots, \beta_{s-1}, \beta_s)$, and an infinity of $(p_s, p_1, \dots, p_{s-1}, p_0) \in S_1^T$ $(p_i' \in C_i,\ i = 0, \dots, s)$, satisfying $|\alpha_i - p_i/p_s| < D_1 p_s^{-\lambda}$, $1 \leq i \leq s-1$, $|\alpha_s - p_0/p_s| < D_1 p_s^{-\lambda}$. Let $\alpha_i = p_i/p_s + \eta_i$, $1 \leq i \leq s-1$, $\alpha_s = p_0/p_s + \eta_s$, $|\eta_i| < D_1 p_s^{-\lambda}$ $(0 \leq i \leq s)$. For $1 \leq i \leq s-1$ we then have

$$\frac{\alpha_i}{\alpha_s} = \frac{p_i}{p_0} \cdot \frac{1 + \eta_i p_s/p_i}{1 + \eta_s p_s/p_0},$$

$$\left|\frac{\alpha_i}{\alpha_s}-\frac{p_i}{p_0}\right|<\frac{p_i}{p_0}(1-D_1p_s^{1-\lambda}/p_0)^{-1}\left(|\eta_1|\frac{p_s}{p_i}+|\eta_s|\frac{p_s}{p_0}\right)$$
$$\leq 2\left(\frac{b_i'}{a_i'}\right)(1-D_1p_s^{1-\lambda}/p_0)^{-1}D_1p_s^{-\lambda}<Dp_0^{-\lambda},$$

if D_1 is sufficiently small. A similar computation shows that $|\alpha_s^{-1}-p_sp_0^{-1}| < Dp_0^{-\lambda}$ for $\bar{D}$ small enough. Thus

$$|\beta_i-p_i/p_0|<Dp_0^{-\lambda}, \qquad i=1,\dots,s,$$

which shows that $\psi(E_1^T)\subset E^T$. Therefore,

$$\dim E^T\geq \dim\psi(E_1^T)=\dim E_1^T\geq\sigma/\lambda.$$

From now on we shall assume $\mu_i\geq\mu_0$ $(1\leq i\leq s)$. We may also assume that every Π_i contains only one prime P_i such that condition I is satisfied, that not all μ_i are 1 because this is Jarník's theorem, and that not all μ_i are zero because then Theorem II is trivial. These assumptions are not essential but permit a simpler exposition.

Let $\delta>0$, $\rho=(\sigma-\delta)/\lambda$. In order to prove that $\rho-m^*(E^T)>0$, we use the following special case of a theorem due to P. A. P. Moran [5].

Lemma 2. *Let s be a positive integer, E a bounded set in R^s and $0\leq\rho\leq s$. A sufficient condition for $\rho-m^*(E)$ to be positive is the existence of a closed subset F of E and an additive function ϕ defined on the ring $\mathfrak{R}$ generated by the semiopen cubes of R^s, satisfying the following properties*:

(a) *ϕ is nonnegative.*

(b) *For every $R\in\mathfrak{R}$ and $R\supset F$ we have $\phi(R)>b>0$ for some fixed b.*

(c) *There exists a positive constant k such that for every semiopen cube R we have $\phi(R)<k\delta(R)^\rho$.*

Lemma 3. *Let $\theta_1,\dots,\theta_s$ be reals such that $1,\theta_1,\dots,\theta_s$ are linearly independent over the rationals, δ, η, $n_0>0$. There exist real numbers b, B such that for every set of real numbers $\alpha_1,\dots,\alpha_s$ there is an $(s+1)$-tuple of integers $(m_0,\dots,m_s)$ satisfying $|m_0\theta_i-m_i-\alpha_i|<\delta$, $1\leq i\leq s$, $n_0<b<m_0<B<(1+\eta)b$.*

Except for the explicit bound on m_0, this is Kronecker's theorem. The bound can be obtained by introducing a slight change in one of the proofs of Kronecker's theorem, for example, Lettenmeyer's proof [4].

Let t' be the number of nonzero μ_i $(1 \leq i \leq s)$, $0 < \mu < \min_{\mu_i \neq 0} \mu_i$. We shall now formulate the main lemma.

Lemma 4. *Let $L < L_0$, θ, η be positive real, $q_0 = q_0(a_i, b_i, \Pi_i, \mu_i, L, \eta)$ a sufficiently large real number. There exist reals a, A such that for every cube $I \subset W$ with edge L, there is a subset $S_I \subset S^T$ with the following properties:*

(i) *If $(p_0, \ldots, p_s) \in S_I$, then $(p_1/p_0, \ldots, p_s/p_0) \in I$, $q_0 < a < p_0 < A < a^{1+\eta}$, $(p_i, p_0) = 1$, $a^{-\mu} < L$, and all the $(p_0, \ldots, p_s) \in S_I$ share the same fixed $(s+1)$-tuple $(p_0', \ldots, p_s')$.*

(ii) *If $p_0^{(1)} \leq p_0^{(2)}$ and $(p_0^i, \ldots, p_s^i) \in S_I$ $(i = 1, 2)$, then there exists at least one j such that*

$$|p_j^{(1)}/p_0^{(1)} - p_j^{(2)}/p_0^{(2)}| \geq (p_0^{(1)})^{-(\sigma/s)-\theta}. \tag{3}$$

(iii) *Let $a^{-\mu} < l \leq L$, I_l any cube with edge length l contained in I, V_l the number of elements $(p_0, \ldots, p_s)$ of S_I such that $(p_1/p_0, \ldots, p_s/p_0) \in I_l$. Then*

$$V_l < K l^{t'} p_0'^{\sigma/(1-\mu_0)}/Y,$$

where

$$Y = \begin{cases} \log p_0' & \text{if } \mu_0 > 0, \\ 1 & \text{if } \mu_0 = 0, \end{cases}$$

K a suitable positive constant depending on S^T, W, λ, D, η, θ.

(iv) *The total number V_L of elements of S_I satisfies*

$$V_L > K L^{t'} \frac{p_0'^{\sigma/(1-\mu_0)}}{Y} \geq K L^{t'} \frac{a^{\sigma}}{X},$$

where

$$X = \begin{cases} \log a & \text{if } \mu_0 > 0, \\ 1 & \text{if } \mu_0 = 0. \end{cases}$$

Remark. The convention on K will be used for the rest of the paper, for the sake of simplicity of notation.

Proof. Let $\epsilon > 0$ be sufficiently small,

$$I = \{(x_1, \ldots, x_s) \mid a_i + \epsilon < y_i \leq x_i \leq y_i + L < b_i,\ 1 \leq i \leq s\}, \tag{4}$$

$$1 < c_0 < c_1 < c, \quad c_1 < 1 + \min_i (\epsilon/a_i), \quad c_1/c_0 < 2, \quad c_0 < 2. \tag{5}$$

Since $\mu_i \geq \mu_0$ and not all μ_i are 1, we have $\mu_0 < 1$. Suppose that $\mu_0, \dots, \mu_h$ $(h \leq s)$ are all the μ_i which are not 1. We assume first $h > 0$. Let

$$\delta = \min_{1\leq i\leq h} \frac{1-\mu_i}{2\log P_i} \log\left(1 + \frac{L}{b_i}\right),$$

$$\theta_i = \frac{(1-\mu_i)\log P_0}{(1-\mu_0)\log P_i}, \quad \xi_i = -\frac{1-\mu_i}{2\log P_i}\log\left(\frac{\gamma_i(\gamma_i+L)}{c_1^2}\right), \qquad 1 \leq i \leq h.$$

Condition I implies that $1, \theta_1, \dots, \theta_h$ are linearly independent over the rationals. By Lemma 3, there exist numbers b, B and an $(h+1)$-tuple of integers $(m_0, \dots, m_h)$ satisfying

$$\text{(6)} \qquad \begin{gathered} (1-\mu_0)\log_{P_0}(q_0/c_0) < b < m_0 < B < (1+\eta)b, \\ |m_0\theta_i - m_i - \xi_i| < \delta, \qquad 1 \leq i \leq h. \end{gathered}$$

This with the definition of δ implies

$$\text{(7)} \qquad \gamma_i < c_1 P_i^{m_i/(1-\mu_i)}/P_0^{m_0/(1-\mu_0)} < \gamma_i + L, \qquad 1 \leq i \leq h.$$

Define a set T_l of $(s+1)$-tuples $(p_0, \dots, p_s)$ of integers with $p_i = p_i^* p_i'$ $(0 \leq i \leq s)$ satisfying:

1. $p_i' = P_i^{m_i}$ $(0 \leq i \leq h)$, where $(m_0, \dots, m_h)$ is a fixed $(h+1)$-tuple of integers satisfying (7), and $p_i' = 1$ for $i > h$.

2. If $\mu_0 > 0$, p_0^* ranges over all primes $> \max_i P_i$ satisfying

$$\text{(8)} \qquad c_0 p_0'^{\mu_0/(1-\mu_0)} \leq p_0^* \leq c_1 p_0'^{\mu_0/(1-\mu_0)}.$$

The existence of such p_0^* is guaranteed if q_0 is sufficiently large. If $\mu_0 = 0$, put $p_0^* = 1$.

3. If $\mu_i > 0$, p_i^* ranges over all integers satisfying

$$\text{(9)} \qquad \gamma_i \frac{p_0}{p'} < p_i^* < (\gamma_i + L)\frac{p_0}{p'}, \qquad (p_i^*, p_0 p_i') = 1, \quad 1 \leq i \leq s.$$

Since every interval of length ≥ 5 contains an integer relatively prime to the product of three given primes, integers p_i^* satisfying (9) will exist if Lp_0/p_i' > 6. By (7) this condition is easily seen to hold if q_0 is sufficiently large. If $\mu_i = 0$, put $p_i^* = 1$.

Now assume $h = 0$. Choose $b = m_0 - 1 > (1-\mu_0)\log_{P_0}(q_0/c_0)$, $B = m_0 + 1$, $p_0' = P_0^{m_0}$, $p_i' = 1$ $(1 \leq i \leq s)$, and p_0^*, p_i^* as above. It is clear that such $p_i^* = p_i$ satisfying (9) do in fact exist. Moreover, for q_0 sufficiently large, (6) holds.

The definition of T_l implies that if $(p_0, \dots, p_s) \in T_l$, then $a_i < p_i/p_0 < b_i$, and $(p_i, p_0) = 1$ $(1 \le i \le s)$. This follows from (9) if $h\mu_i > 0$ or $h = 0$. If $h > 0$, $\mu_i = \mu_0 = 0$, it follows from (7) and (5). For $h > 0$, $\mu_i = 0$, $\mu_0 > 0$, we have by (4), (5), (7) and (8),

$$a_i < \frac{a_i + \epsilon}{c_1} < \frac{\gamma_i}{c_1} < \frac{p_i}{p_0} < \frac{\gamma_i + L}{c_0} < \gamma_i + L.$$

Let $a = c_0 P_0^{b/(1-\mu_0)}$, $A = c_0 P_0^{B/(1-\mu_0)}$. If q_0 is sufficiently large, we obtain, by (6), (8) and (5) $(\mu_0 \ge 0,\ h \ge 0)$,

$$q_0 < a < p_0 < A < a^{1+\eta}, \qquad a^{-\mu} < L.$$

For $\mu_0 > 0$, (8) implies $p_0^{\mu_0} < c_0^{1-\mu_0} p_0^{\mu_0} \le p_0^* < c_1 p_0^{\mu_0} < c p_0^{\mu_0}$, and for $\mu_0 = 0$, $p_0^* = p_0^{\mu_0}$. To prove that $T_l \subset S^T$ it remains to show that

$$(10) \qquad p_i^{\mu_i} \le p_i^* < c p_i^{\mu_i}, \qquad 1 \le i \le s.$$

We may assume $0 < \mu_i < 1$ $(1 \le i \le s)$, because otherwise (10) is trivial. If $\mu_0 > 0$, we obtain, from (7), (8), (9),

$$(c_0 c_1)^{1-\mu_i} \frac{\gamma_i}{\gamma_i + L} p_i^{\mu_i} < p_i^* < c_1^{(1-\mu_i)^2} \frac{\gamma_i + L}{\gamma_i} p_i^{\mu_i},$$

and for $\mu_0 = 0$, we obtain, from (7) and (9),

$$\frac{\gamma_i}{\gamma_i + L} c^{1-\mu_i} p_i^{\mu_i} < p_i^* < \frac{\gamma_i + L}{\gamma_i} c_1^{1-\mu_i} p_i^{\mu_i}.$$

Therefore (10) will hold by choosing L to satisfy

$$0 < L < L_0 < \min_{1 \le i \le s} (a_i(c/c_1 - 1),\ a_i(c_1^{1-\mu_i} - 1)).$$

We thus proved that $T_l \subset S^T$. Let

$$I_l = \{(x_1, \dots, x_s) | \gamma_i < \beta_i \le x_i \le \beta_i + l \le \gamma_i + L,\ 1 \le i \le s\}, \qquad a^{-\mu} < l \le L.$$

Let p_0 be fixed. For $\mu_i > 0$ $(i > 0)$, denote by $W_l^i(p_0)$ the number of integers p_i^* relatively prime to $p_0^* P_0 P_i$, which satisfy $\beta_i p_0/p_i' < p_i^* < (\beta_i + l) p_0/p_i'$. Lemma 4 of [2] implies

$$\left(\frac{lp_0}{p_i'}-1\right)\left(1-\frac{1}{P_i}\right)\left(1-\frac{1}{P_0}\right)\left(1-\frac{1}{p_0^*}\right)-2^3<W_l^i(p_0)$$
$$<\left(\frac{lp_0}{p_i'}+1\right)\left(1-\frac{1}{P_i}\right)\left(1-\frac{1}{P_0}\right)\left(1-\frac{1}{p_0^*}\right)+2^3,$$

except that the factor $1-1/p_0^*$ is dropped if $\mu_0=0$. Since $l>a^{-\mu}>p_0^{-\mu}$, (9) and (10) imply $lp_0/p_i'>Kp_0^{\mu_i-\mu}$. Since $\mu_i-\mu>0$, 1 is absorbed by lp_0/p_i'. Thus

$$Klp_0^{\mu_i}<W_l^i(p_0)<Klp_0^{\mu_i}. \tag{11}$$

For fixed p_0, denote by $W_l(p_0)$ the number of elements $(p_0, \dots, p_s)\in T_l$ such that $(p_1/p_0, \dots, p_s/p_0)\in I_l$. Multiplying together the t' inequalities (11) and defining $W_l^i(p_0)=1$ for $\mu_i=0$, we obtain

$$Kl^{t'}p_0^{\sigma-\mu_0}<W_l(p_0)<Kl^{t'}p_0^{\sigma-\mu_0}. \tag{12}$$

It is easily seen that if $s=1$, the set T_l satisfies all the conditions of the lemma for S_l. For $s>1$, however, condition (ii) is not necessarily satisfied. Let $(p_0, p_1^{(1)}, \dots, p_s^{(1)})$ and $(p_0, p_1^{(2)}, \dots, p_s^{(2)})$ be two distinct elements of T_l with the same p_0. By (9) and (10),

$$\left|\frac{p_i^{(1)}}{p_0}-\frac{p_i^{(2)}}{p_0}\right|=\frac{p_i'}{p_0}\left|p_i^{*(1)}-p_i^{*(2)}\right|\geq\frac{p_i'}{p_0}>Kp_0^{-\mu_i}.$$

There exists j such that

$$\mu_j\leq\frac{1}{s}\sum_{i=1}^{s}\mu_i\leq\frac{\sigma}{s}<\frac{\sigma}{s}+\theta;$$

hence

$$\left|\frac{p_j^{(1)}}{p_0}-\frac{p_j^{(2)}}{p_0}\right|\geq Kp_0^{-\mu_j}>Kp_0^{-(\sigma/s)-\theta}.$$

Condition (ii) of the lemma is therefore satisfied for two elements of T_l with the same p_0. If $\mu_0=0$, then all the elements of T_l have the same p_0 and we define $S_l=T_l$ in this case. If $\mu_0>0$, we define $S_l\subset T_l$ by excluding all those elements $(p_0, \dots, p_s)$ of T_l for which there exists $p_0^{(1)}<p_0$ and $(p_0^{(1)}, \dots, p_s^{(1)})\in T_l$ such that for $i=1, \dots, s$ we have

(13) $$\left|\frac{p_i^{(1)}}{p_0^{(1)}} - \frac{p_i}{p_0}\right| < (p_0^{(1)})^{-(\sigma/s)-\theta}.$$

Clearly, S_l satisfies condition (ii) of the lemma. We shall now count the number of elements of T_l which are not in S_l. Let $N(p_0, p_0^{(1)})$ be the number of elements of T_l for a fixed p_0 and fixed $p_0^{(1)} < p_0$, for which (13) holds for some i. For fixed p_0, let $N(p_0)$ denote the number of those elements $(p_0, \ldots, p_s)$ of T_l for which there exists an element $(p_0^{(1)}, \ldots, p_s^{(1)})$ of T_l such that (13) holds for every i. Clearly,

$$N(p_0) \le \sum_{p_0^{(1)} \le p_0} \prod_{i=1}^{s} N_i(p_0, p_0^{(1)}).$$

From (13),

$$|p_i^* p_0^{*(1)} - p_i^{*(1)} p_0^*| < p_0^* p_0^{(1)}/p_i' p_0^{(1)(\sigma/s)+\theta}.$$

The expression $p_i^{*(1)} p_0^* - p_0^{*(1)} p_i^*$ can therefore assume at most

$$2p_0 p_0^{(1)}/p_i' p_0^{(1)(\sigma/s)+\theta}$$

different values. Let u be a fixed integer. The equation $p_i^* p_0^{*(1)} - p_i^{*(1)} p_0^* = u$ implies

(14) $$p_i^* p_0^{*(1)} \equiv u \pmod{p_0^*}.$$

Since p_0^* is a prime, this congruence has exactly one solution p_i^* in each interval of length p_0^*. The integer p_i^* is to be chosen in the interval $[\gamma_i p_0/p_i', (\gamma_i + L)p_0/p_i']$ of length $Lp_0/p_i' = KLp_0^{\mu_i}$. Since $p_0^* > c_0^{1-\mu_0} p_0^{\mu_0}$ and $\mu_i \ge \mu_0$, the number of solutions of (14) is $Lp_0/p_0^* p_i' < KLp_0^{\mu_i - \mu_0}$. Therefore

$$N_i(p_0, p_0^{(1)}) \le KL \frac{p_0^* p_0^{(1)} p_0^{\mu_i - \mu_0}}{p_i' p_0^{(1)(\sigma/s)+\theta}} \le KL \frac{p_0^{\mu_i} p_0^{(1)\mu_i}}{(p_0^{(1)})^{(\sigma/s)+\theta}},$$

and hence

$$N(p_0) \le KL^s p_0^{\mu_1 + \cdots + \mu_s} \sum_{p_0^{(1)} < p_0} p_0^{(1)\mu_1 + \cdots + \mu_s} / p_0^{(1)\sigma + \theta s}$$

$$= KL^s p_0^{\mu_1 + \cdots + \mu_s} \sum_{p_0^{(1)} < p_0} p_0^{(1)^{-\mu_0 - \theta s}}$$

$$\le KL^s p_0^{\sigma - \mu_0 - \theta s/2} \sum_{p_0^{(1)} < p_0} p_0^{(1)^{-\mu_0 - \theta s/2}}.$$

The last sum converges as was shown in the proof of Theorem I. Therefore,

$$N(p_0) \le KL^s p_0^{\sigma-\mu_0-\theta s/2}.$$

Let $V_l(p_0)$ denote the number of elements $(p_0, \dots, p_s)$ of S_l such that $(p_1/p_0, \dots, p_s/p_0) \in I_l$ for fixed p_0, and let V_l be the total number of those elements in S_l. By (12),

$$V_l(p_0) \le W_l(p_0) \le Kl^{t'} p_0^{\sigma-\mu_0},$$

$$V_L(p_0) = W_L(p_0) - N(p_0) \ge KL^{t'} p_0^{\sigma-\mu_0}.$$

Therefore,

$$V_l < Kl^{t'} \sum\nolimits^* p_0^{\sigma-\mu_0}, \qquad V_L > KL^{t'} \sum\nolimits^* p_0^{\sigma-\mu_0},$$

where Σ^* denotes summation over all p_0 so that $(p_0, \dots, p_s) \in S_l$. By (8),

$$Kp_0'^{(\sigma-\mu_0)/(1-\mu_0)} \sum_{p_0^*}^{*} 1 < \sum\nolimits^* p_0^{\sigma-\mu_0} < Kp_0'^{(\sigma-\mu_0)/(1-\mu_0)} \sum_{p_0^*}^{*} 1,$$

where $\Sigma^*_{p_0^*} 1 = 1$ if $\mu_0 = 0$. If $\mu_0 > 0$, we obtain from (8) and the Prime Number Theorem,

$$Kp_0'^{(\sigma-\mu_0)/(1-\mu_0)}/\log p_0' < \sum_{p_0^*}^{*} 1 < Kp_0'^{(\sigma-\mu_0)/(1-\mu_0)}/\log p_0'.$$

Therefore we obtain $(\mu_0 \ge 0)$

$$V_l < Kl^{t'} p_0'^{\sigma/(1-\mu_0)}/Y,$$

$$V_L > KL^{t'} p_0'^{\sigma/(1-\mu_0)}/Y > KL^{t'} a^{\sigma}/X,$$

completing the proof of Lemma 4.

7. **Proof of Theorem II.** By (1), $\lambda = \sigma/\min(s, t) + \tau$, for some $\tau > 0$. We shall construct by induction a sequence of closed sets $F_0 \supset F_1 \supset \cdots$ and a sequence of additive functions ϕ_n on $\Re$ such that the set $F = \bigcap_{n=1}^{\infty} F_n \subset E$, and the function $\phi = \lim_{n\to\infty} \phi_n$ satisfy the hypothesis of Lemma 2 with $\rho = (\sigma - \delta)/\lambda$. Let $F_0 = W$, G_0 the set whose unique element is F_0. Let $A_0 > (L_0/D)^{-1/\lambda}$ be sufficiently large. For every $I \in \Re$ and $I \subset W$ we define $\phi_0(I) = V(I)/L_0^s$, where $V(I)$ denotes the s-dimensional volume of I.

Suppose that for $k = 0, \dots, n-1$, a suitable increasing sequence of positive numbers A_k and sets G_k of disjoint closed cubes all with edge $L_k = 2D(2A_k)^{-\lambda}$ have already been defined such that every element of G_k is contained in some element of G_{k-1}. Let F_k be the union of all elements of G_k.

Suppose also that a sequence ϕ_k of additive functions on $\Re$ has already been defined for all $k < n$.

Let $I \in G_{n-1}$, I' the cube concentric with I with edge $L_{n-1}/2$. We apply Lemma 4 with θ, η satisfying $0 < \theta < \min(\delta, r)$, $0 < \eta < \delta/(\sigma - \delta)$, where $0 < \delta < \sigma$; $L = L_{n-1}/2$, A_{n-1} as q_0 and I' as I. There exist reals a_n, A_n and a subset $S_{I'} \subset S^T$ of $(s+1)$-tuples of integers $(p_0, \dots, p_s)$ satisfying

$$(p_1/p_0, \dots, p_s/p_0) \in I', \qquad A_{n-1} < a_n < p_0 < A_n < a_n^{1+\eta},$$

and (3). Let G_n be the set of all closed cubes with centers $(p_1/p_0, \dots, p_s/p_0) \in I'$ and length of edge $2D(2A_n)^{-\lambda}$ where I ranges over all cubes of G_{n-1}. Note that each I' has its own unique p_0', which induces a number of p_0 as specified by (8) (if $\mu_0 > 0$), but by Lemma 3 all of these p_0 satisfy the inequalities of (i) of Lemma 4 for the same $a_n = a$, $A_n = A$.

By (3), all cubes in G_n are disjoint if A_n is sufficiently large, as we shall assume. Let F_n be the union of all cubes in G_n. Then F_n is closed and $F_n \subset F_{n-1}$. If $I \in G_n$, then $I \subset J \in G_{n-1}$. Letting N_J be the number of elements of G_n contained in J, we define $\phi_n(I) = \phi_{n-1}(J)/N_J$. If $I \in \Re$ and $I \subset J \in G_n$, let $\phi_n(I) = \phi_n(J) \cdot V(I)/V(J)$. If $I \subset W$ is an arbitrary element of $\Re$, then $I = \bigcup_h I_h \cup Q$, where $I_h = I \cap J_h$, $J_h \in G_n$, $Q \cap F_n = \emptyset$. In this case we define $\phi_n(I) = \Sigma_h \phi_n(I_h)$. The following properties of the functions ϕ_n are obvious: They are nonnegative finite additive functions on $\Re$, and for $I \in G_{n-1}$, $\phi_n(I) = \phi_{n-1}(I)$. If $I \in \Re$, $I \supset F_n$, then $\phi_n(I) = 1$. Let δ_i, $i = 0, 1, 2, \dots$, be positive reals such that the product $\Pi_{i=0}^{\infty}(1 + \delta_i)$ converges and δ_0, δ_1 sufficiently large. Let $k_n = \Pi_{i=0}^{n}(1 + \delta_i)$. We shall prove by induction on n that the sequence A_i can be chosen such that for every cube $I \subset W$,

$$\phi_n(I)/\delta(I)^\rho < k_n. \tag{15}$$

For $n = 0$,

$$\frac{\phi_0(I)}{\delta(I)^\rho} = \frac{V(I)}{L_0^s \delta(I)^\rho} = S^{-s/2} L_0^{-s} \delta(I)^{s-\rho} \le K L_0^{-\rho} < 1 + \delta_0.$$

Let $\Delta_n = \max_{I \in G_n} \phi_n(I)$. By (iv) of Lemma 4,

$$\Delta_n < K L_{n-1}^{-t'} \Delta_{n-1} X_n a_n^{-\sigma}, \qquad X_n = \begin{cases} \log a_n & \text{if } \mu_0 > 0, \\ 1 & \text{if } \mu_0 = 0. \end{cases}$$

For proving (15) we distinguish several cases.

(a) $I \in G_n$. Then

$$\frac{\phi_n(I)}{\delta(I)^\rho} < \frac{\Delta_n}{L_n^\rho} < KL_{n-1}^{-t'}\Delta_{n-1}X_n a_n^{-\sigma}A_n^{\lambda\rho} < KL_{n-1}^{-t'}\Delta_{n-1}X_n a_n^{-\sigma+(1+\eta)(\sigma-\delta)}.$$

The exponent of a_n is negative. For a_n large enough, $\phi_n(I)/\delta(I)^\rho$ can thus be made as small as desired.

(b) $I \subset J \in G_n$. Then

$$\frac{\phi_n(I)}{\delta(I)^\rho} = \phi_n(J)\frac{V(I)}{V(J)\delta(I)^\rho} = \frac{\phi_n(J)}{\delta(J)^\rho}\left(\frac{\delta(I)}{\delta(J)}\right)^{s-\rho} \leq \frac{\phi_n(J)}{\delta(J)^\rho},$$

which is reduced to the previous case.

(c) $I \subset J \in G_{n-1}$ and the length l of the edge of I is greater than $a_n^{-\mu}$. Let N_I and N_J denote the number of elements of G_n with nonempty intersection with I and J respectively. By (iii) and (iv) of Lemma 4,

$$\frac{\phi_n(I)}{\delta(I)^\rho} \leq \frac{\phi_{n-1}(J)}{N_J} \cdot \frac{N_I}{\delta(I)^\rho} \leq K\frac{\phi_{n-1}(J)}{\delta(I)^\rho}\,\frac{l^{t'}}{L_{n-1}^{t'}}$$

$$\leq K\,\frac{\phi_{n-1}(J)}{\delta(J)^\rho}\left(\frac{\delta(I)}{\delta(J)}\right)^{t'-\rho} < K\frac{\phi_{n-1}(J)}{\delta(J)^\rho},$$

since inequality (1) on λ implies $t' - \rho > 0$. For $n > 1$, the last expression can be made as small as desired if a_{n-1} is large enough, as was shown in case (a). For $n = 1$,

$$\frac{\phi_1(I)}{\delta(I)^\rho} < K\frac{\phi_0(J)}{\delta(J)^\rho} < \frac{K}{L_0^\rho} \leq 1 + \delta_1,$$

if δ_1 is sufficiently large.

(d) $I \subset J \in G_{n-1}$ but the edge l of I is not greater than $a_n^{-\mu}$. The cubes concentric to the cubes of G_n and with edge of length $A_n^{-(\sigma/s)-\theta}$ are disjoint by (3), so the number N_I of cubes of G_n with nonempty intersection with I is at most $N_I \leq K\delta(I)^s A_n^{\sigma+\theta s}$. Therefore,

$$\frac{\phi_n(I)}{\delta(I)^\rho} \leq \frac{N_I\Delta_n}{\delta(I)^\rho} \leq K\Delta_{n-1}L_{n-1}^{-t'}a_n^{-\mu(s-\rho)+(1+\eta)(\sigma+\theta s)}a_n^{-\sigma}X_n.$$

For θ, η small enough and a_n large enough, this can be made as small as desired.

(e) I is an arbitrary cube of edge length l. We may assume $n > 1$, as the case $n = 1$ is settled by the previous cases. We may also assume $l > \frac{1}{2}A_{n-1}^{-(\sigma/s)-\theta}$, since otherwise, for A_{n-1} large enough, I intersects at most one element of G_{n-1}, which is also subsumed by the previous cases. Let J be a cube with the same center as I and edge length $l + 4A_{n-1}^{-\lambda}$. For A_{n-1} large enough we have

$$(\delta(J)/\delta(I))^{\rho} < 1 + \delta_n,$$

$$\frac{\phi_n(I)}{\delta(I)^{\rho}} \leq \frac{\phi_{n-1}(J)}{\delta(I)^{\rho}} = \frac{\phi_{n-1}(J)}{\delta(J)^{\rho}}\left(\frac{\delta(J)}{\delta(I)}\right)^{\rho} < (1 + \delta_n)k_{n-1} = k_n,$$

which proves (15).

Now let ϵ_i, $i \geq 2$, be any sequence of positive integers such that $\sum_{i=2}^{\infty} \epsilon_i$ converges. For every cube $I \in \mathfrak{R}$, we have

$$\phi_n(I) = \phi_0(I) + (\phi_1(I) - \phi_0(I)) + \cdots + (\phi_n(I) - \phi_{n-1}(I)).$$

The difference $\phi_k(I) - \phi_{k-1}(I)$ is contributed by those elements of G_{k-1} which intersect the boundary of I. Let $\overline{N}_k$ be the number of those elements of G_{k-1}. The cubes concentric to the elements of G_{k-1} and whose length of edge is $\frac{1}{2}A_{k-1}^{-(\sigma/s)-\theta}$ are disjoint. Therefore,

$$\overline{N}_k \leq K \max\{\delta(I)^{s-1}A_{k-1}^{((\sigma/s)+\theta)(s-1)}, 1\}, \tag{16}$$

and

$$|\phi_k(I) - \phi_{k-1}(I)| \leq \overline{N}_k\Delta_{k-1}.$$

If the max in (16) is 1, then for a_{k-1} large enough $|\phi_k(I) - \phi_{k-1}(I)| < \epsilon_k$. Otherwise,

$$|\phi_k(I) - \phi_{k-1}(I)| \leq K\delta(I)^{s-1}L_{k-2}^{-t'}\Delta_{k-2}X_{k-1}A_{k-1}^{((\sigma/s)+\theta)(s-1)-\sigma(1+\eta)}.$$

For θ small and A_{k-1} large enough, this is smaller than ϵ_k. This proves that the functions ϕ_n converge on each cube $I \in \mathfrak{R}$. Since the functions ϕ_n are additive, they converge also for every $I \in \mathfrak{R}$. The limit function ϕ is non-negative, finite and additive. If $I \in \mathfrak{R}$, $I \supset F$, there exists n such that $I \supset F_n$ and so $\phi(I) = \phi_n(I) = 1$. For every cube $I \subset W$ there exists n such that

$$|\phi_n(I) - \phi(I)| < \delta(I)^{\rho}, \qquad \frac{\phi(I)}{\delta(I)^{\rho}} < \frac{\phi_n(I) + \delta(I)^{\rho}}{\delta(I)^{\rho}} < k_n + 1 < k.$$

So ϕ, F, ρ satisfy the conditions of Lemma 2, and we have $\rho - m^*E^T > 0$.

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