

THE OSCILLATION OF AN OPERATOR ON L^p

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ABSTRACT. We introduce and discuss the oscillation of an operator T mapping $L^p(S, \Sigma, \mu)$ into a Banach space. We establish results relating the oscillation, a "local norm", to the norm of the operator. Also using the oscillation we define a generalization of the Fredholm operators T with index $\kappa(T) < \infty$ and a corresponding perturbation class which contains the compact operators.

1. **Introduction.** The oscillation of a bounded linear operator mapping $C[0, 1]$ into a Banach space was introduced in [3] and was used there to discuss two classes of operators generalizing the weakly compact operators. These generalizations have been useful in approximation theory [2, §3].

The concept of the oscillation of an operator was explored further in [10] for bounded linear operators mapping $C(S)$, S a compact Hausdorff space, into a Banach space. Several interesting properties of such operators were defined in terms of the oscillation of the operator: a generalized notion of a Fredholm operator T with index $\kappa(T) < \infty$, an associated perturbation class and a basic type of conservative summability method.

Here we will discuss the oscillation of an operator T mapping $L^p(S, \Sigma, \mu)$ into a Banach space X . We will derive two distinct types of results. The first type of result, in §2, deals with the relationship between the uniform norm of an operator and the oscillation, a "local norm". The second type of result, in §3, deals with perturbation theorems which generalize the classical Fredholm result.

In what follows (S, Σ, μ) will be a measure space. To avoid complicating pathology S will be a Hausdorff space, Σ will contain the Borel sets, μ will be a regular measure on Σ , and, further, each point will have finite measure and each open set will have nonzero measure. The last two con-

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ditions insure that each point has a neighborhood U with nonzero functions in L^p which vanish off U . Anyone who is interested in densely defined operators should note that some restriction on the domain $D(T)$ of the operator T is necessary; it will suffice, for example, to have $f\chi_E$ in $D(T)$ for f in $D(T)$ and each set E in Σ . Also X will denote a Banach space.

1.1. **Definition.** Let $T: L^p(S, \Sigma, \mu) \rightarrow X$ be an operator. The oscillation of T at a point s in S , $\omega(T, s)$, is the supremum over all nonnegative α satisfying the following: for every neighborhood U of s there is a nonzero L^p function f (in the domain of T) vanishing off U such that $\|Tf\| \geq \alpha \|f\|$.

This definition is the obvious parallel of the oscillation as defined for operators on $C(S)$, and can be similarly defined for any operator whose domain consists of functions defined on a topological space.

One thing should be kept in mind. Any linear isometry L of $C(S)$ onto itself is, to within a function of modulus one, induced by a homeomorphism Φ on S . It is easy to show that $\omega(L^{-1}TL, \Phi(s)) = \omega(T, s)$. Consequently the oscillation is invariant under different representations of $C(S)$ as $C(S')$. This is not the case for the oscillation of an operator on $L^p(S, \Sigma, \mu)$. The operator P , of Example 3.5, is a projection on $L^2[0, 1]$ with infinite dimensional range whose oscillation is identically zero. It is easy to see how to represent $L^2[0, 1]$ as l^2 and have the oscillation of P equal to 1 at infinitely many points.

The oscillation function approximates the "local" uniform norm behavior of T . To make this precise let $\beta > \omega(T, s)$ be given. There is a neighborhood U of s such that $\|Tf\| \leq \beta \|f\|$ for all f in the domain of T vanishing off U . (It is clear that $\omega(T, t) \leq \beta$ for all t in U and it follows that $\omega(T, \cdot)$ is an upper semicontinuous function.) If $\omega(T, s) > \alpha$, then for each neighborhood U of s there is a function g in the domain of T vanishing off U with $\|Tg\| > \alpha \|g\|$. Combining these two facts shows that

$$\omega(T, s) = \inf \{ \|T|_{L^p(U)}\| : U \text{ is a neighborhood of } s \}$$

where $L^p(U)$ denotes the subspace of functions in $L^p(S, \Sigma, \mu)$ that vanish off U . Since $\|T|_{L^p(U)}\|$ is a decreasing function in U , when the neighborhoods of s are partially ordered by set inclusion, the infimum is really a generalized limit.

The oscillation function $\omega(\cdot, s)$ is, at each s , a pseudonorm and it is dominated by the uniform norm:

$$\omega(\alpha T, s) = |\alpha| \omega(T, s),$$

$$\omega(R + T, s) \leq \omega(R, s) + \omega(T, s) \quad \text{and} \quad \omega(T, s) \leq \|T\|.$$

For a bounded operator A with domain X we have also

$$\omega(AT, s) \leq \|A\| \omega(T, s).$$

2. The oscillation and the operator norm. We will consider the relationship between the oscillation, i.e., local norm, of T and the uniform norm of T . A surprising improvement of the inequality $\sup\{\omega(T, s) : s \text{ in } S\} \leq \|T\|$ is possible for operators on $L^1(S, \Sigma, \mu)$. We examine this case first, and begin with an example.

2.1. Example. Let $k(\cdot, \cdot)$ be a continuous function on $[0, 1] \times [0, 1]$ and let K be the kernel operator on $L^1[0, 1]$ given by $Kf(s) = \int_0^1 k(s, t) f(t) dt$. We will show that $\omega(K, t) = \int_0^1 |k(s, t)| ds$.

Let $\epsilon > 0$ be given. Since k is uniformly continuous there is a $\delta > 0$ for which $|k(s, t) - k(s, t_0)| < \epsilon$, for all s , whenever $|t - t_0| < \delta$. For f in L^1 vanishing off the interval $(t_0 - \delta, t_0 + \delta)$ we have

$$\begin{aligned} \|Kf\|_1 &= \int_0^1 |Kf(s)| ds \leq \int_0^1 \int_{t_0 - \delta}^{t_0 + \delta} |k(s, t)| |f(t)| dt ds \\ &\leq \int_0^1 \int_{t_0 - \delta}^{t_0 + \delta} (|k(s, t_0)| + \epsilon) |f(t)| dt ds = \left[\int_0^1 |k(s, t_0)| ds + \epsilon \right] \|f\|_1. \end{aligned}$$

Then $\omega(K, t_0) \leq \int_0^1 |k(s, t_0)| ds + \epsilon$.

If g is a nonnegative L^1 function of norm one vanishing off the interval $(t_0 - \delta, t_0 + \delta)$ then

$$\begin{aligned} |Kg(s) - k(s, t_0)| &= \left| \int_0^1 k(s, t) g(t) dt - k(s, t_0) \int_0^1 g(t) dt \right| \\ &\leq \int_{t_0 - \delta}^{t_0 + \delta} |k(s, t) - k(s, t_0)| g(t) dt < \epsilon. \end{aligned}$$

Hence $|\|Kg\|_1 - \int_0^1 |k(s, t_0)| ds| \leq \|Kg - k(\cdot, t_0)\|_1 < \epsilon$ and

$$\int_0^1 |k(s, t_0)| ds - \epsilon \leq \|Kg\|_1.$$

Therefore $\omega(K, t_0) = \int_0^1 |k(s, t_0)| ds$.

It is a classical result that the norm of the operator K in Example 2.1 is given by

$$(*) \quad \|K\| = \sup \left\{ \int_0^1 |k(s, t)| ds : 0 \leq t \leq 1 \right\}.$$

For the class of operators K on L^1 given by continuous kernel functions we have shown then that $\|K\| = \sup\{\omega(K, t): 0 \leq t \leq 1\}$; in Theorem 2.2 below we establish this equality for all linear operators (bounded or unbounded) on $L^1(S, \Sigma, \mu)$. The concept of the oscillation of an operator thus enables us to interpret (*) for an arbitrary operator and establish the generalization. This promises to be an interesting technique for generalizing classical results for kernel operators which are stated in terms of intrinsic properties of the kernel.

2.2. Theorem. *Let T be a linear operator mapping $L^1(S, \Sigma, \mu)$ into X . Then $\|T\| = \sup\{\omega(T, s): s \text{ in } S\}$.*

Proof. If $\sup\{\omega(T, s): s \text{ in } S\} = \infty$ it is clear that $\|T\| = \infty$, i.e., T is unbounded. Suppose that $\sup\{\omega(T, s): s \text{ in } S\} = M$ is finite. Let $\epsilon > 0$ and f in the domain of T be given. The set $A_f = \{x: |f(x)| \neq 0\}$ is σ -finite. In a regular measure space a set of finite measure is (almost everywhere) the union of a countable number of compact sets. Therefore $A_f = L \cup N$ where L is Lindelöf (in fact σ -compact) and N is a set of measure zero. For each point s in L there is a neighborhood V_s for which $\|Tg\| \leq (M + \epsilon)\|g\|$ for each g in the domain of T and vanishing off V_s . The collection $\{V_s: s \text{ in } L\}$ is an open cover of L from which we can extract a countable subcover $\{V_{s_i}: i = 1, 2, \dots\}$. Let $W_1 = V_{s_1}$ and $W_k = V_{s_k} - \bigcup_{i=1}^{k-1} V_{s_i}$; $\{W_i\}$ is a cover of L composed of pairwise disjoint measurable sets. Then

$$\begin{aligned} \|Tf\| &= \|T(f\chi_L)\| = \left\| T\left(\sum_{i=1}^{\infty} f\chi_{W_i}\right) \right\| \leq \sum_{i=1}^{\infty} \|T(f\chi_{W_i})\| \\ &\leq \sum_{i=1}^{\infty} (M + \epsilon) \|f\chi_{W_i}\|_1 = (M + \epsilon) \sum_{i=1}^{\infty} \int_{W_i} |f| d\mu = (M + \epsilon) \|f\|_1. \end{aligned}$$

It follows that $\|Tf\| \leq M\|f\|$. Therefore $\|T\| \leq M$ and we know $\|T\| \geq M$. We conclude that $\|T\| = M$.

2.3. Corollary. *Let T be a linear operator defined on $L^1(S, \Sigma, \mu)$. If an L^1 function in the domain of T vanishes off the set $D = \{s: \omega(T, s) = 0\}$ of diffusion points of T then $Tf = 0$.*

Proof. Let $\epsilon > 0$ be given and let L be as in the proof of Theorem 2.2. For each point s in L there is a neighborhood V_s with $\|Tg\| < \epsilon\|g\|_1$ whenever g vanishes off V_s . As in the proof of the previous theorem, $\|Tf\| \leq \epsilon\|f\|$.

The results of Theorem 2.2 fail spectacularly for operators defined on L^p spaces, for $p > 1$. We shall see many examples of this later, but a simple example now may be edifying.

2.4. Example. Let $I: L^p[0, 1] \rightarrow L^1[0, 1]$, $p > 1$, be the injection map defined by $If = f$. For a function f in L^p vanishing off V , Hölder's inequality yields $\|If\| = \|f\|_1 \leq \mu(V)^{1/q} \|f\|_p$. This inequality shows that $\omega(I, s) = 0$ for all s . But note that $\|I\| = 1$. This example shows that the oscillation of an operator on L^p , $p > 1$, can be "much smaller" than the uniform norm of the operator, while Theorem 2.2 shows that the oscillation of an operator on L^1 must have supremum equal to the uniform norm of the operator.

Since $\omega(T, s)$ is the "local norm of T at s " we will make the following definition.

2.5. Definition. T is locally bounded if $\omega(T, s)$ is finite for each s in S .

2.6. Example. Let the closed operator M be densely defined on $L^p(\mathbb{R})$, $p \geq 1$ by $Mf(x) = xf(x)$. Then $\omega(M, x) = |x|$. M is locally bounded but not bounded.

In light of Theorem 2.2 we see that an operator on $L^1(S, \Sigma, \mu)$ which is uniformly locally bounded is also bounded. This result also holds true for operators on $L^p(S, \Sigma, \mu)$, $p > 1$. However we must place a topological condition on S .

2.7. Theorem. Let $T: L^p(S, \Sigma, \mu) \rightarrow X$, $p \geq 1$, be a linear operator. Suppose that S is compact. If T is locally bounded, then it is bounded.

Proof. Since $\omega(T, \cdot)$ is upper semicontinuous and S is compact, $\sup\{\omega(T, s): s \in S\} = M$ is finite. For each point s in S there is a neighborhood V_s with $\|Tf\| \leq (M + 1)\|f\|$ whenever f vanishes off V . From the open cover $\{V_s: s \text{ in } S\}$ we extract a finite subcover V_{s_1}, \dots, V_{s_n} . Let $W_1 = V_{s_1}$ and $W_k = V_{s_k} - \bigcup_{i=1}^{k-1} V_{s_i}$. For f in L^p ,

$$\begin{aligned} \|Tf\| &= \left\| T\left(\sum_{i=1}^n f\chi_{W_i}\right) \right\| \leq \sum_{i=1}^n \|T(f\chi_{W_i})\| \\ &\leq (M + 1) \sum_{i=1}^n \|f\chi_{W_i}\|_p \leq (M + 1)n\|f\|_p. \end{aligned}$$

We conclude that T is bounded.

The oscillation of T in Theorem 2.7 was actually uniformly locally bounded because S was compact. The next example, where S is not com-

pact, shows that a uniformly bounded oscillation function is not enough for an operator on L^p to be bounded.

2.8. Example. Let V be the densely defined operator $Vf(x) = \int_0^x f(t) dt$ mapping $L^p(\mathbb{R})$, $p > 1$, into the Banach space of bounded continuous functions on \mathbb{R} . The domain of V is $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Hölder's inequality shows that $\|Vf\|_\infty \leq \mu(A_f)^{1/q} \|f\|_p$, where $A_f = \{x \in \mathbb{R} \mid f(x) \neq 0\}$. Therefore the oscillation of V is uniformly bounded, in fact it is zero everywhere. However the operator V is not bounded.

3. The oscillation and perturbation. In this section we establish results concerning the perturbation of an operator with "large" oscillation by one with "small" oscillation. With the appropriate definitions of large and small oscillation it is easy to show that the sum has a large oscillation. The major interest in this result lies in the deeper fact that it constitutes a proper generalization of the classical result on perturbing a semi-Fredholm operator by a compact operator.

We carry over from $C(S)$ the following definition.

3.1. Definition. Let $T: L^p(S, \Sigma, \mu) \rightarrow X$ be a linear operator. T is diffuse [3] if $\omega(T, s) = 0$ for all s in S , and T is a c_0 operator [10] if $\omega(T, s) = 0$ except, possibly, for a countable number of points $\{s_n\}$ in S for which we have $\omega(T, s_n)$ converging to zero.

The subadditivity of the oscillation function implies that the sets of diffuse and c_0 operators are subspaces. The inequality $|\omega(T, s) - \omega(L, s)| \leq \omega(T - L, s) \leq \|T - L\|$ shows that these subspaces are closed in the uniform operator topology. If A is a bounded linear operator with domain L^p and T is a diffuse operator (or a c_0 operator) mapping L^p into itself then AT is a diffuse operator (or a c_0 operator) since $\omega(AT, s) \leq \|A\| \omega(T, s)$. In particular the diffuse and c_0 operators are closed left ideals in the algebra of all bounded linear operators on $L^p(S, \Sigma, \mu)$. The example below demonstrates that they are not generally right ideals.

3.2. Example. A bounded linear operator P on $L^2[0, 1]$ which is diffuse, i.e., $\omega(T, s) = 0$ for all s in $[0, 1]$, and a bounded linear operator B on $L^2[0, 1]$ with $\omega(PB, s) = 1$ for all s are given.

Let P be the operator of Example 3.5 below. It is shown there that the oscillation of P is identically zero. The operator P has the form $Pf = \sum_{n=1}^{\infty} (f|f_n) f_n$ where the functions f_n form an (incomplete) orthonormal sequence. Suppose that $\{g_n\}$ is a complete orthonormal sequence for $L^2[0, 1]$; we define B on $L^2[0, 1]$ by $Bf = \sum_{n=1}^{\infty} (f|g_n) f_n$. The composite operator PB is an isometry on $L^2[0, 1]$ so that $\omega(PB, s) = 1$ for all s in $[0, 1]$.

3.3. Theorem. Let $T: L^p(S, \Sigma, \mu) \rightarrow X$ be a compact linear operator, $p > 1$, then T is a c_0 operator. Further, if $\mu(\{s\}) = 0$ then $\omega(T, s) = 0$.

Proof. Suppose that $\mu(\{s\}) = 0$. Since μ is a regular measure there exist open sets V_n containing s with $\mu(V_n) < 1/n$. Assume that $\omega(T, s) = \alpha$. Then there are functions f_n of p -norm one vanishing off V_n with $\|Tf_n\| \geq \alpha/2$. For each function g in L^q we can apply Hölder's inequality to obtain

$$\left| \int f_n g \, d\mu \right| \leq \|f_n\|_p \left\{ \int_{V_n} |g|^q \, d\mu \right\}^{1/q}.$$

The set function ν , defined by $\nu(E) = \int_E |g|^q \, d\mu$, is absolutely continuous with respect to μ . Therefore $\int f_n g \, d\mu$ converges to zero, i.e., $\{f_n\}$ converges weakly to zero. Since T is compact Tf_n converges to zero in norm. We conclude that $\omega(T, s) = 0$.

Assume that T is not a c_0 operator. Then there are a countably infinite number of distinct points $\{s_n\}$ and a positive constant β with $\omega(T, s_n) \geq \beta > 0$. The argument above demonstrates that each s_n must satisfy $\mu(\{s_n\}) > 0$. We first suppose that the sequence $\{s_n\}$ has a cluster point s . Let V be a neighborhood of s of finite measure. There are infinitely many s_n , say t_1, t_2, t_3, \dots , in V . We have $\sum \mu(\{t_n\}) \leq \mu(V) < \infty$ so that $\mu(\{t_n\})$ converges to zero. Let V_n be a neighborhood of t_n with $\mu(V_n) < 1/n + \mu(\{t_n\})$. There are functions f_n of p -norm one vanishing off V_n with $\|Tf_n\| \geq \beta/2 > 0$. The argument of the first paragraph of this proof shows that $\{f_n\}$ converges weakly to zero and, since T is compact, $\|Tf_n\|$ converges to zero, which is a contradiction. Second, suppose that the sequence $\{s_n\}$ has no cluster point. In this case we can obtain a pairwise disjoint sequence $\{V_n\}$ of neighborhoods of the points in the sequence $\{s_n\}$. Let f_n be as above and g be in L^q . Since μ is a regular measure, given $\epsilon > 0$ there is a compact set K off which the q -norm of g is less than ϵ . Applying Hölder's inequality we see that

$$\left| \int g f_n \, d\mu \right| \leq \left\{ \int |g|^q \chi_{K \cap V_n} \, d\mu \right\}^{1/q} + \epsilon$$

Since $\sum \mu(K \cap V_n) \leq \mu(K) < \infty$, the right-hand side of the inequality converges to ϵ as n tends to infinity. Again $\{f_n\}$ converges weakly to zero, and as before this will produce the contradiction that $\{Tf_n\}$ converges to zero. We conclude that T is a c_0 operator.

The next two examples show that the c_0 operators and the diffuse operators are distinct from the compact operators.

3.4. Example. Let T be defined on $L^p[0, 1]$ for $p > 1$ by the equation

$$Tf(x) = \frac{1}{x} \int_0^x f(t) dt.$$

It is a well-known fact, due to Hardy [5], that T is a bounded linear operator on $L^p[0, 1]$ with $\|T\| = p/(p-1)$. We will show that

$$\omega(T, s) = \begin{cases} p/(p-1) & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases}$$

The operator T is a c_0 operator which is not compact.

For $a > 0$, T is easily seen to be a compact operator on $L^p[a, 1]$ since $Vf(x) = \int_a^x f(t) dt$ is compact, $Mg(x) = (1/x)g(x)$ is bounded, and $T = MV$. Therefore Theorem 3.3 shows that $\omega(T, s) = 0$ if $a < s \leq 1$. We now consider the function $f(t) = t^{\epsilon-1/p} \chi_{[0, a]}$ for arbitrary positive ϵ , which has L^p norm $[a^{p\epsilon/p\epsilon}]^{1/p}$. We compute

$$Tf(x) = \begin{cases} \frac{x^{\epsilon-1/p}}{1-1/p+\epsilon} & \text{if } 0 \leq x \leq a, \\ \frac{a^{1-1/p+\epsilon}}{1-1/p+\epsilon} \cdot \frac{1}{x} & \text{if } a \leq x \leq 1. \end{cases}$$

Then

$$\|Tf\|_p \geq \frac{1}{1-1/p+\epsilon} \left\{ \int_0^a (x^{\epsilon p-1}) dx \right\}^{1/p} = \frac{p}{p-1+p\epsilon} \|f\|_p.$$

Since ϵ is arbitrary we find that $\omega(T, 0) \geq p/(p-1)$, and of course $\omega(T, 0) \leq \|T\| = p/(p-1)$. It follows that $\omega(T, 0) = p/(p-1)$.

The operator T in the previous example, if regarded as an operator on $L^p(0, 1]$, is noncompact and has oscillation identically zero. We now give an example of a bounded linear operator P on $L^p(S)$, S compact, which has oscillation identically zero but which is not a compact linear operator.

3.5. Example. A norm one projection P of $L^p[0, 1]$, $p > 1$, onto a subspace isomorphic to l^p with $\omega(P, s) = 0$ for $0 \leq s \leq 1$.

We first construct a sequence $\{E_i\}$ of subsets of $[0, 1]$;

$$E_1 = (1/3, 2/3), \quad E_2 = (1/9, 2/9) \cup (7/9, 8/9),$$

$$E_3 = (1/27, 2/27) \cup (7/27, 8/27) \cup (19/27, 20/27) \cup (25/27, 26/27), \dots$$

The set E_i is the one removed at the i th step of the construction of the Cantor set, and it is a union of 2^{i-1} subintervals of the form $(r/3^i, (r+1)/3^i)$. The collection $\{E_i\}$ is a sequence of pairwise disjoint open sets with $\mu(E_i) = (1/2)(2/3)^i$.

Let V be an interval of form $[0, 1/3^k), (i/3^k, (i - 1)/3^k)$ or $(1 - (1/3^k), 1]$. For $i > k$ suppose V intersects E_i . Then $2^k - 1$ disjoint translates of V intersect E_i in a set of the same measure as $V \cap E_i$ (if $V \cap E_i \neq \emptyset$ then V must be one of the 2^i subintervals of length $1/3^k$ abutting a subinterval of E_k). Therefore $\mu(V \cap E_i) \leq (1/2^k)\mu(E_i)$.

Now define

$$f_n = \frac{1}{\mu(E_n)^{1/p}} \chi_{E_n}, \quad g_n = \frac{1}{\mu(E_n)^{1/q}} \chi_{E_n}$$

and define $x_n^*(f) = \int f g_n d\mu$ on L^p . We have $\|f_n\|_p = \|g_n\|_q = \|x_n^*\| = 1$.

The operator $P: L^p \rightarrow L^p$ is given by $Pf = \sum x_n^*(f) f_n$. To compute the bound on P we have

$$\begin{aligned} \|Pf\|_p^p &= \sum \left| \int_{E_n} f d\mu \right|^p \mu(E_n)^{1-p} \\ &\leq \sum \left(\int_{E_n} |f|^p d\mu \right) \mu(E_n)^{p/q} \mu(E_n)^{1-p} = \sum \int_{E_n} |f|^p d\mu \leq \|f\|_p^p. \end{aligned}$$

This shows simultaneously that the partial sums defining P converge in p -norm and that P is bounded with norm one. Note that P is a projection ($P^2 = P$) with an infinite dimensional range $\overline{\text{span}}(\{f_n\})$ which is linearly isometric to l^p ; it follows that P is definitely not a compact operator.

Now let a point s in $[0, 1]$ and $\epsilon > 0$ be given; and let V be a neighborhood of s . If s is not a "triadic rational", i.e. a number of the form $i/3^k$, then s has a neighborhood of U of the form $[0, 1/3^k), (i/3^k, (i + 1)/3^k)$ or $(1 - 1/3^k, 1]$ which is contained in V for some large k . For this U we have seen that $\mu(U \cap E_i) \leq (1/2^k)\mu(E_i)$ which is less than ϵ for sufficiently large k . If s has the form $i/3^k$ we take the neighborhood U to be $((i - 1)/3^k, (i + 1)/3^k)$ and we have $\mu(U \cap E_i) \leq 2(1/2^k)\mu(E_i)$. In either case for sufficiently large k_0 we have formed a neighborhood U of s with $\mu(U \cap E_i) \leq \epsilon^{q/p}\mu(E_i)$ for $i \geq k_0$.

If f is any function vanishing off U , then, for $i \geq k_0$,

$$\begin{aligned} \|x_i^*(f)/f_i\|_p^p &= |x_i^*(f)|^p = \mu(E_i)^{1-p} \left| \int_{E_i} f d\mu \right|^p \\ &\leq \mu(E_i)^{1-p} \left(\int_{E_i} |f|^p d\mu \right) \mu(E_i \cap U)^{p-1} \leq \epsilon \int_{E_i} |f|^p d\mu. \end{aligned}$$

Also, there is a neighborhood W of s with the property that, for g vanishing off W ,

$$|x_i^*(g)|^p = |x_i^*(g \chi_{E_i})|^p \leq \epsilon \int_{E_i} |g|^p d\mu,$$

$i = 1, 2, \dots, k_0$ because the map $g \rightarrow x_i^*(g)$ is compact. Hence for h vanishing off $U \cap W$ we have

$$\begin{aligned} \|Ph\|_p^p &= \int \left| \sum_{i=1}^{\infty} x_i^*(h) f_i \right|^p d\mu = \sum_{i=1}^{\infty} |x_i^*(h)|^p \int_{E_i} |f_i|^p d\mu \\ &= \sum_{i=1}^{k_0} |x_i^*(h)|^p \int_{E_i} |f_i|^p d\mu + \sum_{i=k_0+1}^{\infty} |x_i^*(h)|^p \int_{E_i} |f_i|^p d\mu \\ &\leq \epsilon \sum_{i=1}^{k_0} \int_{E_i} |h|^p d\mu + \epsilon \sum_{i=1}^{\infty} \int_{E_i} |h|^p d\mu = \epsilon \sum_{i=1}^{\infty} \int_{E_i} |h|^p d\mu = \epsilon \|h\|_p^p. \end{aligned}$$

It follows that $\omega(P, s) = 0$ for all s .

3.6. **Definition.** A linear operator $T: L^p(S, \Sigma, \mu) \rightarrow X$ is concentrated [10] if $\omega(T, s)$ is bounded from zero except, possibly, for a finite number of points where it is zero.

In [10], the concentrated operators were introduced as a perturbation class dual to the class of c_0 operators. We now show that on L^p these operators contain the semi-Fredholm operators T with index $\kappa(T) < \infty$, alias the Φ_+ operators [8], i.e., those operators with closed range and finite dimensional null manifold.

3.7. **Theorem.** Let $T: L^p(S, \Sigma, \mu) \rightarrow X, p \geq 1$, be a linear operator with closed range and finite dimensional null manifold. Then T is a concentrated operator.

Proof. We first suppose that s is a point of S having no atomic neighborhoods. Under this assumption, for any neighborhood V of s , the space $L^p(V)$ is infinite dimensional. Hence there is a nonzero function f_0 in L^p vanishing off V for which $\|f_0\| = \text{dist}(f_0, N(T))$, the distance from f_0 to the null manifold of T [4, p. 110], [6]. Since T has closed range, its minimum modulus [4, p. 96] $\gamma(T)$ is positive. It follows that $\|Tf_0\| \geq \gamma(T)\|f_0\|$, and since V was arbitrary, $\omega(T, s) \geq \gamma(T) > 0$.

Second, we consider the set A of all points in S with atomic neighborhoods. Each point in A is an open set. To see this suppose V is an atomic neighborhood of s with finite measure. Either $\mu(\{s\}) = \mu(V)$ or $\mu(\{s\}) = 0$. If $\mu(\{s\}) = 0$ the regularity of the measure implies that there is a nonempty open subset W of V with $\mu(W) < \frac{1}{2}\mu(V)$. But V is atomic so $\mu(W) = 0$. This contradicts the basic assumption that open sets have positive measure. If $\mu(\{s\}) = \mu(V)$ then $\mu(V/\{s\}) = 0$ and $V/\{s\}$ is the null set since it is open

and has zero measure. Let $f_s = \chi_{\{s\}}/\mu(\{s\})^{1/p}$, then $\omega(T, s) = \|Tf_s\|$. The functions f_s , s in A , are linearly independent so at most finitely many may have $Tf_s = 0$ since they would then lie inside the finite dimensional null manifold of T . Suppose T is not concentrated. Then there exists a sequence of functions $f_{s_n} = f_n$, s_n in A , with $\|Tf_n\| \neq 0$ but $\|Tf_n\|$ converging to zero. Since $\|Tf_n\| \geq \gamma(T) \text{dist}(f_n, N(T))$ we must have $\text{dist}(f_n, N(T))$ converging to zero. Therefore there is a sequence $\{g_n\}$ in $N(T)$ with $\|f_n - g_n\|_p$ converging to zero. The g_n are uniformly bounded and $N(T)$ is finite dimensional so some subsequence $\{g_{n_i}\}$ converges to a function g in $N(T)$. The corresponding subsequence $\{f_{n_i}\}$ must also converge to g and is, therefore, a Cauchy sequence. But $\|f_s - f_t\| = 2^{1/p}$ for $s \neq t$ so we have a contradiction. It follows that except for a finite number of points $\omega(T, s)$ must be bounded away from zero.

If T is a Φ_+ operator, i.e., a semi-Fredholm operator with index $\kappa(T) < \infty$, and K is compact then $T + K$ is a Φ_+ operator [4, Chapter V]. By Theorem 3.3 the compact operators are a subclass of the c_0 operators and by Theorem 3.7 the Φ_+ operators are a subclass of the concentrated operators. Consequently the next theorem, which is easy to establish, generalizes the classical perturbation result.

3.8. Theorem. *If $T: L^p(S, \Sigma, \mu) \rightarrow X$, $p \geq 1$, is a concentrated linear operator, and $K: L^p(S, \Sigma, \mu) \rightarrow X$ is a c_0 operator then $T + K$ is a concentrated operator.*

Proof. Since T is concentrated there is a positive constant $\delta > 0$ with the property that $\omega(T, s) \geq \delta$ for all but a finite number of points. K is a c_0 operator so $\omega(K, s) \leq \delta/2$ for all but a finite number of points. Then, except for a finite number of points, $\omega(T + K, s) \geq |\omega(T, s) - \omega(K, s)| \geq \delta/2$. It follows that $T + K$ is concentrated.

The next two examples show that perturbing even the identity operator by a c_0 operator does not yield a Φ_+ operator. This demonstrates that concentrated operators need not be Φ_+ operators.

3.9. Example. Let P be the projection on $L^p[0, 1]$ of Example 3.5. P is a c_0 operator. The null manifold of $I - P$ is the infinite dimensional range space of P , $\overline{\text{span}}(\{f_n\})$. Theorem 3.8 shows that $I - P$ is concentrated but its null manifold is infinite dimensional so $I - P$ is not a Φ_+ operator.

3.10. Example. Let T be the c_0 operator of Example 3.4. By setting $(\lambda I - T)f = 0$, and differentiating, we find that any eigenfunction must be a multiple of x^α where $x = \lambda^{-1} - 1$. Applying the condition that x^α be in L^p we find $\text{Re}(\lambda^{-1}) > q^{-1}$, [7], [1], [8]. Thus q belongs to the spectrum

of T and $qI - T$ is one-to-one. Since $(qI - T)x^n = (q - 1/(n + 1))x^n$ the range of $qI - T$ contains the polynomials and therefore is dense. The range of $qI - T$ cannot be closed, otherwise the closed graph theorem would imply that the inverse of $qI - T$ would be bounded, contradicting the fact that q belongs to the spectrum of T . Again, $qI - T$ is concentrated but its range is not closed so $qI - T$ is not a Φ_+ operator.

The generalization of the classical perturbation results which we have established for operators defined on L^p , $p > 1$, does not hold for operators on L^1 . The exact nature of this failure is interesting. The basic Theorem 3.8 holds. It follows easily from the definitions and would hold in any setting in which the oscillation was defined. Also Theorem 3.7 holds; a Φ_+ operator on L^1 is concentrated. However Theorem 3.3 fails totally in L^1 as the next example shows.

3.11. Example. Let $T: L^1[0, 1] \rightarrow X$ be a c_0 operator. Then $T = 0$. (The only property of $[0, 1]$ needed is that points have measure zero.) Given f in L^1 there is a g , equal to f almost everywhere with g vanishing off $\{s: \omega(T, s) = 0\}$. Hence by Corollary 2.3, $Tf = Tg = 0$.

We see that the perturbation theorem is uninteresting in $L^1[0, 1]$ because there are no nonzero c_0 operators with which to perturb. That this is true follows immediately from the validity of the equation $\|T\| = \sup\{\omega(T, s): s \text{ in } S\}$ of Theorem 2.2.

One still might hope that the compact operators on L^1 had an oscillation which distinguished them in some way from other operators. Then, perhaps, one could establish an appropriate generalization of the perturbation result. The next example shows that this is not possible. In fact the simplest type of compact operator, one with one dimensional range, turns out to be indistinguishable by its oscillation from a very noncompact multiplication operator.

3.12. Example. Let g be an $L^\infty(S, \Sigma, \mu)$ function of norm one, and let f_0 be of norm one in $L^1(S, \Sigma, \mu)$. Let F be the one dimensional range (compact) operator defined on L^1 by $Tf = (\int f \cdot g \, d\mu)f_0$. Also define M on L^1 by $Mf = f \cdot g$. Let V be a neighborhood of a point s . On $L^1(V, \Sigma \cap V, \mu|_V)$ the linear operators $T|_{L^1(V)}$ and $M|_{L^1(V)}$ have norms both equal to $\|g\chi_U\|_\infty$. Then $\omega(T, s) = \omega(M, s) = \inf\{\|g\chi_U\|_\infty: U \text{ is a neighborhood of } S\}$. The operator M is only compact and nonzero when g is zero except for a countable number of points $\{s_n\}$, each point having positive measure, with $g(s_n)$ converging to zero. Both operators are Φ_+ operators if g is essentially bounded below on S .

For $p = 1$ and $g \in L^\infty$,

$$\|g\chi_U\|_\infty = \text{ess sup}_{x \in U} |g(x)| = \inf\{\sup_{x \in U} |h(x)|: h = g \text{ a.e.}\}$$

We then have

$$\omega(M, s) = \inf_{U \ni s} \inf_{h=g \text{ a.e.}} \sup_{x \in U} |b(x)| = \inf_{h=g \text{ a.e.}} \inf_{U \ni s} \sup_{x \in U} |b(x)|.$$

However $\inf_{U \ni s} \sup_{x \in U} |b(x)|$ is the value of the upper semicontinuous envelope of $|b(x)|$ at s . Therefore $\omega(M, s)$ is the infimum of the upper semicontinuous envelopes of the absolute values of functions in the same equivalence class as g .

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