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EXISTENCE AND UNIQUENESS THEOREMS FOR RIEMANN PROBLEMS

BY

TAI-PING LIU

ABSTRACT. In [2] the author proposed the entropy condition (E) and solved the Riemann problem for general 2×2 conservation laws $u_t + f(u, v)_x = 0$, $v_t + g(u, v)_x = 0$, under the assumptions that the system is hyperbolic, and $f_u \geqslant 0$ and $g_v \leqslant 0$. The purpose of this paper is to extend the above results to a much wider class of 2×2 conservation laws. Instead of assuming that $f_u \geqslant 0$ and $g_v \leqslant 0$, we assume that the characteristic speed is not equal to the shock speed of different family. This assumption is motivated by the works of Lax [1] and Smoller [4].

We consider the 2×2 conservation laws

$$(1) \qquad \begin{aligned} &u_t + f(u, v)_x = 0, \\ &v_t + g(u, v)_x = 0, \quad -\infty < x < \infty, \quad t \geqslant 0, \end{aligned}$$

where $(u, v) = (u, v)(x, t)$ and $f, g \in C^2(U)$ for some open set U in $\mathbf{R}^2$. We are interested in the Riemann problem for (1), that is, the Cauchy problem (1) with initial data

$$(2) \qquad (u(x, 0), v(x, 0)) \equiv (u_0(x), v_0(x)) = \begin{cases} (u_l, v_l) & \text{for } x < 0, \\ (u_r, v_r) & \text{for } x > 0, \end{cases}$$

where (u_l, v_l) and (u_r, v_r) are arbitrary constants in U.

We assume that

$$(3) \qquad f_v < 0, \quad g_u < 0$$

so that (1) is strictly hyperbolic, that is, $d(f, g)$ has real and distinct eigenvalues $\lambda_1 < \lambda_2$. Let r_i be right eigenvectors corresponding to λ, $i = 1, 2$. These can be taken of the form $r_1 = (1, a_1)^t$, $r_2 = (1, a_2)^t$. It can be shown that (3) implies

$$(4) \qquad a_1 < 0 < a_1.$$

Since the solution to (1) is usually discontinuous, we seek the weak solutions to (1) and (2).

DEFINITION 1. The bounded measurable function (u, v) is said to be a weak solution to (1), (2) if

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$$
(5) \qquad \begin{aligned}
&\iint_{t\geq 0} [u\phi_t + f(u, v)\phi_x]\, dx\, dt + \int_{t=0} u_0 \phi\, dx = 0,\\
&\iint_{t\geq 0} [v\phi_t + g(u, v)\phi_x]\, dx\, dt + \int_{t=0} v_0 \phi\, dx = 0.
\end{aligned}
$$

for all functions $\phi \in C_0^\infty(R \times (0, \infty))$.

From (5) it follows that if a weak solution (u, v) is discontinuous along $x = x(t)$, then the following Hugoniot condition is satisfied.

$$
\text{(H)} \qquad \frac{f(u_+, v_+) - f(u_-, v_-)}{u_+ - u_-} = \frac{g(u_+, v_+) - g(u_-, v_-)}{v_+ - v_-} = s
$$

where $(u_+, v_+) = (u, v)(x + 0, t)$, $(u_-, v_-) = (u, v)(x - 0, t)$ and $s = \dot{x}(t)$.

Through any point (u_0, v_0) in U, we define the shock set to be the set

$$
S(u_0, v_0) = \left\{ (u, v) \in U \;\middle|\; \frac{f(u, v) - f(u_0, v_0)}{u - u_0} = \frac{g(u, v) - g(u_0, v_0)}{v - v_0} = \sigma(u, v; u_0, v_0) \right\}
$$

where $\sigma(u, v; u_0, v_0)$ is the shock speed. The Hugoniot condition (H) says that $(u_+, v_+) \in S(u_-, v_-)$ and $s = \sigma(u_+, v_+; u_-, v_-)$. Condition (3) implies that if $u = u_0$ or $v = v_0$, then (u, v) is not in $S(u_0, v_0) - \{(u_0, v_0)\}$. Hereafter, we assume that for any (u_0, v_0) in U,

(6) the shock set $S(u_0, v_0)$ consists of two curves $S_1(u_0, v_0)$ and $S_2(u_0, v_0)$ such that for any (u, v) on $S_1(u_0, v_0)$, $(u, v; u_0, v_0) < \lambda_2(u, v)$ and for any (u, v) on $S_2(u_0, v_0)$, $\sigma(u_0, v_0; u, v) > \lambda_1(u, v)$.

In [1], Lax proved that $S_i(u_0, v_0)$ is tangent to r_i at (u_0, v_0). Therefore we can write $S_i = S_i^+ \cup S_i^-, i = 1, 2$, such that $S_1^+(u_0, v_0) \subset \mathrm{I}(u_0, v_0)$, $S_1^-(u_0, v_0) \subset \mathrm{III}(u_0, v_0)$, $S_2^+(u_0, v_0) \subset \mathrm{IV}(u_0, v_0)$ and $S_2^-(u_0, v_0) \in \mathrm{II}(u_0, v_0)$ (cf. Figure 1), where $\mathrm{I}(u_0, v_0) = \{(u, v) \in U \mid u \geq u_0, v \geq v_0\}$, etc.

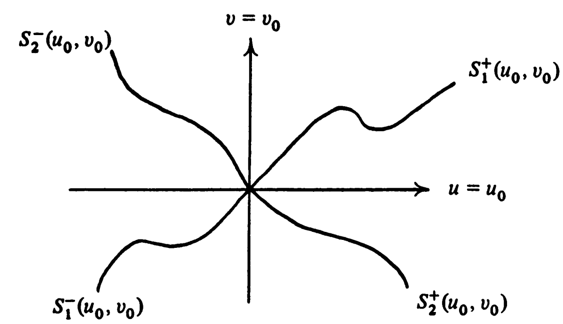

FIGURE 1

Let $h_i = h_i(u_0, v_0; u, v)$ be a nonzero smooth tangent to $S_i(u_0, v_0)$ at (u, v) such that $h_i(u_0, v_0; u_0, v_0) = r_i(u_0, v_0)$, $i = 1, 2$. Let $R_i(u_0, v_0)$ be the integral curve of r_i through (u_0, v_0), $i = 1, 2$. Denote by $d/d\mu_i$ and $d/d\nu_i$ the directional derivatives along S_i and R_i, respectively, i.e.

$$d/d\mu_i = h_i \cdot \nabla \quad \text{and} \quad d/d\nu_i = r_i \cdot \nabla.$$

It is known that certain elementary weak solutions called i-rarefaction waves and i-shock waves can be defined along R_i and S_i curves (see for example [1]).

LEMMA 1. *Assume that* (3) *and* (6) *hold. Set* $h_i(u_0, v_0; u, v) = \Sigma_{j=1}^2 a_{ij} r_j(u, v)$. *Then* $a_{11} > 0$ *and* $a_{22} > 0$.

THEOREM 1. *Assume that* (3) *and* (6) *hold. Then for* $(u, v) \in S_i^+(u_0, v_0)$, *we have*

(i) $d\sigma/d\mu_i > 0$ *if and only if* $\sigma < \lambda_i$ *at* (u, v),
(ii) $d\sigma/d\mu_i < 0$ *if and only if* $\sigma > \lambda_i$ *at* (u, v);

and for $(u, v) \in S_i^-(u_0, v_0)$, *we have*

(iii) $d\sigma/d\mu_i > 0$ *if and only if* $\sigma > \lambda_i$ *at* (u, v),
(iv) $d\sigma/d\mu_i < 0$ *if and only if* $\sigma < \lambda_i$ *at* (u, v).

LEMMA 2. *Assume that* (3) *and* (6) *hold. Let* (u_1, v_1) *and* (u_2, v_2) *be points on* $S_i(u_0, v_0)$, $i = 1$ *or* 2, *such that* $\sigma(u_1, v_1, u_0, v_0) = \sigma(u_2, v_2; u_0, v_0)$. *Then* $(u_1, v_1) \in S_i(u_2, v_2)$ *and* $\sigma(u_1, v_1; u_2, v_2) = \sigma(u_1, v_1; u_0, v_0)$.

Theorem 1 and Lemmas 1 and 2 were proved in [3]. Our next theorem is important in solving the Riemann problem.

THEOREM 2. *Suppose that* (3) *and* (6) *hold. Assume that* $(u_1, v_1) \in S_1(u_0, v_0)$ *and* $(u_2, v_2) \in S_2(u_0, v_0)$. *Then* $\sigma(u_1, v_1; u_0, v_0) < \sigma(u_2, v_2; u_0, v_0)$.

PROOF. We only prove the theorem when $u_1 > u_0$ and $u_2 > u_0$. The other cases are proved similarly.

Suppose, on the contrary, $\sigma(u_1, v_1; u_0, v_0) \geqslant \sigma(u_2, v_2; u_0, v_0)$. Since, by (7), $\sigma(u_2, v_2; u_0, v_0) > \lambda_1(u_0, v_0)$, there exists (u_3, v_3) on $S_1^+(u_0, v_0)$ between (u_0, v_0) and (u_1, v_1) such that $\sigma(u_0, v_0; u_2, v_2) = \sigma(u_0, v_0; u_3, v_3)$. It follows that $(u_2, v_2) \in S(u_3, v_3)$, $(u_3, v_3) \in S(u_2, v_2)$ and $\sigma(u_3, v_3; u_2, v_2) = \sigma(u_0, v_0; u_2, v_2) = \sigma(u_0, v_0; u_3, v_3)$.

If $u_0 < u_3 < u_2$, then $(u_3, v_3) \in S_1^-(u_3, v_3)$. Therefore by Lemma 2, $(u_3, v_3) \in S_2(u_0, v_0)$ which is a contradiction.

If $u_0 < u_2 < u_3$, then $(u_0, v_0) \in S_1^-(u_3, v_3)$ and $(u_2, v_2) \in S_2^-(u_2, v_2)$. Again by Lemma 2, $(u_2; v_2) \in S_1(u_0, v_0)$ which is a contradiction.

This completes the proof of the theorem. Q.E.D.

For simplicity, we make the following assumption:

(7) The intersection of any R_i curve with the set $V_i \equiv \{(u, v) | d\lambda_i(r_i) = 0\}$ does not have any accumulation point.

As was shown in [2], the crucial step in proving the existence and uniqueness theorems for the Riemann problem is to first establish the analogous theorems for i-waves, $i = 1, 2$. For this, we construct a curve $\alpha_i(u_0, v_0)$ such that (u_0, v_0) is connected to any (u, v) on $\alpha_i(u_0, v_0)$ on the right by i-waves. We now describe briefly the construction of the curves $\alpha_i(u_0, v_0)$.

Suppose $d\lambda_i(r_i) < 0$ at (u_0, v_0), then the first segment of $\alpha_i(u_0, v_0)$ is $S_i^+(u_0, v_0)$ and the solution of the Riemann problem $\{(u_0, v_0); (u, v)\}$, $(u, v) \in S_i^+(u_0, v_0)$ and $|u - u_0|$ small, is an i-shock. As (u, v) moves further away from (u_0, v_0) along $S_i^+(u_0, v_0)$, we may have $\sigma(u_1, v_1; u_0, v_0) = \lambda_i(u_1, v_1)$ at some $(u_1, v_1) \in S_i^+(u_0, v_0)$. The curve $\alpha_i(u_0, v_0)$ is then continued from (u_1, v_1) by the rarefaction curve $R_i(u_1, v_1)$, so that the solution is a shock wave connecting (u_0, v_0) to (u_1, v_1) followed by a rarefaction wave connecting (u_1, v_1) to (u, v) on $R_i(u_1, v_1)$. When $R_i(u_1, v_1)$ first leaves the region $\{(u, v) | d\lambda_i(r_i) > 0$ at $(u, v)\}$ at (u_2, v_2), we continue $\alpha_i(u_0, v_0)$ from (u_2, v_2) with a *mixed* curve γ^* corresponding to γ. Here γ is the R_i curve between (u_1, v_1) and (u_2, v_2), and γ^* is defined as follows:

$(u, v) \in \gamma^*$ if and only if there is $(u^*, v^*) \in \gamma$ such that (u, v) is the first point on $S_i^+(u^*, v^*)$ with $\sigma(u, v; u^*, v^*) = \lambda_i(u^*, v^*)$.

For $(u, v) \in \gamma^*$ with corresponding $(u^*, v^*) \in \gamma$, we solve the Riemann problem $\{(u_0, v_0); (u, v)\}$ by connecting (u_0, v_0) to (u_1, v_1) by an i-shock, (u_1, v_1) to (u^*, v^*) by an i-rarefaction wave and (u^*, v^*) to (u, v) by an i-shock. The i-shock $\{(u^*, v^*); (u, v)\}$ has the property that the shock speed σ coincides with λ on either side of the shock, we call such continuity a contact discontinuity. Suppose there is a point (u_2, v_2) on γ^* such that $\sigma(u_2, v_2; u_2^*, v_2^*) = \lambda_i(u_2^*, v_2^*) = \lambda_i(u_2, v_2)$. We then continue $\alpha_i(u_0, v_0)$ from (u_2, v_2) by $R_i(u_2, v_2)$. Continue these processes so that $\alpha_i(u_0, v_0)$ is composed of shock, rarefaction and mixed curves. It is shown that the mixed curve γ^* is tangent to $S_2^+(u^*, v^*)$ at (u, v).

The solution of the Riemann problem $\{(u_0, v_0), (u, v)\}$, $(u, v) \in \alpha_i(u_0, v_0)$, satisfies the following extended entropy condition (E) across any discontinuity (u_-, v_-) and (u_+, v_+):

(E) $\sigma(u, v; u_-, v_-) \geqslant \sigma(u_+, v_+; u_-, v_-)$ for every $(u, v) \in S_i(u_-, v_-)$ between (u_-, v_-) and (u_+, v_+).

It can be shown that condition (E) is equivalent to Lax's shock inequalities [2] when (1) is genuinely nonlinear.

THEOREM 3. *Suppose that* (3), (6) *and* (7) *hold. Then through each point* (u_0, v_0) *in U, there exist smooth curves* $\alpha_i(u_0, v_0)$ *and* $\beta_i(u_0, v_0)$, $i = 1, 2$, *such that for any* (u, v) *on* $\alpha_i(u_0, v_0)$ $(\beta_i(u_0, v_0))$, (u_0, v_0) *can be connected to* (u, v) *on the right (left) by i-shocks, i-rarefaction waves and i-contact discontinuities such that condition* (E) *is satisfied across any discontinuity. Conversely, if* (u, v) *is any point in U which can be connected to* (u_0, v_0) *on the left (right) by i-waves satisfying condition* (E), *then* $(u, v) \in \alpha_i(u_0, v_0)$ $(\in \beta_i(u_0, v_0))$, $i = 1, 2$, *and the solution has a unique form.*

The proof of Theorem 3 is rather complicated. However, using Theorem 1, we can prove Theorem 3 by essentially the same techniques used in the proof of Theorems 2.1 and 3.1 in [2]. We omit the proof.

THEOREM 4. *Suppose that there exists* (u_m, v_m) *such that* $(u_m, v_m) \in \alpha_1(u_l, v_l) \cap \beta_2(u_r, v_r)$. *Then the Riemann problem* $\{(u_l, v_l); (u_r, v_r)\}$ *can be solved by connecting* (u_l, v_l) *to* (u_m, v_m) *by* 1-*waves and* (u_m, v_m) *to* (u_r, v_r) *by* 2-*waves such that condition* (E) *is satisfied across any discontinuity.*

PROOF. The theorem is an immediate consequence of Theorem 3. We have only to show that the 1-waves connecting (u_l, v_l) and (u_m, v_m), and the 2-waves connecting (u_m, v_m) and (u_r, v_r) do not overlap and are separated by the constant (u_m, v_m). Indeed by (6) and Theorem 2 we know that the 1-waves and 2-waves do not overlap in the $x - t$ plane. Q.E.D.

When condition (6) fails, and so does Theorem 2, then 1-waves may overlap 2-waves and the Riemann problem cannot be solved by our techniques.

Given arbitrary points (u_l, v_l) and (u_r, v_r), a counterexample was given in [4] to show that the point (u_m, v_m) in Theorem 4 may not exist even if (1) takes rather simple form. In [2] certain conditions on (1) were given to guarantee the existence of (u_m, v_m). In the next theorem we prove that the solution to the Riemann problem is always unique.

THEOREM 5. *Suppose that* (3), (6) *and* (7) *hold. Then there exists at most one solution to the Riemann problem* $\{(u_l, v_l), (u_r, v_r)\}$ *in the class of shocks, rarefaction waves and contact discontinuities which satisfies the entropy condition* (E) *across any discontinuity.*

PROOF. Suppose the Riemann problem $\{(u_l, v_l), (u_r, v_r)\}$ can be solved by connecting (u_l, v_l) to (u_m, v_m) by 1-waves and (u_m, v_m) to (u_r, v_r) by 2-waves, and can also be solved by connecting (u_l, v_l) to $(\bar{u}_m, \bar{v}_m)$ by 1-waves and $(\bar{u}_m, \bar{v}_m)$ to (u_r, v_r) by 2-waves (cf. Figure 2). By Theorem 3, (u_m, v_m) and $(\bar{u}_m, \bar{v}_m)$ both belong to $\alpha_1(u_l, v_l) \cap \beta_2(u_r, v_r)$ and the proof of Theorem 5 will be complete if we can show that $(u_m, v_m) = (\bar{u}_m, \bar{v}_m)$.

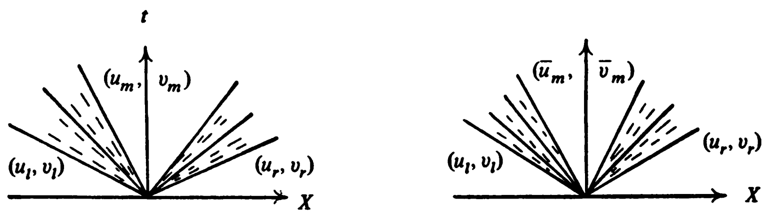

FIGURE 2

Suppose, $(u_m, v_m) \neq (\bar{u}_m, \bar{v}_m)$. By Theorem 4, we know $(u_r, v_r) \in \alpha_2(u_m, v_m) \cap \alpha_2(\bar{u}_m, \bar{v}_m)$. Choose (u^1, v^1) on $\alpha_1(u_l, v_l)$ between (u_m, v_m) and $(\bar{u}_m, \bar{v}_m)$. Then by Lemma 1, $\alpha_2(u^1, v^1)$ must intercept either $\alpha_2(u_m, v_m)$ or $\alpha_2(\bar{u}_m, \bar{v}_m)$ (cf. Figure 3), say at (u_1, v_1).

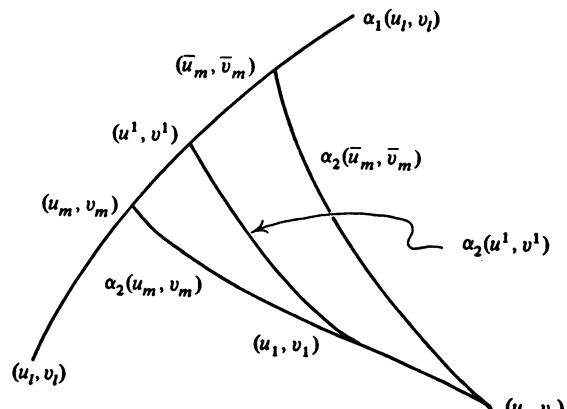

FIGURE 3

Without loss of generality, assume $(u_1, v_1) \in \alpha_2(u^1, v^1) \cap \alpha_2(u_m, v_m)$. It follows that both (u_m, v_m) and (u^1, v^1) belong to $\beta_2(u_1, v_1)$ by Theorem 3. We then take (u^2, v^2) on $\alpha_1(u_l, v_l)$ between (u_m, v_m) and (u^1, v^1). Again by Lemma 1, $\alpha_2(u^2, v^2)$ must intercept either $\alpha_2(u^1, v^1)$ or $\alpha_2(u_m, v_m)$, say at (u_2, v_2). Continuing the process, we get sequences $\{(u^i, v^i)\}$ and $\{(u_i, v_i)\}$, $i = 1, 2, \ldots$, such that $\alpha_2(u^i, v^i)$ intercept either $\alpha_2(u^{i-1}, v^{i-1})$ or $\alpha_2(u_m, v_m)$ at (u_i, v_i). By our constructions, both sequences are contained in a bounded set and $\{(u^i, v^i)\}$ converges to a point, say (u^0, v^0). Let (u_0, v_0) be a limiting point of $\{(u_i, v_i)\}$. Since (u^i, v^i) is in $\beta_2(u_i, v_i)$, we know that $\beta_2(u_0, v_0)$ is tangent to $\alpha_1(u_l, v_l)$ at (u^0, v^0).

Suppose $\alpha_1(u_l, v_l)$ is composed of a shock or a mixed curve at (u^0, v^0). Then there exists (u^*, v^*) on $\alpha_1(u_l, v_l)$ such that $(u^0, v^0) \in S_1(u^*, v^*)$ and $h_1(u^0, v^0; u^*, v^*)$ is a tangent to $\alpha_1(u_l, v_l)$ at (u^0, v^0). Similarly, if $\beta_2(u_0, v_0)$ is composed of a shock or mixed curves at (u^0, v^0), then there exists (u_*, v_*) on $\beta_2(u_0, v_0)$ such that $(u^0, v^0) \in S_2(u_*, v_*)$ and $h_2(u_*, v_*; u^0, v^0)$ is a tangent to $\beta_2(u_0, v_0)$ at (u^0, v^0). Since $\beta_2(u_0, v_0)$ is tangent to $\alpha_1(u_l, v_l)$ at (u^0, v^0), it follows that (cf. [2]).

$$(8)\quad \frac{g_u(u^* - u^0) + (\sigma_1 - f_u)(v^* - v^0)}{f_v(v^* - v^0) + (\sigma_1 - g_v)(u^* - u^0)} = \frac{g_u(u_* - u^0) + (\sigma_2 - f_u)(v_* - v^0)}{f_v(v_* - v^0) + (\sigma_2 - g_v)(u_* - u^0)}$$

where $\sigma_1 = \sigma(u^*, v^*; u^0, v^0)$, $\sigma_2 = \sigma(u_*, v_*; u^0, v^0)$, and f_u, f_v, g_u, g_v are evaluated at (u^0, v^0).

From (8), we have

$$\begin{aligned} &(v^* - v^0)(u_* - u^0)[f_v g_u - (\sigma_1 - f_u)(\sigma_2 - g_v)] \\ (9)\quad &+ (u^* - u^0)(u_* - u^0)g_u(\sigma_1 - \sigma_2) + (v^* - v^0)(v_* - v^0)f_v(\sigma_2 - \sigma_1) \\ &+ (u^* - u^0)(v_* - v^0)[(\sigma_1 - g_v)(\sigma_2 - f_u) - f_v g_u] = 0. \end{aligned}$$

By Theorem 2, $\sigma_2 > \sigma_1$ and so

$$(10)\quad g_u(\sigma_1 - \sigma_2) > 0 \quad \text{and} \quad f_v(\sigma_2 - \sigma_1) < 0.$$

Since $\{(u^*, v^*); (u^0, v^0)\}$ and $\{(u^0, v^0); (u_*, v_*)\}$ both satisfy condition (E), we have, by Theorem 1,

$$(11)\quad \lambda_1(u^0, v^0) \leq \sigma_1 < \sigma_2 \leq \lambda_2(u^0, v^0).$$

If $\sigma_1 - g_v \geq 0$ and $\sigma_2 - f_u \leq 0$, then

$$(12)\quad (\sigma_1 - g_v)(\sigma_2 - f_u) - f_v g_u < 0.$$

If $\sigma_1 - g_v < 0$, then $(\sigma_1 - g_v)(\sigma_2 - f_u) < (\sigma_1 - g_v)(\sigma_1 - f_u) + (\sigma_2 - \sigma_1)(\sigma_1 - g_v) < (\sigma_1 - g_v)(\sigma_1 - f_u)$. On the other hand, since λ_1 and λ_2 are the two solutions of $(\lambda - f_u)(\lambda - g_v) - f_v g_u = 0$, and $\lambda_1 \leq \sigma_1 < \lambda_2$, by (11), it follows that $(\sigma_1 - g_v)(\sigma_1 - f_u) - f_v g_u \leq 0$. Therefore $(\sigma_1 - g_v)(\sigma_2 - f_u) - f_v g_u < 0$ which is (12).

If $\sigma_2 - f_u > 0$, then $(\sigma_1 - g_v)(\sigma_2 - f_u) - f_v g_u < (\sigma_2 - g_v)(\sigma_2 - f_u) - f_v g_u$ which is nonpositive, since $\lambda_1 < \sigma_2 < \lambda_2$.

We have proved that (12) holds in all cases. Similarly, using (11), we can prove that

$$(13)\quad f_v g_u - (\sigma_1 - f_u)(\sigma_2 - g_v) > 0.$$

Suppose that $(v^* - v^0)(u_* - u^0) > 0$. Then $(u^* - u^0)(u_* - u^0) > 0$, $(v^* - v^0)(v_* - v^0) < 0$ and $(u^* - u^0)(v_* - v^0) < 0$ because $(u^0, v^0) \in S_1(u^*, v^*)$ and $(u^0, v^0) \in S_2(u_*, v_*)$. Therefore by (10), (12) and (13), the left-hand side of (9) is positive. This is a contradiction. Similarly, when $(v^* - v^0)(u_* - u^0) <$ 0, then the left-hand side of (9) is negative which is again a contradiction.

This completes the proof of the theorem when both $\alpha_1(u_l, v_l)$ and $\beta_2(u_0, v_0)$ are composed of shock or mixed curves at (u^0, v^0). Analogously, the theorem is proved when either $\alpha_1(u_l, v_l)$ or $\beta_2(u_0, v_0)$ is composed of rarefaction curves at (u^0, v^0). Q.E.D.

Finally, we remark that in [2] it was proved that (1) satisfies assumption (6) if

$$f_v < 0, \quad g_u < 0, \quad f_u \geqslant 0 \quad \text{and} \quad g_v \leqslant 0 \tag{14}$$

(cf. [2, Lemma 1.2]). Therefore this paper extends the results of [2]. It can easily be proved that (1) also satisfies (6) if we take

$$f_v < 0 \quad \text{and} \quad g(u, v) = -u. \tag{15}$$

This is an extention of the gas dynamics equations.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742