TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 213, 1975

A DECOMPOSITION FOR CERTAIN REAL SEMISIMPLE LIE GROUPS

BY

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ABSTRACT. For a class of real semisimple Lie groups, including those for which G and K have the same rank, Kostant introduced the decomposition $G = KN_0K$, where N_0 is a certain abelian subgroup of N, and conjectured that the Jacobian of the decomposition with respect to Haar measure, as well as the spherical functions, would be polynomial in the canonical coordinates of N_0. We compute here the Jacobian, which turns out to be polynomial precisely when the equality of ranks is satisfied. We also compute those spherical functions which restrict to polynomials on N_0.

1. **Some preliminaries concerning root systems.** Let V be a Euclidean space with inner product $\langle\ ,\ \rangle$. Let Δ be a root system in V. For $\alpha \in \Delta$ we indicate by s_α the Weyl reflection with respect to α

$$(s_\alpha(v) = v - 2\langle\alpha, v\rangle\alpha/\langle\alpha, \alpha\rangle).$$

The group generated by $\{s_\alpha | \alpha \in \Delta\}$ is called the Weyl group and will be designated by W.

1.1. PROPOSITION. *Let s be an involutive element of W with ± 1-eigenspaces $V_\pm$, respectively. Then s can be written in the form $s = s_{\gamma_1} \cdots s_{\gamma_n}$, where $\{\gamma_1, \ldots, \gamma_n\}$ is an orthogonal basis of V_- and $\gamma_i \pm \gamma_j \notin \Delta$ for $i, j = 1, \ldots, n$.*

PROOF. Let v be a relatively regular element of V_+; i.e., an element of V_+ for which $\langle\alpha, v\rangle = 0$, $\alpha \in \Delta$, implies $\langle\alpha, V_+\rangle = 0$. Since $s(v) = v$, it follows from [2, Chapter V, §3.3, Proposition 1], that s can be written in the form

$$s = s_{\alpha_1} \cdots s_{\alpha_m}, \tag{1}$$

where $\{\alpha_1, \ldots, \alpha_m\} \subset \Delta \cap V_+^\perp = \Delta \cap V_-$. Now introduce any ordering in V, and let γ_1 be the largest element of $\Delta \cap V_-$ with respect to that ordering. Because of (1), γ_1 exists. Now, having chosen $\gamma_1, \ldots, \gamma_k$, if $s \neq s_{\gamma_1} \cdots s_{\gamma_k}$, let γ_{k+1} be the largest element of $\Delta \cap V_-$ orthogonal to $\gamma_1, \ldots, \gamma_k$. γ_{k+1} exists

Received by the editors July 19, 1973 and, in revised form, July 3, 1974.
AMS (MOS) subject classifications (1970). Primary 17B20, 22E30, 43A05, 43A80, 43A90.
[1] Partially supported by NSF Grant No. P22930.

by (1), applied to the involutive element $ss_{\gamma_1}, \ldots, s_{\gamma_k} \in W$. Since V, hence V_-, is finite-dimensional, the process terminates, and we have $s = s_{\gamma_1} \cdots s_{\gamma_n}$. $\gamma_1, \ldots, \gamma_n$ is an orthogonal basis of V_-. If $\gamma_i + \gamma_j \in \Delta$, $i \leqslant j$, then $\gamma_i + \gamma_j > \gamma_i$ and would have had to be chosen in preference to γ_i. Thus $\gamma_i + \gamma_j \notin \Delta$, and for $i \neq j$, $\gamma_i - \gamma_j = s_{\gamma_j}(\gamma_i + \gamma_j) \notin \Delta$. $\gamma_i - \gamma_i = 0 \notin \Delta$.

A subset of Δ having the property that no sum or difference of two of its elements belongs to Δ and no sum of two of its elements is zero is called a set of *strongly orthogonal roots.* Two distinct strongly orthogonal roots are always orthogonal. For if α and β are nonproportional elements of Δ, then if $\langle\alpha, \beta\rangle < 0$, $\alpha + \beta \in \Delta$, and if $\langle\alpha, \beta\rangle > 0$, $\alpha - \beta \in \Delta$; whereas if α and β are proportional then either $\alpha = \pm\beta$, a case we have excluded, or $\alpha = \pm 2\beta$ (or $\beta = \pm 2\alpha$), whence $\alpha + \alpha$ (resp., $\beta + \beta$) is an element of Δ. We have show that the -1-eigenspace of an involutive element of W has a basis of strongly orthogonal roots. (The most interesting case is, of course, the case $s = -1$, $V_- = V$, in case $-1 \in W$.)

Now let σ be a linear involution of V (not necessarily an element of W) with the property that $\sigma(\alpha) - \alpha \notin \Delta$ for $\alpha \in \Delta$. Let $P = \{\frac{1}{2}(\alpha + \sigma(\alpha)) | \alpha \in \Delta\}$. Then P is a root system [1, Proposition 2.1] in the $+1$-eigenspace of σ. The elements of P will be called "restricted roots", while those of Δ will be called simply "roots". For $\alpha \in P$ the multiplicity of α (denoted m_α) is defined as the number of roots β satisfying $\frac{1}{2}(\beta + \sigma(\beta)) = \alpha$. The following lemma is obvious, since σ acts as an involution on $\{\beta | \frac{1}{2}(\beta + \sigma(\beta)) = \alpha\}$ and fixes β iff $\beta = \alpha$.

1.2. LEMMA. *For $\alpha \in P$, $\alpha \in \Delta$ iff m_α is odd* [1, *Proposition* 2.2].

We now relate sets of strongly orthogonal restricted roots to sets of strongly orthogonal roots.

1.3. PROPOSITION. *Let $\gamma_1, \ldots, \gamma_n$ be restricted roots. Then $\{\gamma_1, \ldots, \gamma_n\}$ is a set of strongly orthogonal roots iff it is a set of strongly orthogonal restricted roots of odd multiplicities.*

PROOF. If the multiplicities of the γ_i are odd, then the γ_i are roots, and $\gamma_i \pm \gamma_j \notin \Delta$ if $\frac{1}{2}[(\gamma_i \pm \gamma_j) + \sigma(\gamma_i \pm \gamma_j)] = \gamma_i \pm \gamma_j \notin P$. Conversely, if $\{\gamma_1, \ldots, \gamma_n\}$ is a set of strongly orthogonal roots contained in P, then each α_i is a restricted root of odd multiplicity, and if $\gamma_i \pm \gamma_j \in P$, then $\gamma_i \pm \gamma_j \neq 0$ and $\gamma_i \pm \gamma_j + v \in \Delta$ for some v with $\sigma(v) = v$. But then

$$0 < \frac{2\langle\gamma_i, \gamma_i \pm \gamma_j + v\rangle}{\langle\gamma_i \pm \gamma_j + v, \gamma_i \pm \gamma_j + v\rangle} < 1,$$

an impossibility because Δ is a root system.

Now assume that -1 belongs to the Weyl group of P, so that the $+1$-eigenspace of σ has a basis $\{\gamma_1, \ldots, \gamma_q\}$ of strongly orthogonal restricted roots.

1.4. PROPOSITION. *Every* $\alpha \in P$ *is of the form* $\alpha = \frac{1}{2}\Sigma_{i=1}^{q} n_i\gamma_i$, *where the* n_i *are integers. If* α *is proportional to* γ_i, *then* $\alpha = \pm\gamma_i$ or $\alpha = \pm\frac{1}{2}\gamma_i$.

PROOF.

$$\alpha = \frac{1}{2}\sum_{i=1}^{q} \frac{2\langle\alpha, \gamma_i\rangle}{\langle\gamma_i, \gamma_i\rangle}\gamma_i,$$

and $2\langle\alpha, \gamma_i\rangle/\langle\gamma_i, \gamma_i\rangle$ is an integer. If α is proportional to γ_i, then $\alpha = \pm\gamma_i$, $\alpha = \pm\frac{1}{2}\gamma_i$, or $\alpha = \pm 2\gamma_i$. But $2\gamma_i = \gamma_i + \gamma_i$ and $-2\gamma_i = s_{\gamma_i}(\gamma_i + \gamma_i)$ are not restricted roots.

Let H be any linear functional on the $+1$-eigenspace of σ. Let $t_i = 2\sinh\frac{1}{2}H(\gamma_i)$, and let

$$J(t_1, \ldots, t_q) = \frac{\prod_{\alpha\in P} |\sinh H(\alpha)|^{\frac{1}{2}m_\alpha}}{\prod_{i=1}^{q} \cosh \frac{1}{2}H(\gamma_i)}. \tag{2}$$

The following lemma will be useful in the study of the function J.

1.5. LEMMA. *The function from* $\mathbf{R}^r$ *to* $\mathbf{R}$ *defined by*

$$f(h_1, \ldots, h_r) = (-1)^{2^{r-1}} \prod \sinh(\pm n_1h_1 \pm \cdots \pm n_rh_r),$$

where $n_1, \ldots, n_r$ *are fixed integers and the product is extended over all combinations of signs, is a polynomial in* $\sinh h_1, \ldots, \sinh h_r$. *It is the square of a polynomial in all cases except the case* $r = 1$, n_1 *even.*

PROOF. $f = g^2$, where

$$g(h_1, \ldots, h_r) = \prod \sinh(n_1h_1 \pm \cdots \pm n_rh_r),$$

where the product is extended over all combinations of signs. For $r = 1$, it is well known that g^2 is a polynomial in $\sinh h$ and g is a polynomial in $\sinh h_1$ iff n_1 is odd. For $r \geqslant 2$,

$$\sinh(a_1 + \cdots + a_r) = \sum f_1(a_1)\cdots f_r(a_r),$$

$$f_i \in \{\sinh, \cosh\}, \quad \{i | f_i = \sinh\} \text{ of odd cardinality.}$$

Thus $\Pi \sinh(a_1 \pm \cdots \pm a_r)$ is a sum of terms of the form

$$\prod_{i=1}^{r} (\sinh a_i)^{k_i}(\cosh a_i)^{2^{r-1}-k_i}. \tag{3}$$

Because of the evenness in each a_i, k_i is even for each i in each term (3). But then $2^{r-1} - k_i$ is even, and $\Pi \sinh(a_1 \pm \cdots \pm a_r)$ is a polynomial in

$\sinh^2 a_1, \ldots, \sinh^2 a_r$. Thus g is a polynomial in $\sinh^2 n_1 h_1, \ldots, \sinh^2 n_r h_r$ and hence in $\sinh h_1, \ldots, \sinh h_r$.

1.6. PROPOSITION. *J is the absolute value of a polynomial iff m_{γ_i} is odd for $i = 1, \ldots, q$.*

PROOF. We may partition P into orbits of the subgroup of its Weyl group generated by the s_{γ_i}. The orbit of $\alpha \in P$, where, by Proposition 1.4, $\alpha = \frac{1}{2}\sum_{i=1}^q n_i\gamma_i$ has the form $\{\frac{1}{2}\sum_{i=1}^q \pm n_i\gamma_i\}$ for all combinations of signs. By Lemma 1.5, if $\{i_1, \ldots, i_r\}$ is the subset of $\{1, \ldots, q\}$ on which $n_i \neq 0$,

$$\prod |\sinh(\pm n_{i_1}\gamma_{i_1} \pm \cdots \pm n_{i_r}\gamma_{i_r})|^{1/2}$$

is the absolute value of a polynomial in the t_i except in the case $r = 1$, n_i even. In that case $n_i = \pm 2$, $\alpha = \pm\gamma_{i_1}$, the orbit of α has the form $\{\pm\gamma_{i_1}\}$, and we consider the factor of J:

$$\frac{|\sinh H(\gamma_{i_1})|^{\frac{1}{2}m_{\gamma_{i_1}}} |\sinh H(-\gamma_{i_1})|^{\frac{1}{2}m_{-\gamma_{i_1}}}}{\cosh \frac{1}{2}H(\gamma_{i_1})} = \frac{|\sinh H(\gamma_{i_1})|^{m_{\gamma_{i_1}}}}{\cosh \frac{1}{2}H(\gamma_{i_1})}$$

$$= |t_{i_1}|^{m_{\gamma_{i_1}}} [\tfrac{1}{2}(t_{i_1}^2 + 4)]^{\frac{1}{2}(m_{\gamma_{i_1}} - 1)}$$

which is the absolute value of a polynomial iff $m_{\gamma_{i_1}}$ is odd. Thus J is the absolute value of a polynomial iff all the m_{γ_i} are odd.

1.7. COROLLARY. *J is the absolute value of a polynomial iff the $+1$-eigenspace of σ has a basis of strongly orthogonal roots.*

We give in the table below the explicit formula for J, for each restricted root system with -1 in its Weyl group, in terms of the multiplicities. We shall use for convenience the following abbreviated notations.

$$\begin{aligned} P(w, x, y, z) = {} & (w^2x^2 + 2w^2 + 2x^2 - y^2z^2 - 2y^2 - 2z^2)^4 \\ & + w^4x^4(w^2 + 4)^2(x^2 + 4)^2 + y^4z^4(y^2 + 4)^2(z^2 + 4)^2 \\ & - 2(w^2x^2 + 2w^2 + 2x^2 - y^2z^2 - 2y^2 - 2z^2)^2 w^2x^2(w^2 + 4)(x^2 + 4) \\ & - 2(w^2x^2 + 2w^2 + 2x^2 - y^2z^2 - 2y^2 - 2z^2)^2 y^2z^2(y^2 + 4)(z^2 + 4) \\ & - 2w^2x^2y^2z^2(w^2 + 4)(x^2 + 4)(y^2 + 4)(z^2 + 4). \end{aligned}$$

$$Q(t, u, v) = [2t^2(t^2 + 4) - u^2v^2 - 2u^2 - 2v^2]^2 - u^2v^2(u^2 + 4)(v^2 + 4).$$

We have listed in the table only irreducible types. Clearly, for a reducible root system, J is the product over the irreducible direct factors.

The degree d_i in t_i of the polynomial whose absolute value is J (or, more generally, half the degree of J^2) can be read off in each case from Table 1. We prefer, however, to relate d_i to the structures of the root systems P and Δ.

1.8. PROPOSITION. *J^2 is a polynomial in $t_1, \ldots, t_q$ whose degree in t_i is*

$$2d_i = -2 + 2 \sum_{\alpha \in P} \frac{m_\alpha |\langle \gamma_i, \alpha \rangle|}{\langle \gamma_i, \gamma_i \rangle} = -2 + 2 \sum_{\beta \in \Delta} \frac{|\langle \gamma_i, \beta \rangle|}{\langle \gamma_i, \gamma_i \rangle}.$$

PROOF.

$$J^2(t_1, \ldots, t_q) = \frac{\prod_{\alpha \in P} |\sinh H(\alpha)|^{m_\alpha}}{\prod_{i=1}^{q} \cosh^2 \tfrac{1}{2} H(\gamma_i)}.$$

where the numerator is a product of factors of the form

$$\prod \left| \sinh \left(\pm \frac{\langle \gamma_{i_1}, \alpha \rangle}{\langle \gamma_{i_1}, \gamma_{i_1} \rangle} H(\gamma_{i_1}) \pm \cdots \pm \frac{\langle \gamma_{i_r}, \alpha \rangle}{\langle \gamma_{i_r}, \gamma_{i_r} \rangle} H(\gamma_{i_r}) \right) \right|^{m_\alpha},$$

such a factor occurring for each orbit in P of the subgroup of the Weyl group of P generated by the s_{γ_i}. Such a factor is of degree $2^{r+1} m_\alpha \langle \gamma_i, \alpha \rangle / \langle \gamma_i, \gamma_i \rangle$ in $t_i = 2 \sinh \tfrac{1}{2}\gamma_i$ and is counted 2^r times in the product over all $\alpha \in P$. The denominator is of degree 2 in t_i. The first equality of the proposition is now proven. To prove the second equality we note that, for $\beta \in \Delta$, $\langle \gamma_i, \beta \rangle = \langle \gamma_i, \tfrac{1}{2}(\beta + \sigma(\beta)) \rangle$.

Note that the formula for d_i depends only on Δ and γ_i, not on σ.

Now assume that the m_{γ_i} are odd, so that J is the absolute value of a polynomial. Its degree in t_i is

$$d_i = -1 + \sum_{\beta \in \Delta} \frac{|\langle \gamma_i, \beta \rangle|}{\langle \gamma_i, \gamma_i \rangle}.$$

Assume further that Δ is irreducible and reduced. We can then express d_i in terms of the coefficients of the highest root of Δ in terms of a simple system (with respect to some ordering).

1.9. PROPOSITION. *If the highest root of Δ is expressed in terms of the simple system $\{\alpha_1, \ldots, \alpha_n\}$ as $\Sigma_{j=1}^n k_j \alpha_j$, then*

$$d_i = -1 + 2 \sum_{j=1}^{n} k_j \frac{\langle \alpha_j, \alpha_j \rangle}{\langle \gamma_i, \gamma_i \rangle}.$$

PROOF. [2, proof of Proposition 31, Chapter VI, §1, 1.11].

TABLE 1

Type of P	Notation for γ_i	Notation for t_i	Weyl group orbits in P	Notation for m_α
BC_q (including $C_q[s = 0]$)	$\gamma_1, \ldots, \gamma_q$	$t_1, \ldots, t_q$	$\{\pm\frac{1}{2}\gamma_i\}$ $\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j \mid i < j\}$ $\{\pm\gamma_i\}$	s m l
B_{2r} (including $D_{2r}[s = 0]$)	$\delta_1, \ldots, \delta_r$ $\epsilon_1, \ldots, \epsilon_r$	$u_1, \ldots, u_r$ $v_1, \ldots, v_r$	$\{\pm\frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_i\}$ $\{\pm\delta_i\} \cup \{\pm\epsilon_i\} \cup \{\pm\frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_i \pm \frac{1}{2}\delta_j \pm \frac{1}{2}\epsilon_j \mid i < j\}$	s l
B_{2r+1}	γ $\delta_1, \ldots, \delta_r$ $\epsilon_1, \ldots, \epsilon_r$	t $u_1, \ldots, u_r$ $v_1, \ldots, v_r$	$\{\pm\gamma\} \cup \{\pm\frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_i\}$ $\{\pm\delta_i\} \cup \{\pm\epsilon_i\} \cup \{\pm\gamma \pm \frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_i\}$ $\cup\ \{\pm\frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_i \pm \frac{1}{2}\delta_j \pm \frac{1}{2}\epsilon_j \mid i < j\}$	s l
E_7	γ $\delta_1, \delta_2, \delta_3$ $\epsilon_1 = \epsilon_4, \epsilon_2 = \epsilon_5, \epsilon_3$	t u_1, u_2, u_3 $v_1 = v_4, v_2 = v_5, v_3$	$\{\pm\gamma\} \cup \{\pm\delta_i\} \cup \{\pm\epsilon_i\} \cup \{\pm\frac{1}{2}\gamma \pm \frac{1}{2}\delta_1 \pm \frac{1}{2}\delta_2 \pm \frac{1}{2}\delta_3\}$ $\cup\ \{\pm\frac{1}{2}\gamma \pm \frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_{i+1} \pm \frac{1}{2}\epsilon_{i+2}\}$ $\cup\ \{\pm\frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_i \pm \frac{1}{2}\delta_j \pm \frac{1}{2}\epsilon_j \mid i < j\}$	l
E_8	$\delta_1 = \delta_5, \delta_2 = \delta_6, \delta_3 = \delta_7, \delta_4$ $\epsilon_1 = \epsilon_5, \epsilon_2 = \epsilon_6, \epsilon_3 = \epsilon_7, \epsilon_4$	$u_1 = u_5, u_2 = u_6, u_3 = u_7, u_4$ $v_1 = v_5, v_2 = v_6, v_3 = v_7, v_4$	$\{\pm\delta_i\} \cup \{\pm\epsilon_i\} \cup \{\frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_{i+1} \pm \frac{1}{2}\epsilon_{i+2} \pm \frac{1}{2}\epsilon_{i+3}\}$ $\cup\ \{\pm\frac{1}{2}\epsilon_i \pm \frac{1}{2}\delta_{i+1} \pm \frac{1}{2}\delta_{i+2} \pm \frac{1}{2}\delta_{i+3}\}$ $\cup\ \{\pm\frac{1}{2}\delta_i \pm \frac{1}{2}\epsilon_i \pm \frac{1}{2}\delta_j \pm \frac{1}{2}\epsilon_j \mid i < j\}$	l
F_4	$\gamma_1, \ldots, \gamma_4$	$t_1, \ldots, t_4$	$\{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j \mid i < j\}$ $\{\pm\gamma_i\} \cup \{\pm\frac{1}{2}\gamma_1 \pm \frac{1}{2}\gamma_2 \pm \frac{1}{2}\gamma_3 \pm \frac{1}{2}\gamma_4\}$	s l
G_2	γ δ	t u	$\{\pm\delta\} \cup \{\pm\frac{1}{2}\gamma \pm \frac{1}{2}\delta\}$ $\{\pm\gamma\} \cup \{\pm\frac{1}{2}\gamma \pm \frac{3}{2}\delta\}$	s l

TABLE 1 (Continued)

$J(t_1, \ldots, t_q)$
$(\tfrac{1}{2})^{qs+q(l-1)/2+q(q-1)m}\left\vert\prod_{i=1}^{q} t_i^{s+l}(t_i^2+4)^{(l-1)/2}\cdot\prod_{1\le i<j\le q}(t_i^2-t_j^2)^m\right\vert.$
$(\tfrac{1}{2})^{r(6r-5)l-r+r(r-1)s}\left\vert\prod_{1\le i<j\le r}P(u_i,v_i,u_j,v_j)^l\right\vert\cdot\left\vert\prod_{i=1}^{q}u_i^lv_i^l(u_i^2+4)^{(l-1)/2}(v_i^2+4)^{(l-1)/2}(u_i^2-v_i^2)^s\right\vert.$
$(\tfrac{1}{2})^{r(6r+1)l-r+r(r-1)s+(s-1)/2}\left\vert t^s(t^2+4)^{(s-1)/2}\prod_{1\le i<j\le r}P(u_i,v_i,u_j,v_j)\right\vert$ $\cdot\left\vert\prod_{i=1}^{r}u_i^lv_i^l(u_i^2+4)^{(l-1)/2}(v_i^2+4)^{(l-1)/2}(u_i^2-v_i^2)^sQ(t,u_i,v_i)^l\right\vert.$
$(\tfrac{1}{2})^{195l/2-7/2}\left\vert t^l(t^2+4)^{(l-1)/2}P(t,u_1,u_2,u_3)^l\prod_{1\le i<j\le 3}P(u_i,v_i,u_j,v_j)^l\right\vert$ $\cdot\left\vert\prod_{i=1}^{3}u_i^lv_i^l(u_i^2+4)^{(l-1)/2}(v_i^2+4)^{(l-1)/2}P(t,u_i,v_{i+1},v_{i+2})\right\vert.$
$(\tfrac{1}{2})^{172l-4}\left\vert\prod_{1\le i<j\le 4}P(u_i,v_i,u_j,v_j)\right\vert$ $\cdot\left\vert\prod_{i=1}^{4}u_i^lv_i^l(u_i^2+4)^{(l-1)/2}(v_i^2+4)^{(l-1)/2}P(u_i,v_{i+1},v_{i+2},v_{i+3})P(v_i,u_{i+1},u_{i+2},u_{i+3})\right\vert.$
$(\tfrac{1}{2})^{14l-2+12s}\left\vert P(t_1,t_2,t_3,t_4)^l\prod_{i=1}^{4}t_i^l(t_i^2+4)^{(l-1)/2}\cdot\prod_{1\le i<j\le 4}(t_i^2-t_j^2)^s\right\vert.$
$(\tfrac{1}{2})^{9l/2+9s/2-1}\vert t^lu^s(t^2+4)^{(l-1)/2}(u^2+4)^{(s-1)/2}(t^2-u^2)^s[t^2-(u^3+3u)^2]^l\vert.$

1.10. COROLLARY. *If h is the Coexeter number of Δ and γ_i is a root of minimal length, then $d_i = 2h - 3$.*

PROOF. [2, loc. cit.].

2. **Application to Lie algebras.** Let $\mathfrak{g}$ be a noncompact real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal commutative subspace of $\mathfrak{p}$. $\mathfrak{a}$ can be extended to a maximal commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and such an $\mathfrak{h}$ has the form $\mathfrak{h} = \mathfrak{h}_+ + \mathfrak{a}$, where $\mathfrak{h}_+ \subset \mathfrak{k}$ [4, p. 221]. The nonzero eigenvalues of the adjoint representation of $\mathfrak{h}$ on the complexification $\mathfrak{g}C$ of $\mathfrak{g}$ form a reduced root system Δ in $i\mathfrak{h}_+^* + \mathfrak{a}^*$, with inner product $\langle\ ,\ \rangle$ dual to the killing form B of $\mathfrak{g}$. (The stars denote real dual vector spaces, and $C\mathfrak{h}_+^* + C\mathfrak{a}^*$ is naturally identified with $C\mathfrak{h}^*$.) We let σ be the linear involution of $i\mathfrak{h}_+^* + \mathfrak{a}^*$ which is -1 on $i\mathfrak{h}_+^*$ and $+1$ on $\mathfrak{a}^*$. Then the restricted root system P defined by σ is the set of nonzero eigenvalues of the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$, and the multiplicity m_α of $\alpha \in P$ is equal to the dimension of its eigenspace in $\mathfrak{g}$. (For details of the above, see e.g. [1].)

$B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite, while $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite. Let θ be the symmetry; i.e., the linear involution of $\mathfrak{g}$ equal to $+1$ on $\mathfrak{k}$, to -1 on $\mathfrak{p}$. θ is an algebra automorphism of $\mathfrak{g}$. For $\alpha \in P$ let $H_\alpha \in \mathfrak{a}$ be the unique element such that $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{a}$.

Now let $\{\gamma_1, \ldots, \gamma_r\}$ be a set of strongly orthogonal restricted roots. Let X_i be an element of the eigenspace of γ_i in $\mathfrak{g}$ such that $-B(X_i, \theta X_i) = 2/\gamma_i(H_{\gamma_i})$. Let $Y_i = -\theta X_i$, $Z_i = 2H_{\gamma_i}/\gamma_i(H_{\gamma_i})$.

2.1. PROPOSITION. *For the X_i, Y_i, Z_i, we have the following multiplication table*:

$$[X_i, X_j] = [Y_i, Y_j] = [Z_i, Z_j] = 0, \qquad [Z_i, X_j] = 2\delta_{ij}X_j,$$

$$[X_i, Y_j] = \delta_{ij}Z_j, \qquad [Z_i, Y_j] = -2\delta_{ij}Y_j.$$

Furthermore, $X_i - Y_i \in \mathfrak{k}$, $X_i + Y_i \in \mathfrak{p}$, $Z_i \in \mathfrak{p}$.

PROOF (as in [4, Chapter VI, Lemma 3.1]). $Z_i \in \mathfrak{a}$, which is commutative; for $i \neq j$, $[X_i, Y_j]$, $[X_i, X_j]$, and $[Y_i, Y_j]$ belong to $(\pm\gamma_i \pm \gamma_j)$-eigenspaces of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$, which are all $\{0\}$, and $[Z_i, X_j] = \gamma_j(Z_i)X_j = 0 = -\gamma_j(Z_i)Y_j = [Z_i, Y_j]$. For $i = j$,

$$[Z_i, X_i] = \gamma_i(Z_i)X_i = 2X_i, \qquad [Z_i, Y_i] = -\gamma_i(Z_i)Y_i = -2Y_i,$$

and $[X_i, Y_i]$ belongs to the 0-eigenspace of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$. Also

$$\theta([X_i, Y_i]) = [\theta X_i, \theta Y_i] = [-Y_i, -X_i] = [Y_i, X_i] = -[X_i, Y_i].$$

Therefore $[X_i, Y_i] \in \mathfrak{p}$, and so $[X_i, Y_i] \in \mathfrak{a}$, by maximality of $\mathfrak{a}$ in $\mathfrak{p}$. Now, for $H \in \mathfrak{a}$,

$$B(H, [X_i, Y_i]) = B([H, X_i], Y_i) = \gamma_i(H)B(X_i, Y_i) = 2\gamma_i(H)/\gamma_i(H_i).$$

Therefore $[X_i, Y_i] = Z_i$. $\theta(X_i - Y_i) = -Y_i + X_i = X_i - Y_i$. Therefore $X_i - Y_i \in \mathfrak{k}$. $\theta(X_i + Y_i) = -Y_i - X_i = -(X_i + Y_i)$. Therefore $X_i + Y_i \in \mathfrak{p}$. Finally, $Z_i \in \mathfrak{a} \subset \mathfrak{p}$.

2.2. COROLLARY. *$X_1, \ldots, X_r, Y_1, \ldots, Y_r$, and $Z_1, \ldots, Z_r$ generate (as a vector space) a subalgebra of $\mathfrak{g}$ isomorphic to the Lie algebra direct sum of r copies of $\mathfrak{sl}(2, R)$ and having a Cartan decomposition compatible with that of $\mathfrak{g}$. Specifically, the Lie algebra generated by X_i, Y_i, and Z_i is mapped isomorphically onto $\mathfrak{sl}(2, R)$ by the linear mapping defined on the given basis by*

$$X_i \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_i \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z_i \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

and the Cartan decomposition

$$\mathfrak{sl}(2, R) = \mathfrak{so}(2) + \left\{ \begin{pmatrix} t & u \\ u & -t \end{pmatrix} \right\}$$

is compatible with the Cartan decomposition of $\mathfrak{g}$.

PROOF. Direct computation.

We now determine a necessary and sufficient condition on $\mathfrak{g}$ for $\mathfrak{a}^*$ to have a basis of strongly orthogonal roots. (If we require only a basis of strongly orthogonal restricted roots, a necessary and sufficient condition is simply that -1 belong to the Weyl group of P.)

2.3. PROPOSITION. *$\mathfrak{a}^*$ has a basis of strongly orthogonal roots if and only if $\mathfrak{k}$ contains a maximal commutative subalgebra of $\mathfrak{g}$.*

PROOF. Let $\{\gamma_i, \ldots, \gamma_q\}$ be a basis of strongly orthogonal roots for $\mathfrak{a}^*$, and let X_i, Y_i, $\mathfrak{h}$, and $\mathfrak{h}_+$ be as above.

CLAIM. A maximal commutative subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$ is given by

$$\exp\left(\mathrm{ad}_{\mathfrak{g}^C}\left(\frac{\pi i}{4} \sum_{i=1}^{q} (X_i + Y_i)\right)\right)(\mathfrak{h}_+ + i\mathfrak{a}).$$

PROOF OF CLAIM. X_i and Y_i commute with $\mathfrak{h}_+$ because the γ_i vanish on $\mathfrak{h}_+$. For $H = \sum_{i=1}^{q} h_i Z_i$, a typical element of $\mathfrak{a}$,

$$\exp\left(\mathrm{ad}_{\mathfrak{g}^C}\left(\frac{\pi i}{4} \sum_{i=1}^{q} (X_i + Y_i)\right)\right)(H)$$

$$= \sum_{i=1}^{q}\left[\sum_{k=0}^{\infty} \frac{(\pi i/2)^{2k}}{(2k)!}(X_i + Y_i) - \sum_{k=0}^{\infty} \frac{(\pi i/2)^{2k+1}}{(2k+1)!}(X_i - Y_i)\right] \tag{4}$$

$$= \sum_{i=1}^{q}\left[\left(\cos\frac{\pi}{2}\right)(X_i + Y_i) - i\left(\sin\frac{\pi}{2}\right)(X_i - Y_i)\right] = -i\sum_{i=1}^{q}(X_i - Y_i) \in i\mathfrak{k}.$$

The commutativity and maximality follow from the same properties for $\mathfrak{h}$.

To prove the converse (along the lines of [4, Chapter VIII, Proposition 7.4]), we assume that $\widetilde{\mathfrak{h}} \subset \mathfrak{k}$ is a maximal commutative subalgebra of $\mathfrak{g}$. We let $\widetilde{\Delta}$ be the root system of nonzero eigenvalues of $\mathrm{ad}_{\mathfrak{g}_C}(C\,\widetilde{\mathfrak{h}})$. $\widetilde{\Delta} = \widetilde{\Delta}_c \cup \widetilde{\Delta}_n$, where the eigenspaces of $\widetilde{\Delta}_c$ are contained in $C\,\mathfrak{k}$, while those of $\widetilde{\Delta}_n$ are contained in $C\,\mathfrak{p}$. Introduce an ordering in the span of $\widetilde{\Delta}_n$, and choose $\widetilde{X}_\beta$ in the β-eigenspace for each $0 < \beta \in \widetilde{\Delta}_n$. Let $\widetilde{Y}_\beta = \sigma\widetilde{X}_\beta$, where σ is the linear involution of $\mathfrak{g}_C$ which is $+1$ on $\mathfrak{g}$ and -1 on $i\mathfrak{g}$. Since $\widetilde{\Delta} \subset i\widetilde{\mathfrak{h}}^*$, $\widetilde{Y}_\beta$ belongs to the $-\beta$-eigenspace. Clearly $\widetilde{X}_\beta + \widetilde{Y}_\beta \in \mathfrak{g}$. Since $0 \neq [\widetilde{X}_\beta, \widetilde{Y}_\beta] \in C\,\widetilde{\mathfrak{h}} \subset C\mathfrak{k}$, $\widetilde{Y}_\beta \notin C\,\mathfrak{k}$. Therefore $\widetilde{Y}_\beta \in C\,\mathfrak{p}$; $\widetilde{X}_\beta + \widetilde{Y}_\beta \in C\mathfrak{p} \cap \mathfrak{g} = \mathfrak{p}$. In fact $\mathfrak{p} = \Sigma_{\beta\in\widetilde{\Delta}_n} R(\widetilde{X}_\beta + \widetilde{Y}_\beta)$.

Now let γ_1 be the highest root in $\widetilde{\Delta}_n$, and, given $\gamma_1, \ldots, \gamma_k$, let γ_{k+1} be the highest root in $\widetilde{\Delta}_n$ such that $\{\gamma_1, \ldots, \gamma_{k+1}\}$ is a strongly orthogonal set (if such a root exists; if not, the process terminates). Let $\{\gamma_1, \ldots, \gamma_q\}$ be the full sequence of strongly orthogonal roots obtained in this manner. Let $\widetilde{\mathfrak{a}} = \Sigma_{i=1}^{q} R(\widetilde{X}_{\gamma_i} + \widetilde{Y}_{\gamma_i})$. Clearly $\widetilde{\mathfrak{a}}$ is commutative. To show that $\widetilde{\mathfrak{a}}$ is maximal commutative in $\mathfrak{p}$, consider any element X of $\mathfrak{p}$.

$$X = \sum_{\beta\in\widetilde{\Delta}_n} t_\beta(\widetilde{X}_\beta + \widetilde{Y}_\beta),$$

and assume that X commutes with $\widetilde{\mathfrak{a}}$ but $X \notin \widetilde{\mathfrak{a}}$. Let r be the smallest index such that $t_\beta \neq 0$ for some β with $\{\gamma_1, \ldots, \gamma_r, \beta\}$ not strongly orthogonal. Then in $[X, \widetilde{X}_{\gamma_r} + \widetilde{Y}_{\gamma_r}] = 0$ we must have

$$t_\beta[\widetilde{X}_\beta, \widetilde{X}_{\gamma_r}] = t_{2\gamma_r-\beta}[\widetilde{X}_{2\gamma_r-\beta}, \widetilde{Y}_{\gamma_r}] \neq 0.$$

But then either $\gamma_r < \beta \in \widetilde{\Delta}_n$ or $\gamma_r < 2\gamma_r - \beta \in \widetilde{\Delta}_n$. Thus either $\{\gamma_1, \ldots, \gamma_{r-1}, \beta\}$ or $\{\gamma_1, \ldots, \gamma_{r-1}, 2\gamma_r - \beta\}$ is a set of roots which is not strongly orthogonal. But we assumed that r was the minimal index for which such a set could be constructed.

Now we can show by a computation similar to (4) that

$$\exp\left(\mathrm{ad}_{\mathfrak{g}C}\left(\frac{\pi i}{4}\sum_{i=1}^{q} B(\widetilde{X}_{\gamma_i}, \widetilde{Y}_{\gamma_i})^{-1/2}(\widetilde{X}_{\gamma_i} - \widetilde{Y}_{\gamma_i})\right)\right)(\widetilde{\mathfrak{a}}) \subset i\widetilde{\mathfrak{h}}.$$

We can therefore view the γ_i as roots of the conjugate of $i\widetilde{\mathfrak{h}}$,

$$\exp\left(\mathrm{ad}_{\mathfrak{g}C}\left(-\frac{\pi i}{4}\sum_{i=1}^{q} B(\widetilde{X}_{\gamma_i}, \widetilde{Y}_{\gamma_i})^{-1/2}(\widetilde{X}_{\gamma_i} - \widetilde{Y}_{\gamma_i})\right)\right)(i\widetilde{\mathfrak{h}}),$$

which is of the form $\widetilde{\mathfrak{a}} + i\mathfrak{h}_+$, $\mathfrak{h}_+ \subset \mathfrak{k}$. The γ_i vanish on $\mathfrak{h}_+$ and can therefore be regarded as forming a basis of $\widetilde{\mathfrak{a}}^*$. Any given maximal commutative subspace $\mathfrak{a}$ of $\mathfrak{p}$ is $\mathrm{Int}(\mathfrak{k})$-conjugate to $\widetilde{\mathfrak{a}}$.

3. **The Horn-Thompson-Kostant decomposition.** Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, and $\mathfrak{a}^*$ be as in §2, and assume that $\mathfrak{a}^*$ has a basis of strongly orthogonal restricted roots (not necessarily roots). Let X_i and Z_i be as in Proposition 2.1, and let $\mathfrak{n}_0 = \sum_{i=1}^q RX_i$. Then $\mathfrak{n}_0$ is a commutative subalgebra of $\mathfrak{g}$.

Now let G be any analytic group having Lie algebra $\mathfrak{g}$. Let K, A, and N_0 be the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}_0$, respectively.

3.1. Proposition. *The element* $\exp \sum_{i=1}^q h_i Z_i$ *of* A *belongs to the same coset in* $K\backslash G/K$ *as the element* $\exp 2\sum_{i=1}^q \sinh h_i X_i$ *of* N_0.

Proof. Because of Corollary 2.2, it is enough to prove the proposition for $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{R})$. Because the center of G is contained in K, it is enough to prove the proposition for one analytic group having Lie algebra $\mathfrak{sl}(2, \boldsymbol{R})$; say, for $G = SL(2, \boldsymbol{R})$.

In $SL(2, \mathfrak{R})$, since

$$(\exp hZ)^t(\exp hZ) = \begin{pmatrix} e^h & 0 \\ 0 & e^{-h} \end{pmatrix}\begin{pmatrix} e^h & 0 \\ 0 & e^{-h} \end{pmatrix} = \begin{pmatrix} e^{2h} & 0 \\ 0 & e^{-2h} \end{pmatrix}$$

is similar to

$$(\exp 2 \sinh X)^t(\exp 2 \sinh hX) = \begin{pmatrix} 1 & 2\sinh h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 \sinh h & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 2\sinh^2 h & 2\sinh h \\ 2 \sinh h & 1 \end{pmatrix},$$

$\exp hZ$ and $\exp 2 \sinh X$ belong to the same double coset of $K = SO(2)$.

3.2. Corollary. *We have the decomposition* (*announced in* [8])

$$G = KN_0K. \tag{5}$$

Proof. The corollary follows from Proposition 3.1 and the well-known decomposition of Cartan $G = KAK$ [8, (4.2.8)].

The decomposition (5) was called by Barker the Thompson-Kostant decomposition. Kostant later added the name Horn upon discovering that Thompson's result for $SL(2, \boldsymbol{R})$, later generalized by Kostant, had previously been discovered by Horn.

3.3. Corollary. *The Haar integral on* G *is given* (*up to normalization by a constant factor*) *by the formula*

$$\int_G f(g)\, dg = \int_K \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_K f\left(k_1 \exp \sum_{i=1}^q t_i X_i k_2\right) \cdot J(t_1, \ldots, t_q)\, dk_1 dt_1 \cdots dt_q dk_2, \tag{6}$$

where $dk_1 = dk_2$ is the Haar measure on K and J is defined by (2) (*and given, for G simple, by Table* 1).

PROOF. The corollary follows from Proposition 3.1 and the well-known formula [4, Chapter X, Proposition 1.17]

$$\int_G f(g)\,dg = \int_K \int_{\mathfrak{a}} \int_K f(k_1 \exp Hk_2) \prod_{\alpha \in P} |\sinh \alpha(H)|^{\frac{1}{2} m_\alpha} dk_1\, dH\, dk_2,$$

where dH is Lebesgue measure on the Euclidean space $\mathfrak{a}$.

Kostant conjectured that the Jacobian appearing in (6) would be a polynomial. We see from Corollary 1.7 and Proposition 2.3 that Kostant's conjecture is true precisely when $\mathfrak{k}$ contains a maximal commutative subalgebra of $\mathfrak{g}$, the case for which Kostant stated in [8, (5.1.1)] the decomposition (5).

We conclude this section by computing the radial part on N_0 of the Casimir operator Ω of G, which will be useful in the next section.

3.4. COROLLARY. *If f is any smooth K-bi-invariant function on G, then*

$$\Omega f\left(\exp \sum_{i=1}^{q} t_i X_i\right) = \left[\sum_{i=1}^{q} \frac{\langle \gamma_i, \gamma_i \rangle}{4} \left\{(t_i^2 + 4)\frac{\partial^2}{\partial t_i^2} + \left[2t_i + (t_i^2 + 4)\frac{\partial \log J}{\partial t_i}\bigg|_{\exp \Sigma_{i=1}^{q} t_i X_i}\right] \frac{\partial}{\partial t_i}\right\}\right] \cdot \left(f\left(\exp \sum_{i=1}^{q} t_i X_i\right)\right),$$

wherever $J(t_1, \ldots, t_q) \neq 0$.

PROOF. The corollary follows from Proposition 3.1 and Helgason's formula for the radial part of Ω on A (as in [5, Theorem 3.3]); namely, for $H \in \mathfrak{a}$,

$$\begin{aligned} \Omega f(\exp H) &= D(\exp H)^{-\frac{1}{2}} \Delta_{\mathfrak{a}} [D(\exp H)^{\frac{1}{2}} f](\exp H) \\ &\quad - D(\exp H)^{-\frac{1}{2}} \Delta_{\mathfrak{a}} [D(\exp H)^{\frac{1}{2}}] f(\exp H), \end{aligned} \tag{7}$$

where $D(\exp H) = \Pi_{\alpha \in P} |\sinh \alpha(H)|^{\frac{1}{2} m_\alpha}$ and $\Delta_{\mathfrak{a}}$ is the Laplacian of the Euclidean space $\mathfrak{a}$. Formula (7) is valid wherever $D(\exp H) \neq 0$.

4. **Spherical polynomials.** Assume that G has finite center, so that K is compact.

Kostant conjectured in [8, Remark 5.1.1], that the (G, K)-spherical functions, which, due to Corollary 3.2, are determined by their values on N_0, might have a polynomial nature there. In case P is of type C_q or BC_q, we do indeed find a sequence of spherical functions whose restrictions to N_0 are polynomials in the canonical coordinates $t_1, \ldots, t_q$. These polynomials can all be expressed in terms of the hypergeometric function F. For other simple types we find that the only spherical polynomial is the constant 1.

4.1. LEMMA. *If f is a K-bi-invariant eigenfunction of Ω whose restriction to N_0 is a polynomial in the canonical coordinates $t_1, \ldots, t_q$, and if $f|_{N_0}$ has an extremal term of the form $at_1^{2n_1} \cdots t_q^{2n_q}$, then*

$$\Omega f = \sum_{i=1}^{q} [n_i^2 + \tfrac{1}{2}(d_i + 1)n_i]\langle\gamma_i, \gamma_i\rangle f,$$

where d_i is as in Proposition 1.8.

PROOF. Apply Corollary 3.4 and equate coefficients of $t^{2n_1} \cdots t^{2n_q}$.

We now introduce in $\mathfrak{a}^*$ the lexicographic ordering with respect to the ordered basis $(\gamma_1, \ldots, \gamma_q)$. With respect to that ordering we let $G = KAN$ be the Iwasawa decomposition; $\mathfrak{a}_+$ and $\mathfrak{a}_+^*$ be the positive Weyl chambers in $\mathfrak{a}$ and $\mathfrak{a}^*$, respectively; and ρ be the half-sum of the positive restricted roots with multiplicities.

4.2. LEMMA.

$$d_i + 1 \geqslant \langle 4\rho, \gamma_i\rangle/\langle\gamma_i, \gamma_i\rangle.$$

Equality holds for $i = 1$. If G is simple, equality holds only for $i = 1$.

PROOF. The inequality, as well as the equality for $i = 1$, follows from Proposition 1.8. If G is simple, then for $i \in \{1, \ldots, q\}$ there exists a finite sequence $(\delta_1, \ldots, \delta_r)$ from Δ such that $\delta_1 = \gamma_1$, $\delta_r = \gamma_i$, and $\langle\delta_j, \delta_{j+1}\rangle \neq 0$. Now let $(\delta_1, \ldots, \delta_r)$ be such a sequence of minimal length. $\langle\delta_j, \delta_{j+2}\rangle = 0$; otherwise we could obtain a shorter sequence by omitting δ_{j+1}. But now there exists a root of the form $\delta_{j+1} \pm \delta_{j+2}$, and $\langle\delta_j, \delta_{j+1} \pm \delta_{j+2}\rangle = \langle\delta_j, \delta_{j+1}\rangle \neq 0$, $\langle\delta_{j+1} \pm \delta_{j+2}, \delta_{j+3}\rangle = \pm\langle\delta_{j+2}, \delta_{j+3}\rangle \neq 0$; so we may obtain a shorter sequence by substituting $\delta_{j+1} \pm \delta_{j+2}$ for δ_{j+1} and δ_{j+2} whenever $2 \leqslant j + 1 < j + 2 \leqslant r - 1$. Therefore $r = 3$, and δ_2 is not orthogonal to either γ_1 or γ_i. By applying Weyl reflections with respect to γ_1 and γ_i, we may assume that $\langle\gamma_1, \delta_2\rangle > 0 > \langle\gamma_i, \delta_2\rangle$. Then $\delta_2 > 0$, and

$$\langle\gamma_i, \rho\rangle = \frac{1}{2}\sum_{\beta>0} \langle\gamma_i, \beta\rangle < \frac{1}{2}\sum_{\beta<0} |\langle\gamma_i, \beta\rangle| = \frac{1}{4}(d_i + 1)\langle\gamma_i, \gamma_i\rangle.$$

4.3. COROLLARY. *If f is a K-bi-invariant function whose restriction to N_0 is a polynomial in $t_1, \ldots, t_q$, and if $\Omega f = cf$, then*

$$c = \sum_{i=1}^{q} \left[n_i^2 + \tfrac{1}{2}(d_i + 1)n_i\right]\langle\gamma_i, \gamma_i\rangle$$

$$\geqslant \left\langle\rho + \sum_{i=1}^{q} n_i\gamma_i, \rho + \sum_{i=1}^{q} n_i\gamma_i\right\rangle - \langle\rho, \rho\rangle,$$

where $f|_{N_0}$ has an extremal term of the form $at_1^{2n_1} \cdots t_q^{2n_q}$ as in Lemma 4.1. *Equality holds if $n_i = 0$ for $i \geqslant 2$, and for G simple only in that case.*

PROOF. The corollary follows from Lemmas 4.1 and 4.2.

4.4. LEMMA. *If f is a K-bi-invariant function on G such that $f|_{N_0}$ is a polynomial in $t_1, \ldots, t_q$ and $e^{-\mu(H)}f(\exp H)$ is bounded away from 0 and ∞ for H in the closure of $\mathfrak{a}_+$, where μ is some element in the closure of $\mathfrak{a}_+^*$; then $\mu = 2\Sigma_{i=1}^q n_i\gamma_i$ for some nonnegative integers $n_1, \ldots, n_q$, and*

$$f\left(\exp \sum_{i=1}^q t_iX_i\right) = a_{n_1,\ldots,n_q}t_1^{2n_1} \cdots t_q^{2n_q} + \sum a_{m_1,\ldots,m_q}t_1^{2m_1} \cdots t_q^{2m_q}$$

$$\left\{m_1, \ldots, m_q \middle| \left\langle \sum_{i=1}^q (n_i - m_i)\gamma_i, \mathfrak{a}_+ \right\rangle \geqslant 0, \quad (m_1, \ldots, m_q) \neq (n_1, \ldots, n_q)\right\}$$

for some coefficients $a_{m_1,\ldots,m_q}$.

PROOF. Since f is invariant under the Weyl reflection with respect to each γ_i, $f(\exp \Sigma_{i=1}^q t_iX_i)$ is even in each t_i. The degree follows from Proposition 3.1.

We now apply Corollary 4.3 and Lemma 4.4 to the problem of determining which spherical functions have polynomial restrictions to N_0. The spherical functions on G are indexed by $\mathfrak{a}_C^*$ (modulo the Weyl group of P) and given by the formula

$$\phi_\lambda(g) = \int_K e^{(i\lambda-\rho)(H(gk))}\,dk,$$

for $\lambda \in \mathfrak{a}_C^*$, where $H(g)$ is the element of $\mathfrak{a}$ such that $g \in K \exp(H(g))N$. If $i\lambda \in \mathfrak{a}_+^* + i\mathfrak{a}^*$ we can transform the integral formula for $\phi_\lambda(a)$ (for $a \in \exp \mathfrak{a}_+$) to an integral over $\bar{N}$, the analytic subgroup of G corresponding to the sum of the negative restricted root spaces. We have, as in [6, Lemma 2.3],

$$\phi_\lambda(a) = \exp[(i\lambda - \rho)(\log a)] \int_{\bar{N}} \exp[(i\lambda - \rho)(H(a\bar{n}a^{-1}))] \exp[(-i\lambda - \rho)(H(\bar{n}))]\,d\bar{n},$$

where $d\bar{n}$ is the Haar measure on $\bar{N}$ such that $\int_{\bar{N}} \exp[-2\rho(H(n))]d\bar{n} = 1$. We see that for $e^{-\mu(\log a)}\phi_\lambda(a)$ to be bounded away from 0 and ∞ on the closure of $\mathfrak{a}_+$, we must have $\mu \in i\lambda - \rho + i\mathfrak{a}^*$. In case $i\lambda - \rho$ is in the closure of $\mathfrak{a}_+^*$, we have indeed

$$0 < c(\lambda) = \int_{\bar{N}} \exp[(-i\lambda - \rho)(H(\bar{n}))]\,d\bar{n} \leqslant \exp[(-i\lambda + \rho)(\log a)]\,\phi_\lambda(a)$$

$$\leqslant \int_{\bar{N}} \exp[-2\rho(H(\bar{n}))]\,d\bar{n} = 1.$$

Furthermore, $\Omega\phi_\lambda = (-\langle\lambda, \lambda\rangle - \langle\rho, \rho\rangle)\phi_\lambda$.

Now assume that $\phi_\lambda|_{N_0}$ is a polynomial in $t_1, \ldots, t_q$. By Lemma 4.4,

$$\phi_\lambda\left(\exp \sum_{i=1}^{q} t_i X_i\right) = a_{n_1 \ldots n_q} t_1^{2n_1} \cdots t_q^{2n_q} + \text{“lower order” terms,}$$

where $i\lambda - \rho = \Sigma_{i=1}^q n_i\gamma_i$ is in the closure of $\mathfrak{a}_+^*$. (We may have $i\lambda - \rho = 0$, $\phi_\lambda = \phi_{-i\rho} \equiv 1$.) Furthermore,

$$-\langle\lambda, \lambda\rangle - \langle\rho, \rho\rangle = \left\langle\rho + \sum_{i=1}^{q} n_i\gamma_i, \rho + \sum_{i=1}^{q} n_i\gamma_i\right\rangle - \langle\rho, \rho\rangle$$
$$= \sum_{i=1}^{q}\left[n_i^2 + \tfrac{1}{2}(d_i + 1)n_i\right]\langle\gamma_i, \gamma_i\rangle.$$

By Corollary 4.3 we must have, for G simple, $n_i = 0$ for $i \geqslant 2$.

Now by considering the asymptotic behavior at ∞ of ϕ_λ in all Weyl chambers, we conclude that $\phi_\lambda|_{N_0}$ must have an extremal term of the form $a\Pi_{i=1}^q t_i^{k_i n_i}$ whenever $\beta = \tfrac{1}{2}\Sigma_{i=1}^q k_i\gamma_i$ belongs to the Weyl group orbit of γ_1. But if P is of a simple type other than C_q or BC_q, then we may set $\beta = \tfrac{1}{2}\gamma_i + \tfrac{1}{2}\gamma_j + \tfrac{1}{2}\gamma_k + \tfrac{1}{2}\gamma_l$ for some choice of i, j, k, l. (We have assumed for convenience that γ_1 is of maximal length.) The number of the indices i, j, k, l equal to $r \in \{1, \ldots, q\}$ is either 0 or $\langle\gamma_1, \gamma_1\rangle/\langle\gamma_r, \gamma_r\rangle$. Then we must have, by comparison of eigenvalues of Ω, that

$$[n_1^2 + \tfrac{1}{2}(d_1 + 1)n_1]\langle\gamma_1, \gamma_1\rangle$$
$$= n_1^2 + \tfrac{1}{4}n_1(4 + d_i\langle\gamma_i, \gamma_i\rangle + d_j\langle\gamma_j, \gamma_j\rangle + d_k\langle\gamma_k, \gamma_k\rangle + d_l\langle\gamma_l, \gamma_l\rangle)$$
$$\geqslant [n_1^2 + (\tfrac{1}{2}d_1 + 1)n_1]\langle\gamma_1, \gamma_1\rangle,$$

whence $n_1 = 0$. (We have used that $d_1 \leqslant \min[d_i, d_j, d_k, d_l]$ and that i, j, k, l are not all equal.) We have proven the following

4.5. THEOREM. *If P is of a simple type other than C_q or BC_q, then the only spherical function on G restricting on N_0 to a polynomial in $t_1, \ldots, t_q$ is $\phi_{-i\rho} \equiv 1$.*

In case P is of type C_q or BC_q, we find the polynomial solution

$$p_n\left(\exp \sum_{i=1}^{q} t_i X_i\right)$$
$$= \frac{-2m(q-1)^2}{q(s+l+1) + 2(q-1)m} + \frac{s + l + 2(q-1)m + 1}{q(s+l+1) + 2(q-1)m}$$
$$\cdot \sum_{i=1}^{q} F(-n, \tfrac{1}{2}s + l + (q-1)m + n; \tfrac{1}{2}s + \tfrac{1}{2}l + (q-1)m + \tfrac{1}{2}; -\tfrac{1}{4}t_i^2)$$

to the differential equation on N_0 for a K-bi-invariant eigenfunction of Ω with eigenvalue $[n^2 + \frac{1}{2}(d_1 + 1)n]\langle\gamma_1, \gamma_1\rangle$. (Here s, m, and l are as in Table 1.) Since p_n is even in each t_i and symmetric in the t_i, it extends to a K-bi-invariant function on G. Now I claim that $p_n(n_0) = \phi_{-i(n\gamma_1+\rho)}(n_0)$ for $n_0 \in N_0$. For p_n is a K-bi-invariant function satisfying

$$\frac{\Omega p_n}{p_n} = \frac{\Omega\phi_{-i(n\gamma_1+\rho)}}{\phi_{-i(n\gamma_1+\rho)}} \quad \text{and} \quad 0 \leqslant p_n \leqslant \phi_{-i(n\gamma_1+\rho)}.$$

Since $\phi_{-i(n\gamma_1+\rho)}$ is a minimal K-bi-invariant eigenfunction of Ω (see [7]), $p_n = k\phi_{-i(n\gamma_1+\rho)}$ for some $k \in [0, 1]$. But $p_n(e) = \phi_{-i(n\gamma_1+\rho)}(e) = 1$. Therefore $k = 1$. We have proven the following theorem.

4.6. THEOREM. *If P is of type C_q or BC_q, then the spherical functions on G restricting on N_0 to polynomials in $t_1, \ldots, t_q$ are precisely*

$$\phi_{-i(n\gamma_1+\rho)}\left(\exp \sum_{i=1}^{q} t_i X_i\right)$$

$$= \frac{-2m(q-1)^2}{q(s+l+1)+2(q-1)m} + \frac{s+l+2(q-1)m+1}{q(s+l+1)+2(q-1)m}$$

$$\cdot \sum_{i=1}^{q} F(-n, \tfrac{1}{2}s + l + (q-1)m + n;\ \tfrac{1}{2}s + \tfrac{1}{2}l + (q-1)m + \tfrac{1}{2}; -\tfrac{1}{4}t_i^2).$$

The formula of the theorem is valid (by the same proof) for all $n \geqslant 0$ and by analytic continuation for all $n \in \boldsymbol{C}$, although $\phi_{-i(n\gamma_1+\rho)}$ is polynomial in $t_1, \ldots, t_q$ only for n a nonnegative integer. Our result includes in particular Harish-Chandra's formula for all spherical functions on a rank-one symmetric space [3].

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