ON THE DOUBLE SUSPENSION HOMOMORPHISM

BY

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ABSTRACT. This paper studies the family of unstable Adams spectral sequences, $E_2^{s,t}(S^{2n+1})$. The main results deal with the range of filtrations for which these groups stabilize and for which the groups $E_2^{s,t}(\Omega^2 S^{2n+1}, S^{2n-1})$ stabilize.

This paper is concerned only with the 2-primary homotopy of spheres and all spaces should be considered as being localized at the prime 2.

Let W_n be the fiber of the map $S^{2n-1} \to \Omega^2 S^{2n+1}$. Using the unstable Adams spectral sequence theory of Kan et al. [1], we can construct a spectral sequence for W_n. We wish to normalize so that $E_2^{1,2}(W_n) = Z_2$ and is the first nonzero group, i.e., $E^{0,t}(W_n) = 0$ for all t and $E_2^{s,t}(W_n) = 0$ for all s if $t = 0, 1$. Our main result is:

THEOREM 1. *There are natural maps*

$$E_2^{s,t}(W_1) \xrightarrow{f_1} E_2^{s,t}(W_2) \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} E_2^{s,t}(W)$$

$$\xrightarrow{f_n} \cdots \xrightarrow{f} \operatorname{Ext}_A^{s-1,t-1}(\widetilde{H}^*(P^2), Z_2)$$

so that f_n is an isomorphism for $6s > t + 20 - 4n$.

In [3], $\operatorname{Ext}_A^{s,t}(\widetilde{H}^*(P^2), Z_2)$ is calculated for $6s > t + 14$ and the E_∞ of the Adams spectral sequence is also obtained for the same range.

This result fits into the general picture in the following way. The Frudenthal suspension theorem says that $\pi_k(S^n)$ depends only on $k - n$ for $k < 2n - 1$. The EHP sequence $\pi_k(S^n) \to \pi_{k+1}(S^{n+1}) \to \pi_{j+1}(S^{2n-1})$ and the suspension theorem assert that the amount by which the suspension is not an isomorphism is stable provided $k < 4n - 3$. If we think in terms of the Adams spectral sequence then the analog of the Frudenthal result is the following result which is due to Curtis.

THEOREM (CURTIS). $E_2^{s,t}(S^n) \to E_2^{s,t}(S^{n+2})$ *is an isomorphism for $3s > t + 6 - n$ and n odd.*

This stability has implications for $\pi_k(S^n)$ for arbitrary k. An obvious cor-

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ollary of our Theorem 1 is the following analog of the "meta-stable" stability implied by the EHP sequence.

THEOREM 2. *The measure of the amount by which* $f\colon E_2^{s,t}(S^{2n-1}) \to E_2^{s,t}(S^{2n+1})$ *fails to be an isomorphism, i.e.,* $E_2^{s,t}(\Omega^2 S^{2n+1}, S^{2n-1})$, *is independent of n for* $6s > t + 20 - 4n$.

2. **Preliminary definitions.** The key tool used in this paper is the Λ-algebra [1] and [5]. We recall its definition and some of its properties. $\Lambda(n)$ is a graded Z_2-algebra generated by monomials in λ_i, $i = 0, 1, \ldots$, such that if $\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_j}$ is a basis element then (1) $i_1 \leqslant n$, (2) $2i_k \geqslant i_{k+1}$. Such monomials are called admissible. A multiplication is defined by juxtaposition. The following relations are used to change such a product into a sum of admissible monomials. Let D be a derivation with $D\lambda_i = \lambda_{i+1}$ and $D^t = D(D^{t-1})$. Then $D^t(\lambda_i\lambda_{2i+1}) = 0$ for all i and $t \geqslant 0$. In $\Lambda(n)$ there is a boundary operator d such that

$$d\lambda_n = \sum_{i=0}^{[n/2]} \binom{n-i}{i} \lambda_{n-i}\lambda_{i-1}.$$

We can bigrade $\Lambda(n)$ by assigning filtration $(1, i+1)$ to λ_i. Then $H_{s,t}(\Lambda(n), d) = E_2^{s,t}(S^{n+1})$.

The EHP sequence produces short exact sequences

$$0 \to \Lambda^{s,t}(n-1) \to \Lambda^{s,t}(n) \to \Lambda^{s-1,t-n-2}(2n) \to 0$$

where the first is the obvious inclusion and the second map is defined by forgetting λ_n when it occurs as a leading element in a basis element.

To prove Theorem 1 we wish to look at the double suspension. Let $\Lambda(W_n) = \kappa_2\Lambda(4n) \oplus \kappa_1\Lambda(4n-2)$ and assign κ_i filtration $(1, i+1)$. Then we have

$$(2.1) \qquad 0 \to \Lambda^{s,t}(2n-2) \to \Lambda^{s,t}(2n) \xrightarrow{p} \Lambda^{s,t-2n+2}(W_n) \to 0$$

where the first map is the obvious inclusion and the second map satisfies $p(\lambda_{2n}\lambda_I) = \kappa_2\lambda_I$, $p(\lambda_{2n-1}\lambda_I) = \kappa_1\lambda_I$ and, if the basis monomial λ_I starts with λ_i for $i < 2n-1$, then $p(\lambda_I) = 0$. From (2.1) we can define a boundary operator, d, in $\Lambda(W_n)$, so that the sequence (2.1) is a short exact sequence of chain complexes. Theorem 1 will be proved explicitly by proving the following.

THEOREM 3. *There is a natural sequence of chain maps*

$$\Lambda^{s,t}(W_1) \subset \Lambda^{s,t}(W_2) \subset \cdots \subset \Lambda^{s,t}(W_n) \to \cdots \to \kappa_2\Lambda \oplus \kappa_1\Lambda,$$

where the last term is associated with $\Lambda(P^2)$ *such that* $\Lambda(W_n)/\Lambda(W_{n+1})$ *has zero homology if* $6s > t + 14 - 4n$, *for* $n \geqslant 1$.

3. The chain complex $\Lambda(W_n)$. The first step in proving Theorem 3 is to determine the differential in $\Lambda(W_n)$.

PROPOSITION 3.1. $d(\kappa_2\lambda_I \oplus \kappa_1\lambda_J) = \kappa_2 d\lambda_I \oplus \kappa_1(\lambda_0\lambda_I + d\lambda_J + a)$ where $a = 0$ if $\lambda_I \in \Lambda(4n-1) \subset \Lambda(4n)$ and $a = (d\lambda_{4n+1})\lambda_{I'}$ if $\lambda_I = \lambda_{4n}\lambda_{I'}$.

The proof is long and so we delay it until the end of this section.

DEFINITION 3.2. The map $f_n \colon \Lambda(W_n) \longrightarrow \Lambda(W_{n+1})$ is given by $f_n(\kappa_2\lambda_I \oplus \kappa_1\lambda_J) = \kappa_2\lambda_I \oplus \kappa_1(\lambda_J \oplus \epsilon\lambda_{4n+1}\lambda_{I'})$ where $\epsilon = 0$ if $\lambda_I \in \Lambda(4n-1)$ and $\epsilon = 1$ if $\lambda_I = \lambda_{4n}\lambda_{I'}$.

PROPOSITION 3.3. f_n is a chain mapping.

PROOF. The proof is clear from 3.1. Indeed,

$$f(d(\kappa_2\lambda_I \oplus \kappa_1\lambda_J)) = f(\kappa_2 d\lambda_I \oplus \kappa_1(d\lambda_J \oplus \lambda_0\lambda_I \oplus \epsilon d\lambda_{4n+1}\lambda_{I'}))$$

$$= \kappa_2 d\lambda_I \oplus \kappa_1(\epsilon\lambda_{4n+1}d\lambda_{I'} \oplus d\lambda_J \oplus \lambda_0\lambda_I \oplus \epsilon(d\lambda_{4n+1})\lambda_{I'})$$

$$= \kappa_2 d\lambda_I \oplus \kappa_1(d(\lambda_J \oplus \epsilon\lambda_{4n+1}\lambda_{I'}) \oplus \lambda_0\lambda_I)$$

$$= df_n(\kappa_2\lambda_I \oplus \kappa_1\lambda_J).$$

SKETCH OF THE PROOF OF THEOREM 3. We will construct the chain complex $\Lambda(W_{n+1}/W_n)$ and find an artificial complex $\Lambda(C_n)$ which maps into $\Lambda(W_{n+1}/W_n)$. This map will be shown, using an induction hypothesis, to induce an isomorphism. This is done in §4. Then in §5 we look at an A_1 free stable complex X and show that $\Lambda(C_n)$ maps into $\Lambda(X)$ and also induces an isomorphism in an appropriate range of dimensions. Finally, we recall that $\text{Ext}_A^{s,t}(A_1, Z_2) = 0$ for a certain range of dimensions and this completes the proof.

PROOF OF PROPOSITION 3.1. We need to calculate the differential in $\kappa_2\Lambda(4n) \oplus \kappa_1\Lambda(4n-2)$. The differential is evaluated by the following sequence of maps:

$$(3.4) \quad \kappa_2\Lambda(4n) \oplus \kappa_1\Lambda(4n-2) \longrightarrow \Lambda(2n) \xrightarrow{d} \Lambda(2n) \longrightarrow \kappa_2\Lambda(4n) \oplus \kappa_1\Lambda(4n-2)$$

where the first map is given by $\kappa_i \longrightarrow \lambda_{2n+i-2}$ and the last map is p of 3.1. Thus we need to put in admissible form $(d\lambda_{2n})\lambda_I$ and $(d\lambda_{2n+1})\lambda_J$ and determine the coefficient of λ_{2n-1}. We need the following lemma.

LEMMA 3.5. $\lambda_i\Lambda(k) \subset \Lambda(k-i-1) \cup \Lambda(i)$.

PROOF. We wish to look at $\lambda_i\lambda_J$ where $\lambda_J \in \Lambda(k)$. If λ_J is also in $\Lambda(2i)$ then $\lambda_i\lambda_J$ is admissible and in $\Lambda(i)$. We suppose that $\lambda_i\lambda_J$ is not admissible as it stands. If $|J| = 1$ then $\lambda_i\lambda_j = \Sigma_{k \geqslant 2i+1} a_k\lambda_{i+j-k}\lambda_k$ and this is in $\Lambda(j-i-1)$. Suppose we have established the lemma for all i and J such that $|J| < n$. Suppose $j_1 = 2l$. Then

$$\lambda_{l-1}\lambda_J = \lambda_{l-1}\lambda_{j_1}\lambda_{J'} = \lambda_l\lambda_{j_1-1}\lambda_{J'}$$

$$\epsilon\ \lambda_l\lambda_{j_1-1}\Lambda^{n-1}(2j_1) \subset \lambda_l\Lambda^n(j_1) \subset \Lambda^{n+1}(l)$$

and this is the lemma. Suppose $j_1 = 2l + 1$. Then $\lambda_{l-1}\lambda_{j_1}\lambda_{J'} = \lambda_{l+1}\lambda_{j_1-2}\lambda_{J'} \in \lambda_{l+1}\Lambda^n(j_1 + 1) \in \Lambda(l + 1)$ which is the lemma. Now suppose we have established the lemma for $\lambda_{\bar{l}}\lambda_{\bar{J}}$ if $|\bar{J}| < n$ or if $|\bar{J}| = n$ then $\bar{j}_1 - 2\bar{i} - 1 < q$. Suppose $|J| = n$ and $j_1 - 2i - 1 = q$. Then

$$\lambda_i\lambda_j = \sum_{j_1 > k > 2i+1} a_k\lambda_{i+j_1-k}\lambda_k\lambda_{J'} + \lambda_{j_1-i-1}\lambda_{2i+1}\lambda_{J'}.$$

If $j_1 \geqslant k > 2i + 1$ then

$$\lambda_{i+j_1-k}\lambda_k\Lambda^{n-1}(2j_1) \subset \lambda_{i+j_1-k}\Lambda^n(2j_1 - k - 1).$$

Since $2j_1 - k - 1 - 2(i + j_1 - k) - 1 < q$ the induction hypothesis implies that the last expression is in $\Lambda^{n+1}(j_1 - i - 2)$. Finally

$$\lambda_{j_1-i-1}\lambda_{2i+1}\Lambda^{n-1}(2j_1) \subset \lambda_{j_1-i-1}\Lambda(2j_1 - 2i - 2) \subset \Lambda(j_1 - i - 1).$$

This completes the double induction and the proof of the lemma.

Now we can compute the coefficient of λ_{2n-1} in $(d\lambda_{2n})\lambda_I$. If I begins with $i_1 < 4n$ then

$$d(\lambda_{2n})\lambda_I = \sum_{i > 1} \binom{2n - i}{i} \lambda_{2n-i}\lambda_{i-1}\lambda_I\lambda_{2n-1}\lambda_0\lambda_I$$

$$+ \sum_{i > 1} \binom{2n - i}{i}\lambda_{2n-i}\lambda_{i-1}\lambda_I$$

and

$$\lambda_{2n-i}\lambda_{i-1}\Lambda(4n - 1) \subset \lambda_{2n-1}\Lambda(4n - i - 1) \subset \Lambda(2n - 2).$$

Thus if $\lambda_I \in \Lambda(4n - 1)$, $d(\kappa_2\lambda_I) = \kappa_2 d\lambda_I + \kappa_1\lambda_0\lambda_I$.

Using 3.5 in a similar way, we see that if $i_1 = 4n$ then

$$(d\lambda_{2n})\lambda_I = \sum_{i > 1} \binom{2n - i}{i}\lambda_{2n-i}\lambda_{i-1}\lambda_{4n}\lambda_{I'}$$

$$= \sum_{i > 1} \binom{2n - i}{i}\lambda_{2n-1}\lambda_{4n-2i+1}\lambda_{2i-1}\lambda_{I'} + c + \lambda_{2n-1}\lambda_0\lambda_I$$

where $c \in \Lambda(2n - 2)$.

LEMMA 3.6.

$$d\lambda_{4n+1} = \sum_{i > 1} \binom{2n - i}{i}\lambda_{4n-2i+1}\lambda_{2i-1}.$$

PROOF.

$$d\lambda_{4n+1} = \sum_{j \geqslant 1} \binom{4n+1-j}{j} \lambda_{4n-j}\lambda_{j-1}.$$

Thus we need to show that $\binom{4n+1-j}{j} \equiv 0 \pmod 2$ if $j \equiv 1 \pmod 2$ and $\binom{4n+1-2i}{2i} \equiv \binom{2n-i}{i} \bmod 2$. Note that if α generates $H^1(RP)$ then $Sq^j\alpha^{4n+1-j} = \binom{4n+1-j}{j}\alpha^{4n+1}$ and if κ generates $H^2(CP)$ then $Sq^{2i}\kappa^{2n-i} = \binom{2n-i}{i}\kappa^{2n}$. Since $\alpha^{4n+1} \notin \operatorname{im} Sq^1 \binom{4n+1-j}{j} \equiv 0 \ (2)$ for $j \equiv 1 \ (2)$. Since there is a stable map $f: \Sigma CP \to RP$ so that $f^*(\alpha^{2i+1}) = \kappa^i$ we see that

$$\binom{4n+1-2i}{2i} \equiv \binom{2n-i}{i} \bmod 2.$$

We return to the proof of 3.1. Note that $\lambda_{2n-1}\lambda_{4n-1}\lambda_1\lambda_{I'} = 0$ since $2(2n-1) + 1 = 4n - 1$. Thus

$$\sum_{i \geqslant 2} \binom{2n-i}{i} \lambda_{2n-1}\lambda_{4n-2i+1}\lambda_{2i-1}\lambda_{I'} = \lambda_{2n-1}(d\lambda_{4n+1})\lambda_{I'}$$

by 4.3. Hence $d(\lambda_{2n})\lambda_I = \lambda_{2n-1}(\lambda_0\lambda_I + (d\lambda_{4n+1})\lambda_{I'}) + c'$ where $c' \in \Lambda(2n-2)$. Putting all of this together, we see that $d(\lambda_{2n}\lambda_I + \lambda_{2n-1}\lambda_J) = \lambda_{2n-1}(d\lambda_J + \lambda_0\lambda_I + a) + \lambda_{2n}d\lambda_I + c''$ where $c'' \in \Lambda(2n-2)$ and this proves the proposition.

4. The chain complex $\Lambda(F_n) = \Lambda(W_{n+1}/W_n)$. Let $\Lambda(F_n)$ be the quotient chain complex of the map f_n. Then $\Lambda(F_n) = \kappa_2 \bigoplus_{i=1}^4 \lambda_{4n+i}\Lambda(8n+2i) \oplus \kappa_1\bigoplus_{i=-1}^2 \lambda_{4n+1}\Lambda(8n+2i)$ and $\Lambda(F_n)$ receives a differential from $\Lambda(W_{n+1})$. The differential is calculated by the following composite $\Lambda(F_n) \xrightarrow{i} \Lambda(W_{n+1}) \xrightarrow{d} \Lambda(W_{n+1}) \xrightarrow{p} \Lambda(F_n)$ where i is the vector space inclusion and p is the projection. The exact form of this differential is very complicated and we will not need it.

Let

$$\Lambda(C_n) = \kappa_2 \oplus [(\lambda_{4n+1} \oplus \lambda_{4n+3})\Lambda(8n-2) \oplus (\lambda_{4n+2} + \lambda_{4n+4})\Lambda(8n)]$$
$$+ \kappa_1 \oplus [(\lambda_{4n-1} \oplus \lambda_{4n+1})\Lambda(8n-2) \oplus (\lambda_{4n} + \lambda_{4n+2})\Lambda(8n)].$$

Let $g: \Lambda(C_n) \to \Lambda(F_n)$ be given by

$$g(\kappa_i(\lambda_{4n+2j}\lambda_I + \lambda_{4n+2j-1}\lambda_J)) = \kappa_i(\lambda_{4n+2j}\lambda_I \ \lambda_{4n+2j-1}(\lambda_{8n+1}\lambda_{I'} + \lambda_J))$$

for $j = 1, 2, i = 2$ and $j = 1, i = 1$ and where $\lambda_{I'}$ is zero unless $\lambda_I = \lambda_{8n}\lambda_{I'}$; $\lambda_I \in \Lambda(8n)$ and $\lambda_J \in \Lambda(8n-2)$;

$$g(\kappa_1(\lambda_{4n}\lambda_J + \lambda_{4n-1}\lambda_I)) = \kappa_1(\lambda_{4n}\lambda_J + \lambda_{4n-1}\lambda_I).$$

LEMMA 4.1. $dg \subset \operatorname{im} g$.

PROOF. The d in $\Lambda(F_n)$ is calculated by retracting $\Lambda(F_n)$ into $\Lambda(W_{n+1})$, calculating d in $\Lambda(W_{n+1})$ and projecting back to $\Lambda(F_n)$. When this is done for the image of g we get the following formulae:

$$dg\kappa_2\lambda_{4n+4}\lambda_I = \kappa_2(\lambda_{4n+4}d\lambda_I + \lambda_{4n+3}(\lambda_0\lambda_I + (d\lambda_{8n+1})\lambda_{I'}$$

$$+ \lambda_{8n+1}d\lambda_{I'}) + \lambda_{n+2}\lambda_1\lambda_I)$$

$$+ \kappa_1(\lambda_{4n+2}(\lambda_2\lambda_I + \lambda_1\lambda_{8n+1}\lambda_{I'}) + \epsilon_n\lambda_{4n+1}\lambda_3\lambda_I$$

$$+ \lambda_{4n}(\lambda_4\lambda_I + \lambda_3\lambda_{8n+1}\lambda_{I'}))$$

where $\epsilon_n \equiv (m) \mod 2$ and where $\lambda_{I'} = 0$ unless $\lambda_I = \lambda_{8n}\lambda_{I'}$.

$$d(g\kappa_2\lambda_{4n+3}\lambda_J) = \kappa_1(\lambda_{4n+2}\lambda_1 + \lambda_{4n}\lambda_3)\lambda_J + \kappa_2\lambda_{4n+3}d\lambda_J,$$

$$d(g\kappa_2\lambda_{4n+2}\lambda_I) = \kappa_2(\lambda_{4n+2}d\lambda_I + \lambda_{4n+1}(\lambda_0\lambda_I + (d\lambda_{8n+1})\lambda_{I'} + \lambda_{8n+1}d\lambda_{I'}))$$

$$+ \kappa_1(\lambda_{4n+1}\lambda_1\lambda_I + \lambda_{4n}(\lambda_2\lambda_I + \lambda_1\lambda_{8n+1}\lambda_{I'}) + \lambda_{4n-1}\lambda_3\lambda_J),$$

$$d(g\kappa_2\lambda_{4n+1}\lambda_J) = \kappa_2\lambda_{4n+1}d\lambda_J + \kappa_1\lambda_{4n}\lambda_1\lambda_J,$$

$$d(g\kappa_1\lambda_{4n+2}\lambda_I) = \kappa_1(\lambda_{4n+1}(\lambda_0\lambda_I + d(\lambda_{8n+1})\lambda_{I'}) + \lambda_{4n-1}\lambda_2\lambda_I + \lambda_{4n+2}d\lambda_I)$$

$$+ \lambda_{4n-1}(\lambda_1\lambda_{8n+1} + d\lambda_{8n+3})\lambda_{I'},$$

$$d(g\kappa_1\lambda_{4n+1}\lambda_J) = \kappa_1(\lambda_{4n+1}d\lambda_J + \lambda_{4n-1}\lambda_1\lambda_J),$$

$$d(g\kappa_1\lambda_{4n}\lambda_I) = \kappa_1(\lambda_{4n}d\lambda_I + \lambda_{4n-1}(\lambda_0\lambda_I + d\lambda_{8n+1}\lambda_{I'})),$$

$$d(g\kappa_1\lambda_{4n-1}\lambda_J) = \kappa_1\lambda_{4n-1}d\lambda_J.$$

To see this observe that $d(\kappa_j\lambda_{4n+i})\lambda_I$ in $\Lambda(W_{n+1})$ involves terms of the form $\kappa_j\lambda_{4n-p}\lambda_{p+i-1}\lambda_I$ or, if $j = 2$, terms like $\kappa_{j-1}\lambda_{4n-p}\lambda_{p+i}\lambda_I$. If $j \neq 1$ and i is neither 0 nor 2, terms such as these, when made admissible, project to zero in $\Lambda(E_n)$. Indeed, $\lambda_{4n-p}\lambda_{p+i}\lambda_I \subset \Lambda(4n - i - 1)\cup\Lambda(4n - p) \subset \Lambda(4n - 2)$ except for the above exceptions. In the case of exceptions when $i = 0$ the argument is just that of §3. When $i = 2$ we see that

$$p\kappa_1 d\lambda_{4n+1}\lambda_{8n+1} = \sum \binom{4n + 1 - i}{i}k_1\lambda_{4n-1}\lambda_{8n-2i+3}\lambda_{2i-1}.$$

Also $d\lambda_{8n+3} = \Sigma\binom{8n+3-i}{i}\lambda_{8n+3-i}\lambda_{i-1}$. The argument from Lemma 3.5 shows that these are the same and thus $pk_1 d(\lambda_{4n+1})\lambda_{8n+1} = \kappa_1\lambda_{4n-1}(\lambda_1\lambda_{8n+1} + d(\lambda_{8n+3}))$. Thus we see that in all cases the above formulae describe what happens.

It is a simple direct verification now that $dg \subset \text{im } g$. We will do the first one term by term. Suppose that $\lambda_I = \lambda_{8n}\lambda_{I'}$. The other case is easier. Consider

$$\kappa_2(\lambda_{4n+4}(d\lambda_{8n})\lambda_{I'} + \lambda_{8n}d\lambda_{I'}) + \lambda_{4n+3}(\lambda_0\lambda_{8n}\lambda_{I'} + d\lambda_{8n+1}\lambda_{I'} + \lambda_{8n+1}d\lambda_{I'}).$$

The classes $(d\lambda_{8n})\lambda_{I'} \subset \Lambda(8n-1)$ by (3.5). The class

$$\lambda_0\lambda_{8n}\lambda_{I'} + (d\lambda_{8n+1})\lambda_{I'} \subset \Lambda(8n-2)$$

by (3.4). Thus the above term is $g(\kappa_2\lambda_{4n+4}(a + \lambda_{8n}d\lambda_{I'}) + \lambda_{4n+3}(b)))$ where $a \in \Lambda(8n-1)$ and $b \in \Lambda(8n-2)$. The class $\kappa_2\lambda_{4n+2}\lambda_1\lambda_I$ is handled by noting that $\lambda_1\lambda_I \subset \Lambda(8n-2) \subset \Lambda(8n)$. Continuing with the terms of $dg\kappa_2\lambda_{4n+4}\lambda_I$ we see $\lambda_2\lambda_I \subset \Lambda(8n-3) \subset \Lambda(8n)$; $\lambda_1\lambda_{8n+1}\lambda_{I'} \subset \lambda_1\Lambda(8n+1) \subset \Lambda(8n-1)$; $\lambda_3\lambda_I \subset \Lambda(8n-4) \subset \Lambda(8n-2)$; $\lambda_4\lambda_I \subset \Lambda(8n-5) \subset \Lambda(8n)$; $\lambda_3\Lambda(8n+1) \subset \Lambda(8n-3) \subset \Lambda(8n)$. All the other cases are similarly handled. This proves the lemma.

A key step in the proof of 2.2 is the following result.

LEMMA 4.2. *For a fixed* t, *if Theorem 3 is true for all* $t' < t$, *then* g *induces an isomorphism in homology for* $6s > t + 3 - 12n$.

PROOF. We will filter the map g in the following fashion.

$$A_1 = \kappa_1(\lambda_{4n-1}\Lambda(8n-2) \oplus \lambda_{4n}\Lambda(8n)) \xrightarrow{g} \kappa_1(\lambda_{4n-1}\Lambda(8n-2) \oplus \lambda_{4n}\Lambda(8n)) = B_1,$$

$$\cap \qquad\qquad\qquad\qquad \cap$$

$$A_2 = A_1 \oplus \kappa_1(\lambda_{4n+1}\Lambda(8n-2) \oplus \lambda_{4n+2}\Lambda(8n)) \rightarrow \kappa_1 \bigoplus_{i=-1}^{2} \lambda_{4n+i}\Lambda(8n+2i) = B_2,$$

$$A_3 = A_2 \oplus \kappa_2(\lambda_{4n+1}\Lambda(8n-2) \oplus \lambda_{4n+2}\Lambda(8n))$$

$$\rightarrow B_2 \oplus \kappa_2 \bigoplus_{i=1} \lambda_{4n+i}\Lambda(8n+2i) = B_3,$$

$$A_4 = \Lambda(C_n) \rightarrow \Lambda(F_n) = B_4.$$

For the resulting spectral sequence we see that $E_0^{s,t,i}(C) = (A_{i+1}/A_i)^{s,t} = \Lambda^{s-1,t-4n-\epsilon_i}(W_{2n})$ where $\epsilon_i = 0, -2, -3, -5$ for $i = 1, 2, 3, 4$ respectively. Also $E_0^{s,t,i}(F) = (B_{i+1}/B_i)^{s,t} = \Lambda^{s-1,t-4n-\epsilon_i}(W_{2n+\delta_i})$ where ϵ_i is as above and $\delta_i = 0, 1, 1, 2$ for $i = 1, 2, 3, 4$ respectively. The map g induces

$$g_i: E_0^{s,t,i}(C) \rightarrow E_0^{s,t,i}(F)$$

and g_1 is an isomorphism, g_2 and g_3 are f_{2n} and g_4 is $f_{2n+1} \circ f_{2n}$. These are quite easily seen but let us look at g_4.

$$g_4(\kappa_2\lambda_{4n+3}a + \lambda_{4n+4}\lambda_{8n+8}b + \lambda_{4n+4}c)$$

$$= \kappa_2(\lambda_{4n+3}a + \lambda_{4n+4}\lambda_{8n+8} + \lambda_{4n+3}\lambda_{8n+9}b + \lambda_{4n+4}c)$$

and this is just what f_{2n} does. The second inclusion is just the identity.

If Theorem 3 is true for $t' < t$, then g induces an isomorphism at the E_1

level, $g_\#E_1^{s,t,i}(C) \simeq E_1^{s,t,i}(F)$ for all i if $6s > t - 12n + 24$. Thus g_∞ is an iso-morphism for all i, if $6s > t - 12n + 30$.

This proves the lemma.

5. **A second auxilary complex.** The complex $\Lambda(C_n)$ does not represent any-thing geometric that has been identified. It was introduced because it also is comparable with an identifiable stable complex.

The following is an easy exercise in stable homotopy.

PROPOSITION 5.1. *Let A_1 be the subalgebra of A, the Steenrod algebra, gen-erated by Sq^1 and Sq^2. There is a space X such that $H^*(X)$ is a free module over A_1 on one generator x.*

PROOF. Take $K(Z_2, n)$ for $n \geqslant 6$ and kill Sq^4, $Sq^4 Sq^2$ and everything in dimension above $n + 6$. The resulting space is X.

There is a choice of X so that $Sq^4 Sq^2 x = 0$ and one in which $Sq^4 Sq^2 x = Sq^3 Sq^3 x$. Let X_1 have $Sq^4 Sq^2 x \neq 0$ and X_2 have $Sq^4 Sq^2 x = 0$. In both X_k we require $Sq^6 x = 0$.

PROPOSITION 5.2. $\Lambda(X_k) = \bigoplus_{i=1}^{2} \bigoplus_{j=2i-3}^{2i} \kappa_{i,j}\Lambda$ *with*

$$d(\kappa_{2,4}) = \kappa_{2,3}\lambda_0 + \kappa_{2,2}\lambda_1 + \kappa_{1,2}\lambda_2 + (k)_{\mathrm{mod}\,2}\kappa_{1,1}\lambda_3 + \kappa_{1,0}\lambda_4,$$

$$d(\kappa_{2,3}) = \kappa_{1,2}\lambda_1 + \kappa_{1,0}\lambda_3,$$

$$d(\kappa_{2,2}) = \kappa_{2,1}\lambda_0 + \kappa_{1,1}\lambda_1 + \kappa_{1,0}\lambda_2 + \kappa_{1,-1}\lambda_3,$$

$$d(\kappa_{2,1}) = \kappa_{1,0}\lambda_1,$$

$$d(\kappa_{1,2}) = \kappa_{1,1}\lambda_0 + \kappa_{1,-1}\lambda_2,$$

$$d(\kappa_{1,1}) = \kappa_{1,-1}\lambda_1,$$

$$d(\kappa_{1,0}) = \kappa_{1,-1}\lambda_0.$$

PROOF. The L.C.S. E_1 term for a stable complex is given by $H_*(X; Z_2) \otimes \Lambda$ and the differential is given by $d(a \otimes 1) = \Sigma_i Sq_*^i a \otimes \lambda_{i-1}$ where Sq_*^i: $H_j(X; Z_2) \longrightarrow H_{j-i}(X; Z_2)$ is the dual Steenrod square. A direct check of the squaring operations in A_1 gives the result. The following picture may help the reader. Each 0 represents a cell and 0–0 represents Sq^1 and $\overset{\frown}{0\ 0}$ represents Sq^2. Note that there are several other Sq^4's nonzero in the complex. Since $Sq^5 + Sq^4 Sq^1 = Sq^2 Sq^3$ and $Sq^2 Sq^3 \neq 0$ and $Sq^1 Sq^4 x = 0$, we see that $Sq^4 Sq^1 x \neq 0$. Since $Sq^6 = Sq^5 Sq^1 + Sq^2 Sq^4$ and $Sq^5 Sq^1 \neq 0$ we see that $Sq^6 = 0$ im-plies $Sq^4 x \neq 0$. These are reflected in the differentials given above.

Let $\bar{g}: \Lambda(C_n) \longrightarrow \Lambda(X_{(n)})$, where (n) is the congruence class of n mod 2, be given by:

$$\bar{g}\kappa_j\lambda_{4n+2i}\lambda_I = \kappa_{j,2i}\lambda_I + \kappa_{2,2i-1}\lambda_{8n+1}\lambda_{I'}, \quad j = 2, i = 1, 2; j = 1, i = 0,$$

$$\bar{g}\kappa_1\lambda_{4n+2}\lambda_I = \kappa_{1,2}\lambda_I + \kappa_{1,1}\lambda_{8n+1}\lambda_{I'} + \kappa_{1,-1}\lambda_{8n+3}\lambda_{I'},$$

where $\lambda_{I'} = 0$ unless $\lambda_I = \lambda_{8n}\lambda_{I'}$; and $\bar{g}\kappa_j\lambda_{4n+2i-1}\lambda_I = \kappa_{j,2i-1}\lambda_I$.
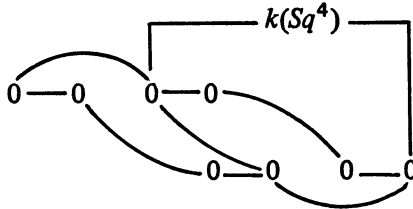

FIGURE 1. $\tilde{H}^*(X)$

PROPOSITION 5.3. $\bar{g}$ *is a chain map.*

PROOF. This is a direct comparison of the two sets of formulae. Analogously to Lemma 4.2 we have

LEMMA 5.4. *For a fixed* t, *if Theorem 3 is true for* $t' < t$, *then* $\bar{g}$ *induces an isomorphism in homology for* $6s > t + 3 - 12n$.

The proof follows closely to that of Lemma 4.2.

6. Proof of Theorem 3. The last step in the proof of Theorem 3 is the following:

PROPOSITION 6.1 [3, COROLLARY 4]. $H_{s,t}(\Lambda(X_{(n)}, d)) = 0$ *if* $6s > t - 4n + 14$.

Now the proof of Theorem 3 follows easily. First note that Theorem 3 is true if $t = 1$. Then note that the case $n = 1$ is not needed in the induction and thus

$$\{(s, t); 6s > t + 30 - 12n\} \supset \{(s, t); 6s > t - 14n + 14\} \quad \text{if } n > 1.$$

The first is when $H_{s,t}(\Lambda(X_{(n)}, d)) \simeq E_2^{s,t}(F_n)$ (4.2 and 5.4) and the second is when the left-hand side is isomorphic to zero (6.1).

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