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AN ORDER TOPOLOGY IN ORDERED TOPOLOGICAL VECTOR SPACES(1)

BY

LYNE H. CARTER

ABSTRACT. An order topology Ω that can be defined on any partially-ordered space has as its closed sets those that contain the (o)-limits of all their (o)-convergent nets. In this paper we study the situation in which a topological vector space with a Schauder basis is ordered by the basis cone. In a Fréchet space (E, τ), we obtain necessary and sufficient conditions both for $\tau \subset \Omega$ and for $\tau = \Omega$. Characterizations of (o)- and Ω-convergence and of Ω-closed sets are obtained. The equality of the order topology with the strong topology in certain dual Banach spaces is related to weak sequential completeness through the concept of a shrinking basis.

I. Definitions. Throughout this paper, (E, τ) will denote a real, Hausdorff topological vector space (t.v.s.). A sequence $\{x_i\}_{i=1}^{\infty} \subset E$ is called a $(\tau$-$)$ *basis* for E if each vector $x \in E$ is uniquely expressible as a τ-convergent sum $x = \sum_{i=1}^{\infty} a_i x_i$, with real coefficients a_i. From the uniqueness of this expansion, we may define the *ith coefficient functional* $f_i(x) = a_i$; $\{x_i, f_i\}_{i=1}^{\infty}$ is then a *biorthogonal system*, i.e. $f_i(x_j) = \delta_{ij}$. For brevity, we shall refer to "the basis $\{x_i, f_i\}$." If each $f_i \in E'$ (the topological dual), we call $\{x_i, f_i\}$ a *Schauder basis.* Where E has the basis $\{x_i, f_i\}$, the *basis cone* K of E is the set $K = \{x \in E : f_i(x) \geqslant 0 \text{ for all } i\}$; we will order E by this cone, i.e. we define $x \geqslant y$ iff $f_i(x) \geqslant f_i(y)$ for all i. With this ordering E is an *ordered vector space* (o.v.s.); for definitions and basic terminology see [13] or [17]. We will call K an *unconditional cone* if $\sum_{i=1}^{\infty} f_i(x) x_i$ converges unconditionally to x for each $x \in K$. McArthur [10, Lemma 6] shows that if (E, τ) is a locally convex t.v.s. ordered by the *normal* cone K of a Schauder basis, then K is unconditional; in Fréchet spaces, the unconditionality of K implies its normality [10, Theorems 1 and 4]. (Recall that in a locally convex t.v.s. (E, τ) ordered by a cone K, K is τ-normal iff τ is generated by "K-monotone" seminorms $\{p_\alpha\}$, i.e. seminorms with the property that $\theta \leqslant x \leqslant y$ implies $p_\alpha(x) \leqslant p_\alpha(y)$ for each α [13, p. 63].) (θ will always denote the zero vector.)

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(1) The results in this paper were obtained during the author's doctoral study under the direction of Professor C. W. McArthur.

A net $\{x_\alpha : \alpha \in A\}$ in an o.v.s. E is said to *(o)-converge* to a vector $x \in E$ (denoted $x_\alpha \xrightarrow{(o)} x$) iff there is an increasing net $\{z_\gamma : \gamma \in C\}$ and a decreasing net $\{y_\beta : \beta \in B\}$ satisfying (i) $\sup_\gamma z_\gamma = x = \inf_\beta y_\beta$ (denoted "$z_\gamma \uparrow x$" and "$y_\beta \downarrow x$"), and (ii) for each $\beta \in B$ and $\gamma \in C$, there exists $\alpha_0 \in A$ with $z_\gamma \leqslant x_\alpha \leqslant y_\beta$ for $\alpha \geqslant \alpha_0$. Several facts about (o)-convergence in a general o.v.s. follow (for proofs in Riesz spaces see [17]).

LEMMA 1. (i) *(o)-limits are unique.*

(ii) *If $x_\alpha \xrightarrow{(o)} x$, any cofinal subnet of $\{x_\alpha\}$ also (o)-converges to x.*

(iii) *$x_\alpha \xrightarrow{(o)} x$ if $(x_\alpha - x) \xrightarrow{(o)} \theta$ and $cx_\alpha \xrightarrow{(o)} cx$ for each real number c.*

(iv) *If $\{x_\alpha\}$ is increasing (resp. decreasing), then $x_\alpha \xrightarrow{(o)} x$ iff $x_\alpha \uparrow x$ (resp. $x_\alpha \downarrow x$).*

(v) *If $x_\alpha \downarrow x$ and $y_\alpha \downarrow y$, then $(x_\alpha + y_\alpha) \downarrow (x + y)$. (The "dual" statement for increasing nets is also true.)*

II. Comparison of the order topology with the original topology in a t. v. s. with a basis. Using (i) and (ii) of Lemma 1, Vulih [17] shows that one gets a (T_1) topology Ω in any o.v.s. E as follows: a subset $F \subset E$ is called Ω-closed (or *order-closed*) iff every (o)-convergent net of elements of F (o)-converges to an element of F. We shall call Ω the *order topology* on E. It should be noted that although in some cases (o)-convergence coincides with convergence with respect to Ω (henceforth denoted Ω-convergence), in general all that is true is that the (o)-convergence of a net implies its Ω-convergence. We introduce an intermediate concept: a net $\{x_\alpha\} \subset E$ is said to *star-converge* to $x \in E$ (denoted $x_\alpha \xrightarrow{(*)} x$) provided every cofinal subnet of $\{x_\alpha\}$ has a cofinal subnet which (o)-converges to x. From (ii) of Lemma 1, (o)-convergence implies $(*)$-convergence, and Vulih [17, p. 35] proves that $(*)$-convergence implies Ω-convergence. We will see that for *sequences* in a t.v.s. E ordered by a basis cone, $(*)$- and Ω-convergence coincide. First, we state the following useful lemma whose proof is straightforward.

LEMMA 2. *Let (E, τ) be a t.v.s. ordered by the cone K of a basis $\{x_i, f_i\}$. Then for a decreasing net $\{y_\alpha\}_{\alpha \in A} \subset K$, we have $y_\alpha \downarrow \theta$ iff $f_i(y_\alpha) \underset{\alpha}{\rightarrow} 0$ for each i.*

PROPOSITION 1. *With notation as in Lemma 2, let $y_\alpha \downarrow \theta$; then there is an increasing sequence $\{\alpha_n\}_{n=1}^\infty \subset A$ with $y_{\alpha_n} \downarrow \theta$.*

PROOF. From Lemma 2, $f_i(y_\alpha) \downarrow 0$ for each i. Choose α_1 so that $f_1(y_{\alpha_1}) < 1$. Inductively, if $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n$ are chosen so that $f_i(y_{\alpha_k}) < 1/k$ for $1 \leqslant i \leqslant k$, $1 \leqslant k \leqslant n$, choose $\alpha_{n+1} \geqslant \alpha_n$ so that $f_i(y_{\alpha_{n+1}}) < 1/(n+1)$ for $1 \leqslant i \leqslant n+1$. Then we claim that $y_{\alpha_n} \downarrow \theta$: If $\epsilon > 0$ and i are fixed, choose $n_0 > i$

so that $1/n_0 < \epsilon$. Then $f_i(y_{\alpha_n}) < 1/n \leqslant 1/n_0 < \epsilon$ for $n \geqslant n_0$. Therefore $f_i(y_{\alpha_n}) \xrightarrow[n]{} 0$ for each i, so $y_{\alpha_n} \downarrow \theta$ from Lemma 2.

COROLLARY 1. *For (E, τ) as above, if $x_\alpha \xrightarrow{(o)} x$, there is an increasing sequence $\{\alpha_n\}$ such that $x_{\alpha_n} \xrightarrow{(o)} x$.*

COROLLARY 2. *For (E, τ) as above, the Ω-closed sets are those which contain the (o)-limits of their (o)-convergent sequences.*

Luxemburg and Zaanen [9] *define* the Ω-closed sets in an o.v.s. in terms of sequences as in Corollary 2. As they show, the open sets for Ω are then those sets U which have the following property: whenever $x_n \xrightarrow{(o)} x \in U$, there exists n_0 with $\{x_n : n \geqslant n_0\} \subset U$. Also, it is not hard to see that the Ω-continuous real-valued functions f on E are those with the property "$x_n \xrightarrow{(o)} x \Rightarrow f(x_n) \longrightarrow f(x)$." The following is another very useful fact in our setting:

PROPOSITION 2. *In any t.v.s. ordered by a basis cone, Ω-convergence and $(*)$-convergence coincide for sequences.*

PROOF. Vulih [17, pp. 159–160] proves this result where the space involved is a Dedekind complete Riesz space of countable type; this hypothesis is only used to guarantee the situation described in Corollary 2 to Proposition 1 above.

An example may be appropriate to distinguish between (o)- and $(*)$-convergence. Letting $\{e_n\}$ denote the usual unit vector basis for l^1 (the Banach space of all real, absolutely summable sequences) and giving l^1 the corresponding basis ordering, it is easy to see that the sequence $\{(1/n)e_n\}$ cannot (o)-converge in l^1 (for it is not bounded above). However, $(1/n)e_n \xrightarrow{(*)} \theta$: for any subsequence $\{(1/n_k)e_{n_k}\}$, choose a summable subsequence $\{1/n_{k_j}\}$ of $\{1/n_k\}$ and it follows that $(1/n_{k_j})e_{n_{k_j}} \xrightarrow{(o)} \theta$ (a "dominating", decreasing sequence is $\{\Sigma_{j=i}^\infty (1/n_{k_j})e_{n_{k_j}} : i = 1, 2, \ldots\}$).

Recall that the *positive part, negative part,* and *absolute value* of an element x of an o.v.s. E are given by, resp., $x^+ = x \vee \theta$, $x^- = (-x)^+ = -(x \wedge \theta)$, and $|x| = x^+ + x^-$, whenever these elements exist in E. Given their existence, we also have $x = x^+ - x^-$.

LEMMA 3. *Let (E, τ) be a sequentially complete, locally convex t.v.s. ordered by a Schauder basis $\{x_i, f_i\}$ having an unconditional basis cone K. For any $x \in E$ satisfying $z \leqslant x \leqslant y$ for some $y \in K$, $z \in (-K)$, we have that x^+ and x^- exist in E and $x^+ \leqslant y$, $x^- \leqslant -z$.*

PROOF. Since $x = \Sigma_{i=1}^\infty f_i(x)x_i$, the existence of x^+ is implied by the convergence of the series $\Sigma_{i=1}^\infty [f_i(x) \vee 0]\, x_i$, for this will then be the basis expansion for x^+. Since K is unconditional and for each i we have $0 \leqslant f_i(x) \vee 0 \leqslant f_i(y)$, for each i there exists $\epsilon_i \in [0, 1]$ with $f_i(x) \vee 0 = \epsilon_i f_i(y)$, and $\Sigma_{i=1}^{\infty} \epsilon_i f_i(y) x_i$ converges [10, Theorem 1]. Clearly $x^+ \leqslant y$. The statement for x^- now follows, since $-x \leqslant -z$ and $x^- = (-x)^+$.

In 1951, Kantorovič, Vulih, and Pinsker [7] asked for sufficient conditions on an ordered t.v.s. (E, τ) so that the topology Ω generated by the ordering will coincide with the "original" topology τ. Ceĭtlin's 1966 paper [1] gave an answer to this question in our basis-ordered setting (our Theorem 2). In comparing Ω with the original vector topology in a space, the following lemma is fundamental.

Lemma 4. *In an o.v.s. E, Ω is the finest topology τ on E for which (o)-convergence is stronger than τ-convergence, i.e. such that $x_\alpha \xrightarrow{(o)} \theta$ implies that $x_\alpha \xrightarrow{\tau} \theta$. (If E is a basis-ordered t.v.s., a sequence $\{x_n\}$ can be used here instead of the net $\{x_\alpha\}$.)*

Proof. Let τ be any such topology and F any τ-closed set. Let $\{x_\alpha\} \subset F$ with $x_\alpha \xrightarrow{(o)} x$. Then $x_\alpha \xrightarrow{\tau} x$ by assumption, so $x \in F$. Therefore F is Ω-closed. Therefore $\tau \subset \Omega$. (Corollary 2 to Proposition 1 allows us to use sequences here if E is a basis-ordered space.)

A theorem of Ceĭtlin [1, Theorem 1] states that if (E, τ) is a sequentially complete, bornological t.v.s. ordered by the cone of an unconditional Schauder basis, then $\tau \subset \Omega$. Our next two results generalize this theorem.

Theorem 1. *Let (E, τ) be a sequentially complete, locally convex t.v.s. ordered by the τ-normal cone K of a Schauder basis $\{x_i, f_i\}$. Then τ is weaker than Ω, i.e. $\tau \subset \Omega$. The converse is true for Fréchet spaces, i.e. if $\tau \subset \Omega$, then K is τ-normal.*

Proof. Let $\{p_\alpha\}_{\alpha \in A}$ be a defining family of K-monotone seminorms for τ, and first assume $y_n \downarrow \theta$. Fix $\epsilon > 0$ and $\alpha \in A$. Choose n so that

$$p_\alpha\left(\sum_{i=n+1}^{\infty} f_i(y_1) x_i\right) < \frac{\epsilon}{2}.$$

Note that for all k we have $\Sigma_{i=n+1}^{\infty} f_i(y_k) x_i \leqslant \Sigma_{i=n+1}^{\infty} f_i(y_1) x_i$, and therefore $p_\alpha(\Sigma_{i=n+1}^{\infty} f_i(y_k) x_i) < \epsilon/2$. Also there exists k_0 such that $k \geqslant k_0, 1 \leqslant i \leqslant n, p_\alpha(x_i) \neq 0$ implies that $|f_i(y_k)| \leqslant \epsilon/2np_\alpha(x_i)$. Then $k \geqslant k_0$ implies that

$$p_\alpha(y_k) \leqslant p_\alpha\left(\sum_{i=1}^{n} f_i(y_k) x_i\right) + p_\alpha\left(\sum_{i=n+1}^{\infty} f_i(y_k) x_i\right) < \sum_{i=1}^{n} \frac{\epsilon}{2n} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $p_\alpha(y_k) \xrightarrow[k]{} 0$ for each α.

Now assume only that $w_n \xrightarrow{(o)} \theta$, so there are sequences $z_n \uparrow \theta, y_n \downarrow \theta$ such that for each n, there exists n_0 with $z_n \leqslant w_k \leqslant y_n$ for $k \geqslant n_0$. From the first case above, $p_\alpha(-z_n) = p_\alpha(z_n) \xrightarrow[n]{} 0$ and $p_\alpha(y_n) \xrightarrow[n]{} 0$ for each α. Fix $\epsilon > 0$, $\alpha \in A$, and choose n so that $p_\alpha(z_n) < \epsilon/2$, $p_\alpha(y_n) < \epsilon/2$. From Lemma 3, there exists n_0 such that w_k^+ and w_k^- exist and satisfy $w_k^+ \leqslant y_n$, $w_k^- \leqslant -z_n$, for $k \geqslant n_0$. Then for $k \geqslant n_0$ we have $p_\alpha(w_k) = p_\alpha(w_k^+ - w_k^-) \leqslant p_\alpha(w_k^+) + p_\alpha(w_k^-) \leqslant p_\alpha(y_n) + p_\alpha(z_n) < \epsilon$, so $p_\alpha(w_k) \xrightarrow[k]{} 0$. From Lemma 4, we conclude $\tau \subset \Omega$.

To see the converse for Fréchet spaces, let F denote the family of finite subsets of the positive integers (ordered by inclusion) and let $x \in K$. Clearly $x = \sup\{\Sigma_{i\in\sigma} f_i(x)x_i : \sigma \in F\}$ and since $\{\Sigma_{i\in\sigma} f_i(x)x_i : \sigma \in F\}$ is an increasing net in E, we have $\Sigma_{i\in\sigma} f_i(x)x_i \xrightarrow[\sigma]{(o)} x$, which implies $\Sigma_{i\in\sigma} f_i(x)x_i \xrightarrow[\sigma]{\tau} x$ by Lemma 4. This means that $\Sigma_{i=1}^\infty f_i(x)x_i$ is unordered (and therefore unconditionally) convergent to x, for each $x \in K$. Therefore K is an unconditional cone and so K is τ-normal since (E, τ) is a Fréchet space [10, Theorems 1 and 4].

Ceĭtlin [1, Proposition 1] proves that if (E, τ) is a sequentially complete, locally convex t.v.s. ordered by the cone of an unconditional Schauder basis, then E is a Dedekind complete Riesz space, and if (E, τ) is also a bornological space, then τ is defined by *lattice seminorms* $\{p_\alpha\}_{\alpha\in A}$, i.e. if $|x| \leqslant |y|$ then $p_\alpha(x) \leqslant p_\alpha(y)$ for each $\alpha \in A$ [1, Proposition 2]. Hofler [4, Theorem 1] gives a proof of the following generalization of this last result; we give an original, more constructive proof based on Ceĭtlin's argument.

PROPOSITION 3. *Let (E, τ) be a sequentially complete, barrelled t.v.s. ordered by the cone K of an unconditional Schauder basis $\{x_i, f_i\}$. Then τ is defined by lattice seminorms.*

PROOF. Let $\{p_\alpha\}_{\alpha\in A}$ be a family of seminorms defining τ. For each $x \in E$, define $p'_\alpha(x) = \sup\{p_\alpha(y) : |y| \leqslant |x|\}$. It is not hard to see that each p'_α is a lattice seminorm on E that dominates p_α, so the topology τ' defined by the system $\{p'_\alpha\}_{\alpha\in A}$ is stronger than τ. To show $\tau' \subset \tau$, we will show that the seminorms p'_α are all τ-continuous. We will utilize the family of operators $T_{(\epsilon_i)}$, where (ϵ_i) is always a sequence of real numbers with $|\epsilon_i| \leqslant 1$ for all i, defined by $T_{(\epsilon_i)}(x) = \Sigma_{i=1}^\infty \epsilon_i f_i(x)x_i$. For each n, define $T^n_{(\epsilon_i)} = T_{(\epsilon'_i)}$, where $\epsilon'_i = \epsilon_i$ for $i \leqslant n$ and $\epsilon'_i = 0$ for $i > n$. Then $\{T^n_{(\epsilon_i)}\}_{n=1}^\infty$ is pointwise bounded in (E, τ) for each (ϵ_i), since it is pointwise weakly bounded: $|f(T^n_{(\epsilon_i)}(x))| \leqslant \Sigma_{i=1}^\infty |f_i(x)f(x_i)|$ for each $f \in E'$. So since (E, τ) is barrelled, the Banach-Steinhaus theorem yields that $T_{(\epsilon_i)} = \lim_n T^n_{(\epsilon_i)}$ is continuous for all (ϵ_i) and that the family of all $T_{(\epsilon_i)}$ is equicontinuous. For each neighborhood V of θ, choose a neighborhood U_V of θ such that $T_{(\epsilon_i)}(U_V) \subset V$ for all (ϵ_i). Then for $\alpha \in A$, $\epsilon > 0$, there is a neighborhood V_ϵ of θ with $p_\alpha(V_\epsilon) \subset (-\epsilon, \epsilon)$, so $\sup\{p_\alpha(T_{(\epsilon_i)}(x)) : x \in U_{V_\epsilon}, (\epsilon_i) \text{ arbitrary}\} \leqslant \epsilon$. But for $x \in U_{V_\epsilon}$, we have $p'_\alpha(x) = \sup_{(\epsilon_i)} p_\alpha(T_{(\epsilon_i)}(x))$ so we have $p'_\alpha(U_{V_\epsilon}) \subset [-\epsilon, \epsilon]$. This verifies the τ-continuity of each p'_α and completes the proof.

Combining the above with Theorem 1, we get the

COROLLARY. *Let (E, τ) be a sequentially complete, barrelled t. v. s. ordered by the cone of an unconditional Schauder basis. Then $\tau \subset \Omega$.*

The following useful lemma results from Proposition 2 and the implication between $(*)$- and Ω-convergence.

LEMMA 5. *Let E be an o. v. s. on which is defined a first-countable topology τ. Then to have $\Omega \subset \tau$, it is sufficient that for sequences $x_n \xrightarrow{\tau} \theta$ we also have $x_n \xrightarrow{(*)} \theta$. The condition is also necessary in case E is ordered by a basis cone.*

EXAMPLE. Let (E, τ) be the space φ of all finitely nonzero sequences, with sup norm, ordered by the unit vector basis. Here τ is strictly weaker than Ω. (We have $\tau \subset \Omega$ from Lemma 4, since $\|x_n\| \longrightarrow 0$ whenever $x_n \downarrow \theta$; $\Omega \neq \tau$ by Lemma 5 since, e.g., $(1/n)e_n \xrightarrow{\tau} \theta$ but $(1/n)e_n \overset{(*)}{\not\rightarrow} \theta$, since no subsequence of $\{(1/n)e_n\}$ is order-bounded in φ.)

Following Schaefer [15], we define a *Fréchet lattice* to be a Riesz space E which is also a Fréchet space (E, τ), with a system $\{p_n\}_{n=1}^{\infty}$ of lattice seminorms defining τ. In particular, from Proposition 3, a Fréchet space ordered by the cone of an unconditional basis is a Fréchet lattice. Vulih [17, pp. 177, 200–201] proves

PROPOSITION 4. *In a Fréchet lattice (E, τ), we have $\Omega \subset \tau$.*

EXAMPLE. Let (E, τ) be the space (m) of all bounded sequences of real numbers ordered coordinatewise (i.e. ordered by the cone of the weak*-basis of unit vectors), where τ is generated by the sup norm. Here Ω is strictly weaker than τ. (We have $\Omega \subset \tau$ from Proposition 4, and $\Omega \neq \tau$ since, e.g.,

$$x_n = (\underbrace{0, 0, \ldots, 0}_{n}, 1, 1, 1, \ldots) \xrightarrow[n]{(o)} \theta$$

but $\|x_n\| = 1 \not\rightarrow 0$. Lemma 4 now yields $\tau \not\subset \Omega$.)

Combining Proposition 4 and the corollary to Proposition 3, we obtain an original proof of a result of Ceĭtlin [1, Theorem 4]:

THEOREM 2. *If (E, τ) is a Fréchet space ordered by the cone of an unconditional Schauder basis, then $\Omega = \tau$.*

COROLLARY. *$\Omega = \tau$ in the following classical spaces with their natural basis orderings*: $l^p (p \geqslant 1)$, (c_0), (c), (s), $L^p[0, 1]$ $(p > 1$; *the Haar bases*).

REMARK. The example following Lemma 5 shows that the assumption of sequential completeness in Theorem 2 is necessary; to see that the assumption of metrizability cannot be discarded, consider l^1 with the weak topology $\sigma(l^1, m)$.

This space is sequentially complete and the unit vectors form an unconditional $\sigma(l^1, m)$-Schauder basis, but $\sigma(l^1, m)$ is strictly weaker than the order topology Ω induced by the basis cone since Ω coincides with the norm topology on l^1.

It would be interesting, however, to know whether the assumptions concerning the space could be replaced by assumptions concerning the basis in Theorem 2.

It is interesting that the converse of Theorem 2 is also true; we prove this and add several equivalences in

THEOREM 3. *Let (E, τ) be a Fréchet space ordered by the cone K of a Schauder basis $B = \{x_i, f_i\}$. The following are equivalent*:

(i) *B is unconditional*;

(ii) *K is generating and τ-normal*;

(iii) $\Omega = \tau$;

(iv) *K is generating and $\tau \subset \Omega$*;

(v) *K is τ-normal and $\Omega \subset \tau$.*

PROOF. (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) is known; see, e.g., [10, Theorem 2] for (i) $\Longleftrightarrow$ (ii), and (i) $\Longrightarrow$ (iii) is Theorem 2 above. We show (iii) $\Longrightarrow$(i): From Theorem 1, K is τ-normal. Suppose that B is not unconditional; from [10, Theorem 2], there exists $x \in E$ such that at least one of x^+ and x^- fails to exist (since K cannot be generating). Clearly $(1/n)x \xrightarrow[n]{\tau} \theta$, so also $(1/n)x \xrightarrow{(*)} \theta$, since $\Omega = \tau$. But if $(1/n_i)x \xrightarrow[i]{(o)} \theta$ for some subsequence $\{(1/n_i)x\}$, this subsequence is eventually bounded above by elements of K and below by elements of $(-K)$; Lemma 3 now guarantees the existence of x^+ and x^-, a contradiction to the choice of x. Therefore B is an unconditional basis.

(iv) $\Longrightarrow$ (i) since if $\tau \subset \Omega$ then K is unconditional from Theorem 1, and if K is also generating, then basis expansions of all elements of E converge unconditionally, since they are differences of unconditional expansions. Clearly (ii) and (iii) imply (iv) and (v); if we assume (v), Theorem 1 yields $\tau \subset \Omega$, which with $\Omega \subset \tau$ implies (iii).

EXAMPLE. If (E, τ) is $C[0, 1]$, with sup norm, ordered by the cone K of *Schauder's basis* (Singer [16, p. 11]), or any basis consisting of functions that are nonnegative on $[0, 1]$, then Ω is strictly stronger than τ. (Since K is τ-normal for such a basis and $C[0, 1]$ has no unconditional basis (Karlin [8]), the result follows from Theorems 1 and 3.)

III. Characterizations of (o)-convergence and the Ω-closure of a set in a space with an unconditional cone.

PROPOSITION 5. *Let (E, τ) be a sequentially complete, locally convex t. v. s. ordered by the unconditional cone K of a Schauder basis $\{x_i, f_i\}$. Then for a* *sequence* $\{y_n\}_{n=1}^{\infty} \subset K$ *to* (o)-*converge to* θ, *it is necessary and sufficient that* $f_i(y_n) \xrightarrow[n]{} 0$ *for each* i *and that each* $y_n \leqslant y$ *for some* $y \in K$.

PROOF. *Necessity.* If $y_n \xrightarrow{(o)} \theta$, it follows from Lemma 2 that $f_i(y_n) \xrightarrow[n]{} 0$ for each i. Also, for some $w \in K$ and some n_0, we have $y_n \leqslant w$ for all $n \geqslant n_0$; letting $y = w + \Sigma_{k=1}^{n_0} y_k$, we have $y_n \leqslant y$ for all n.

Sufficiency. We will utilize the embedding $x \mapsto x' = (f_i(x))_{i=1}^{\infty}$ of E into (s) (the space of all real sequences ordered by the cone of the unit vector basis) discussed by McArthur [11]. The unconditionality of K gives its image K' the following "solid" property: if $0 \leqslant a_i \leqslant f_i(x)$ for some $x \in K$ and each i, then $(a_i) \in K'$. Since $f_i(y_n) \xrightarrow[n]{} 0$ for each i, we have that $y_n' \xrightarrow{(o)} \theta$ in (s) [17, p. 31]. Then there is a sequence $u_n \downarrow \theta$ in (s) with $y_n' \leqslant u_n$ for each n (e.g., since $y_n' = (f_i(y_n))_{i=1}^{\infty}$, let $u_n = (b_{n,i})_{i=1}^{\infty}$, where $b_{n,i} = \sup\{f_i(y_j) : j \geqslant n\}$ for each n and i). For each n, let $v_n = u_n \wedge y' \in K'$ and let $z_n \in K$ be the preimage of v_n, i.e. $z_n' = v_n$. Then $z_n \downarrow \theta$, and since $y_n' \leqslant z_n'$ for each n, we get $y_n \leqslant z_n$ for each n. Then $y_n \xrightarrow{(o)} \theta$ in E.

COROLLARY 1 (*McArthur* [11]). *Let* (E, τ) *be a sequentially complete, locally convex t. v. s. ordered by the cone* K *of an unconditional Schauder basis* $\{x_i, f_i\}$, *and let* $\{y_n\}_{n=1}^{\infty} \subset E$. *Then in order to have* $y_n \xrightarrow{(o)} \theta$, *it is necessary and sufficient that* $f_i(y_n) \xrightarrow[n]{} 0$ *for each* i *and that* $|y_n| \leqslant y$ *for all* n *and some* $y \in K$.

PROOF. Since $y_n \xrightarrow{(o)} \theta$ iff $|y_n| \xrightarrow{(o)} \theta$ and $f_i(|y_n|) = |f_i(y_n)|$ (E is a Riesz space here), the result follows from Proposition 5.

COROLLARY 2. *For* (E, τ) *as in Proposition* 5, *let* $\{y_n\}_{n=1}^{\infty} \subset E$. *Then in order to have* $y_n \xrightarrow{(o)} \theta$ *it is necessary and sufficient that* $f_i(y_n) \xrightarrow[n]{} 0$ *for each* i *and there exist* $z \in (-K)$, $y \in K$, n_0 *such that* $z \leqslant y_n \leqslant y$ *for* $n \geqslant n_0$.

PROOF. The condition is clearly necessary; for the sufficiency, Lemma 3 yields the existence of y_n^+, y_n^- for $n \geqslant n_0$, which with $f_i(y_n) \xrightarrow[n]{} 0$ for each i yields $y_n^+ \xrightarrow{(o)} \theta, y_n^- \xrightarrow{(o)} \theta$ from Proposition 5. Then $y_n = y_n^+ - y_n^- \xrightarrow{(o)} \theta$ since (from Lemma 1 (v) and Proposition 1) (o)-convergence is additive in any basis-ordered space.

In [13] and [17], *regulator* (or *relative uniform*) convergence is discussed; although this is usually a lattice concept, in our setting we are motivated to define $x_n \xrightarrow{(r)} x$ iff there exists $u \in K$ (called the *regulator* of convergence for $\{x_n\}$) and an integer n_0 such that $|x_n - x|$ exists for $n \geqslant n_0$, and for any $\epsilon > 0$ there exists $n_\epsilon \geqslant n_0$ with $|x_n - x| \leqslant \epsilon \cdot u$ for $n \geqslant n_\epsilon$. (r)-convergence implies (o)-convergence in any Archimedean Riesz space [17, p. 68] and using Lemma 2 this is also easy to see in any t.v.s. ordered by a basis cone. In Fréchet spaces, the converse is true.

PROPOSITION 6. *Let (E, τ) be a Fréchet space ordered by the τ-normal cone K of a Schauder basis. Then for sequences, (o)-convergence is equivalent to (r)-convergence.*

PROOF. From Theorem 1, (o)-convergence implies τ-convergence. Let $\{p_n\}_{n=1}^{\infty}$ be seminorms defining τ, and first assume $y_m \downarrow \theta$. Since $p_n(y_m) \underset{m}{\to} 0$ for each n, let $\{m_k\}$ be an increasing sequence of positive integers with $p_n(y_{m_k}) \leqslant 1/k^3$ for $1 \leqslant n \leqslant k$. Then $y = \Sigma_{k=1}^{\infty} ky_{m_k} \in K$ and $ky_{m_k} \leqslant y$ for each k. Therefore $y_{m_k} \leqslant (1/k)y$ for each k, so $y_{m_k} \xrightarrow{(r)} \theta$. Since $\{y_m\}$ is decreasing, we also have $y_m \xrightarrow{(r)} \theta$.

Now only assume $x_n \xrightarrow{(o)} \theta$, so from Proposition 1 there are sequences $y_n \downarrow \theta$, $z_n \uparrow \theta$ such that for each n there exists n_0 with $z_n \leqslant x_k \leqslant y_n$ for $k \geqslant n_0$. Then from Lemma 3, $|x_n|$ exists for all sufficiently large n, and also for each n there exists n_0 with $|x_k| \leqslant y_n - z_n$ for $k \geqslant n_0$. Since $(y_n - z_n) \downarrow \theta$ from Lemma 1 and therefore $(y_n - z_n) \xrightarrow{(r)} \theta$ from the first case above, we also have $x_n \xrightarrow{(r)} \theta$. This completes the proof.

Vulih [17, p. 162] proves that in any Riesz space having a certain property (property "R"), one is guaranteed the existence of a *common* regulator of convergence for any of a countable collection of (r)-convergent sequences; in this case, (r)-convergence possesses a "diagonal property": if $x_{m,n} \xrightarrow{(r)} x_m$ as $n \to \infty$ and if $x_m \xrightarrow{(r)} x$, there exists a strictly increasing sequence $\{n_m\}$ with $x_{m,n_m} \xrightarrow{(r)} x$ as $m \to \infty$. These arguments go through in our setting, using

LEMMA 6 (*Property* R). *If (E, τ) is a Fréchet space ordered by the cone K of a Schauder basis $\{x_i, f_i\}$, then for any sequence $\{y_n\}_{n=1}^{\infty} \subset K$ there exist scalars $c_n > 0$ and $y \in K$ with $c_n y_n \leqslant y$ for each n.*

PROOF. Let seminorms $\{p_k\}_{k=1}^{\infty}$ define τ and, for each n, let

$$a_n = \sup\{p_k(y_n) : 1 \leqslant k \leqslant n\}.$$

Define $c_n = 1$ if $a_n = 0$, and let $c_n = 1/n^2 a_n$ if $a_n \neq 0$. Then for each n and $k \leqslant n$ we have $c_n p_k(y_n) \leqslant 1/n^2$, so we may let $y = \Sigma_{n=1}^{\infty} c_n y_n \in K$. $c_n y_n \leqslant y$ for each n, since $f_i(c_n y_n) \leqslant \Sigma_{k=1}^{\infty} c_k f_i(y_k) = f_i(y)$ for each i.

These results, in particular the diagonal property, yield

PROPOSITION 7. *Let (E, τ) be a Fréchet space ordered by the τ-normal cone K of a Schauder basis. Then for any set $A \subset E$, the Ω-closure $\bar{A}$ of A is given by $\bar{A} = \{x \in E : x_n \xrightarrow{(r)} x$ for some sequence $\{x_n\} \subset A\}$.*

PROOF. Clearly $A \subset \bar{A}$ and $\bar{A}$ is contained in the Ω-closure of A. Then it suffices to show that $\bar{A}$ is Ω-closed. So let $\{y_m\} \subset \bar{A}$ with $y_m \xrightarrow{(o)} y \in E$.

Then for each m, there exists $\{y_{m,n}\} \subset A$ with $y_{m,n} \xrightarrow[n]{(o)} y_m$. Thus, by the diagonal property, there is a strictly increasing sequence $\{n_m\}$ with $y_{m,n_m} \xrightarrow{(o)} y$, so $y \in \bar{A}$. Then by Corollary 2 to Proposition 1, $\bar{A}$ is Ω-closed.

This allows a characterization of Ω-convergence for nets analogous to Proposition 2.

PROPOSITION 8. *Let (E, τ) be as in Proposition 7. Then for a net $\{x_\alpha\} \subset E$, we have $x_\alpha \xrightarrow{\Omega} \theta$ iff every cofinal subnet of $\{x_\alpha\}$ contains a sequence $\{y_n\}$ among its terms such that $y_n \xrightarrow{(r)} \theta$.*

PROOF. For the sufficiency, suppose $x_\alpha \overset{\Omega}{\not\to} \theta$, so there is an Ω-open set U containing θ and a cofinal subnet $\{x_{\alpha_\beta}\}$ of $\{x_\alpha\}$ with $x_{\alpha_\beta} \notin U$ for each β. Then $\{x_{\alpha_\beta}\} \subset E \backslash U$ and $E \backslash U$ is Ω-closed, so no sequence from $\{x_{\alpha_\beta}\}$ can (o)-converge to θ.

For the necessity, let $x_\alpha \xrightarrow{\Omega} \theta$ and let $\{x_{\alpha_\beta}\}$ be a cofinal subnet (so that $x_{\alpha_\beta} \xrightarrow{\Omega} \theta$). Let F denote the Ω-closure of the set of terms of $\{x_{\alpha_\beta}\}$. Then $\theta \in F$ since $x_{\alpha_\beta} \xrightarrow{\Omega} \theta$. Therefore there is a sequence from $\{x_{\alpha_\beta}\}$ which (r)-converges to θ, from Proposition 7.

Proposition 6 also allows us to answer a question posed by McArthur [11], namely "When is Ω a *linear* topology?" Schaefer [15, pp. 230 and 253] indicates that Ω is usually *not* linear in Riesz spaces; Potepun [14] gives some sufficient conditions in Riesz spaces for Ω to be linear. From our Proposition 2, it is not too difficult to see that vector addition is *sequentially* Ω-continuous; the question of additivity for Ω-convergent nets seems to be more difficult to answer (hopefully, more information like that contained in Proposition 8 will prove to be useful here). However, the question concerning Ω-continuity of scalar multiplication is easily answered in our current setting, yielding the following result (in which Theorem 2 clearly gives the sufficiency).

PROPOSITION 9. *Let (E, τ) be a Fréchet space ordered by the τ-normal cone K of a Schauder basis B. Then Ω is linear if and only if B is an unconditional basis for E.*

PROOF. Suppose that B is not unconditional. From Theorem 3, since K is normal, K cannot be generating. Therefore there exists $x \in E$ such that $|x|$ does not exist. For Ω to be linear, we must have $(1/n)x \xrightarrow{(*)} \theta$, so for some subsequence $\{1/n_k\}$ of $\{1/n\}$, we have $(1/n_k)x \xrightarrow{(r)} \theta$ by Proposition 6. But then $|(1/n_k)x| = (1/n_k)|x|$ exists for all sufficiently large k, contradicting the choice of x. Therefore Ω is not a linear topology.

Peressini [13] studies an "order topology" that is different from Ω; following Namioka [12], we shall call it the *order-bound topology* τ_0, since it is defined to be the finest *locally convex* topology on an ordered vector space such that all the order-bounded sets are also topologically bounded. In the setting of Theorem 3, any of the five equivalent conditions implies that the three topologies τ, τ_0, and Ω coincide [13, p. 125]. In the setting of Propositions 6 through 9, Lemma 4 says that Ω is the finest topology τ on E for which (r)-convergence implies τ-convergence. Peressini [13, p. 161] mentions that (r)-convergence implies τ_0-convergence, so in this setting we conclude $\tau_0 \subset \Omega$. Of course, Proposition 9 illustrates how these topologies may differ, since Ω need not be linear.

IV. **The order topology in dual spaces.** If $\{x_i, f_i\}$ is a Schauder basis for a locally convex t.v.s. (E, τ), it is well known that $\{f_i, Jx_i\}$ is a $\sigma(E', E)$-Schauder basis for E', where J is the canonical embedding of E into E''. We can then order E' by the cone of this weak*-basis, defining $f \leqslant g$ in E' iff $f(x_i) \leqslant g(x_i)$ for each i. In general, if E is any ordered t.v.s. with positive cone K, the *dual cone* $K' \subset E'$ is the set $K' = \{f \in E' : f(x) \geqslant 0 \text{ for all } x \in K\}$. In particular, when K is the cone of the basis $\{x_i, f_i\}$ in E, the dual cone K' in E' is easily seen to be the set $K' = \{f \in E' : f(x_i) \geqslant 0 \text{ for each } i\}$. We use Lemma 4 in

PROPOSITION 10. *Let (E, τ) be a locally convex t.v.s. ordered by the cone K of a Schauder basis $\{x_i, f_i\}$. Let E' be ordered by the dual cone K', and also assume that K is generating in E. Then the order topology Ω' in E' is stronger than $\sigma(E', E)$.*

PROOF. First assume $g_n \downarrow \theta$ in E'. Fix $x \in K$ and $\epsilon > 0$. Then $g_1(x) = \Sigma_{i=1}^{\infty} f_i(x) g_1(x_i)$, and since this sum converges, there exists n_0 such that $|\Sigma_{i=n_0}^{\infty} f_i(x) g_1(x_i)| < \epsilon/2$. Since $g_n \downarrow \theta$, there exists n_1 such that $n \geqslant n_1$ implies $|\Sigma_{i<n_0} f_i(x) g_n(x_i)| < \epsilon/2$. Thus $|g_n(x)| < \epsilon$ if $n \geqslant n_1$. Then $g_n(x) \longrightarrow 0$ (and since K generates E it follows that $g_n \xrightarrow{w^*} \theta$).

Now if $g_n \xrightarrow{(o)} \theta$, there are sequences $h_n \downarrow \theta$, $h_n' \uparrow \theta$ in E' such that for each n, there exists n_0 with $h_n'(x_i) \leqslant g_k(x_i) \leqslant h_n(x_i)$ for $k \geqslant n_0$ and each i. Then for each $x \in K$, it follows that $h_n'(x) \leqslant g_k(x) \leqslant h_n(x)$ for $k \geqslant n_0$. Since $h_n'(x) \longrightarrow 0$, $h_n(x) \longrightarrow 0$ from the first case above, it follows that $g_n(x) \longrightarrow 0$. Since K is generating, we have $g_n \xrightarrow{w^*} \theta$ in E. This completes the proof.

For the remainder of this paper, let E be a Banach space. We wish to find conditions under which the order topology Ω' is weaker than the topology $\beta(E', E)$. A theorem due to M. Kreĭn [3] will be useful here; it states that if E is a normed space ordered by a normal cone K with the property that there exists a constant C such that $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, $x \leqslant z \leqslant y$ implies $\|z\| \leqslant C$, then each $f \in E'$ is expressible as $f = f^{(1)} - f^{(2)}$, $f^{(1)}, f^{(2)} \in K'$, and $\|f^{(1)}\| + \|f^{(2)}\| \leqslant C \cdot \|f\|$.

THEOREM 4. *Let E be a Banach space ordered by the cone K of an* *unconditional Schauder basis* $\{x_i, f_i\}$, *and let* E' *be ordered by the dual cone* K'. *Then the strong topology* $\beta(E', E)$ *is stronger than* Ω', *i. e.* $\Omega' \subset \beta(E', E)$.

PROOF. First suppose that $\{g_n\} \subset K'$, $\|g_n\| \longrightarrow 0$, and let $\{n_k\}$ be an increasing sequence of positive integers such that $\{\Sigma_{k=1}^{p} kg_{n_k} : p = 1, 2, \ldots\}$ is a Cauchy sequence in E' (as in the proof of Proposition 6). Then $g \equiv \Sigma_{k=1}^{\infty} kg_{n_k} \in K'$ and $kg_{n_k} \leqslant g$ for each k, so $g_{n_k} \xrightarrow{(o)} \theta$. By a similar argument applied to any subsequence of $\{g_n\}$, we have $g_n \xrightarrow{(*)} \theta$.

Now let $\{g_n\}$ be any sequence in E' such that $\|g_n\| \longrightarrow 0$. From Kreĭn's theorem, each $g_n = g_n^{(1)} - g_n^{(2)}$, $g_n^{(i)} \in K'$, and $\|g_n^{(1)}\| + \|g_n^{(2)}\| \leqslant 2 \cdot \|g_n\|$ (since if $x, y \in E$, $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, $x \leqslant z \leqslant y$, then we can assume $\|z\| \leqslant \|x\| + \|y\| \leqslant 2$ from Proposition 3, since $|z| \leqslant |x| + |y|$). Hence $\|g_n^{(i)}\| \longrightarrow 0$, so from the first case above, $g_n^{(i)} \xrightarrow{(*)} \theta$ and hence $g_n = g_n^{(1)} - g_n^{(2)} \xrightarrow{(*)} \theta$. (The additivity of $(*)$-convergence follows from that of (o)-convergence.)

COROLLARY. *In the setting of Theorem* 4, *the order topology* Ω' *in* E' *lies between the weak* and the strong topologies, i. e.* $\sigma(E', E) \subset \Omega' \subset \beta(E', E)$.

REMARKS. (1) It would be interesting to know whether Theorem 4 is a result about Banach spaces only, or if it is true in a more general setting. The difficulty in deciding this seems to lie in the fact that generalizations of Krein's theorem (Jameson [6, p. 29]) seem to lose their applicability to our problem.

(2) The example following Proposition 4 shows that Ω' may be strictly weaker than $\beta(E', E)$, since $(m) = (l^1)'$. The author knows of no example of an infinite-dimensional dual Banach space in which Ω' coincides with $\sigma(E', E)$; it is suspected that sufficient conditions for this coincidence would involve some pathology, since under fairly strong assumptions we will find that Ω' coincides with $\beta(E', E)$ (Theorem 5).

In the following, we will use the fact [13, p. 72] that for any locally convex t.v.s. E ordered by a *normal* cone, the dual cone is generating in E'. Also, recall that a Schauder basis $\{x_i, f_i\}$ for a t.v.s. E is called *shrinking* if $\{f_i\}$ is a strong basis for E'.

PROPOSITION 11. *Let* (E, τ) *be a Banach space ordered by the cone* K *of a Schauder basis* $\{x_i, f_i\}$ *and let* E' *be ordered by the dual cone* K'. *Then*:

(i) *If* $\{x_i, f_i\}$ *is shrinking and* K *is generating, we have* $\beta(E', E) \subset \Omega'$.

(ii) *If* $\beta(E', E) \subset \Omega'$ *and* K *is* τ*-normal, then* $\{x_i, f_i\}$ *is shrinking.*

PROOF. (i) follows from Theorem 1, since if K is generating, then K' is $\beta(E', E)$-normal [10, Theorem 5]. To prove (ii), suppose that $\beta(E', E) \subset \Omega'$ and that K is τ-normal. Then K' is generating in E' so we have only to show that for $f \in K'$, we have $f = \Sigma_{i=1}^{\infty} f(x_i) f_i$ with respect to $\beta(E', E)$, for then $\{f_i, Jx_i\}$ will be a strong basis for E': The sequence $\{\Sigma_{i=1}^{n} f(x_i)f_i : n = 1, 2, \ldots\}$ is increasing and has f as its supremum, so we have $\Sigma_{i=1}^{n} f(x_i)f_i \xrightarrow[n]{(o)} f$. Since $\beta(E', E) \subset \Omega'$, we know that the convergence is with respect to $\beta(E', E)$, as desired.

Recall from Theorem 3 that K is generating and τ-normal in the above setting if $\{x_i, f_i\}$ is unconditional. Day [2, p. 77] has listed several equivalences, given below, essentially due to R. C. James [5]; we add (ii).

THEOREM 5. *Let E be a Banach space ordered by the cone K of an unconditional Schauder basis* $\{x_i, f_i\}$, *and let* E' *be ordered by the dual cone. The following are equivalent*:

(i) $\{x_i, f_i\}$ *is a shrinking basis for E.*

(ii) $\Omega' = \beta(E', E)$.

(iii) *No subspace of E is isomorphic to* l^1.

(iv) E' *is* $\beta(E', E)$*-separable.*

(v) E' *is weakly sequentially complete.*

(An interesting result is an alternate treatment of the example following Proposition 4, since Theorem 4 says that the norm topology on (m) is stronger than Ω', and $(l^1)' = (m)$ implies the inequality.)

Since a result of R. C. James says that a Banach space E with a basis is reflexive iff the basis is shrinking and boundedly complete, we obtain the following

COROLLARY. *Let E be a reflexive Banach space ordered by the cone of an unconditional basis, and let* E' *be ordered by the dual cone. Then* $\Omega' = \beta(E', E)$.

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DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306

Current address: Federal Reserve Bank of Dallas, Station K, Dallas, Texas 75222