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PRINCIPAL QUADRATIC FUNCTIONALS

BY

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ABSTRACT. The general theory of the existence of a minimum limit in the fixed and variable end point problems of singular quadratic functionals of n dependent variables is developed, generalizing the one dimensional results of Marston Morse, Walter Leighton and A. D. Martin and completing a phase of the n-dimensional theory initiated recently by the author.

Introduction. Morse and Leighton [5] initiated the study of singular quadratic functionals for $n = 1$. The classical sufficient conditions of variational theory were found lacking for the singular problem and a condition, termed the "singularity condition" was discovered which together with the classical conditions yielded necessary and sufficient conditions for the singular problem. This singularity condition was shown to be independent of the classical conditions. The singularity condition (as stated) however depended formally on the class of curves admitted. Leighton [3] specialized the quadratic functionals somewhat to principal quadratic functionals and was able to give necessary and sufficient conditions for a minimum which only depended on the "geometry" of the functional rather than on the class of curves admitted. One condition Leighton [3] imposed on the functional J (given below) was that $P(t)$ be positive for large t. Leighton and Martin [4] then studied singular quadratic functionals without assuming $P(t)$ positive for large t. The author [9] extended the results of Morse and Leighton [5] to $n = n$ (these results were duplicated by Morse [8]) and also extended many of the results of Leighton [3] in the two cases where either all the characteristic roots of $\int_a^t R^{-1}(t)\,dt$ are bounded or all are unbounded. In this paper, sufficient conditions for the existence of a minimum in the missing case in [9] are given, completing a phase of the study initiated there. In addition, the techniques given here are sufficiently general to not require the matrix $P(t)$ to be positive definite and to solve the variable end point problem also. When studying the curious effect of infinite focal points on A-minimum limits,

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the condition that $P(t)$ be positive definite for large t is, however, assumed.

Statement of the problem. In this paper a repeated subscript indicates summation from 1 to n. The symbol "*" will indicate the transpose of a vector or matrix. The determinant of a matrix A will be written det A. Whenever the symbols $\lambda_1[A], \ldots, \lambda_n[A]$ are used, A will be a symmetric matrix and $\lambda_1[A], \ldots, \lambda_n[A]$ are the real characteristic roots of A ordered by size; that is, $\lambda_1[A] \leqslant \ldots \leqslant \lambda_n[A]$.

The reader is assumed to be familiar with [9]; the notation is essentially the same. However the problem will be briefly stated here. Let

$$f(t, y, y') = y^{*\prime}R(t)y' + y^*P(t)y$$

where $R(t)$ and $P(t)$ are symmetric $n \times n$ matrices continuous in the real variable t on $[a, \infty)$, $a \geqslant 0$, and $R(t)$ is positive definite on $[a, \infty)$. We consider the functional

$$J(y)|_a^b = \int_a^b f[t, y(t), y'(t)]\, dt, \qquad a \leqslant b < \infty.$$

Integrals employed throughout are Lebesgue integrals and their extensions. We call the vector function $y^*(t) = (y_1(t), \ldots, y_n(t))$ F-admissible on $[a, \infty)$ if

(1) $y(t)$ is continuous on $[a, \infty)$ and $y(a) = 0$;

(2) each $y_i(t)$ is absolutely continuous and every $y_i(t)y_j(t)$ is Lebesgue integrable on each compact subinterval of $[a, \infty)$.

We call the vector function $y(t)$ A-admissible on $[a, \infty)$ if

(1) $y(t)$ is F-admissible;

(2) $\lim_{t=\infty} y(t) = 0$.

Observe that the segment $[a, \infty)$ of the t-axis is A-admissible and that on this segment $J = 0$. Also if $y_1 = y_1(t)$ is A- or F-admissible and $J(y_1) < 0$, then $J(cy_1) = c^2J(y_1)$ and J cannot possess a minimum. We thus seek conditions under which

$$\liminf_{t=\infty} \int_a^t f[s, y(s), y'(s)]\, ds \geqslant 0. \tag{1}$$

If (1) holds for a given class of curves, we say that $[a, \infty)$ affords a minimum limit to J among curves of the given class.

Fixed end point. Before stating the main theorem of this section, an observation is in order. If $A(t)$ is a symmetric matrix, then

$$\lambda_n\left[\int_a^t A(s)\, ds\right] \leqslant \int_a^t \lambda_n[A(s)]\, ds.$$

Thus $\limsup_{t=\infty} \lambda_n[\int_a^t A(s)\,ds] = \infty$ implies that $\limsup_{t=\infty} \int_a^t \lambda_n[A(s)]\, ds = +\infty$. However the converse need not be true if $A(t)$ is not definite. Consider $A(t) = (a_{ij}(t))$ where $a_{11}(t) = \sin t$, $a_{12}(t) = a_{21}(t) = \cos t$, and $a_{22}(t) =$

$-\sin t$. Then $\int_0^t \lambda_2[A(s)]\,ds = t$ but $\lambda_2[\int_0^t A(s)\,ds] = \sqrt{2-2\cos t}$.

We now state a principal theorem.

THEOREM 1. *If the interval $(a, \infty]$ does not contain a point conjugate to $t = \infty$, and if*

$$(2) \qquad -\infty < \liminf_{t=\infty} \int_a^t \lambda_1[P(s)]\,ds \leqslant \limsup_{t=\infty} \int_a^t \lambda_n[P(s)]\,ds < +\infty,$$

then J possesses an A-minimum limit.

To prove this theorem, we require three lemmas. Also stated are the Morse separation and comparison theorems.

THEOREM A (MORSE SEPARATION THEOREM [7]). *The number of zeros (counting multiplicity) of the determinant of any nontrival prepared matrix solution of* (3) *on a given interval (open or closed) differs from that of any other nontrivial prepared matrix solution by at most n.*

By nontrivial is meant a matrix solution of (3) whose determinant does not vanish identically on any open interval, i.e., does not contain the zero vector solution of the vector analog of (3) as a linear combination of the column vectors of the matrix (cf. [6], where prepared is called conjugate).

THEOREM B (MORSE COMPARISON THEOREM [7]). *If $R_1(t)$ and $P_1(t)$ are symmetric and continuous $n \times n$ matrices on $[a, \infty)$, $R_1(t)$ positive definite on $[a, \infty)$, such that $P_1(t) - P(t)$ and $R(t) - R_1(t)$ are both positive definite on $[a, \infty)$, then the first conjugate point of $t = \infty$ of $[R_1(t)Y']' + P_1(t)Y = 0$ follows that of* (3).

LEMMA 1. *Suppose $A(t) = (a_{ij}(t))$ is an $n \times n$ symmetric matrix and each of the $a_{ij}(t)$ is a holomorphic function of the real variable t on some interval L. There then exists a set of holomorphic characteristic roots, $\lambda_1(t), \ldots, \lambda_n(t)$, on L and a corresponding set of holomorphic orthonormal characteristic vectors, $c_1(t), \ldots, c_n(t)$, on L.*

This lemma can be found in [2, p. 121]. As pointed out in [2, p. 124], a symmetric matrix may be infinitely differentiable and have infinitely differentiable characteristic roots but the characteristic vectors may not even be continuous.

LEMMA 2. *Suppose $R(t)$ and $P(t)$ are continuous $n \times n$ matrices on $[a, \infty)$ with $R(t)$ positive definite on $[\alpha, \infty)$, and $R(t)$ and $P(t)$ piecewise holomorphic (that is to say except possibly at isolated points) matrix functions of the real variable t and that the conjugate point of $t = \infty$ of*

$$(3) \qquad [R(t)X']' + P(t)X = 0$$

is not on $[a, \infty]$. *Let* $U(t)$ *be the prepared matrix solution of* (3) *defined by* $U(a) = 0$, $U'(a) = I$ *(the identity matrix), and define* $S(t) = R(t)U'(t)U^{-1}(t)$ *on* (a, ∞). *Then* $S(t)$ *is symmetric, continuous and piecewise holomorphic on* (a, ∞). *Furthermore, if the characteristic roots,* $\lambda_1(t), \ldots, \lambda_n(t)$, *are chosen as continuous piecewise holomorphic functions as in Lemma* 1, *and the corresponding orthonormal characteristic vectors* $c_1(t), \ldots, c_n(t)$, *are piecewise holomorphic, then the n scalar equations*

$$[r_i(t)x']' + p_i(t)x = 0, \qquad i = 1, \ldots, n, \tag{4}$$

where $r_i^{-1}(t) = c_i^*(t)R^{-1}(t)c_i(t)$ *(no summation on i) and* $p_i(t) = c_i^*P(t)c_i(t)$ *(no summation on i), do not have conjugate points of* $t = \infty$ *on* $[a, \infty]$.

We note that the $\lambda_i(t)$ are differentiable everywhere since (as shall be shown momentarily) $S(t)$ is, and also the $\lambda_i(t)$ are piecewise holomorphic. Although $c_i(t)$ are piecewise holomorphic, $c_i(t)$ need not be continuous. Therefore the terms $r_i(t)$ and $p_i(t)$ in (4) may not be continuous.

However $c_i(t)$ are measurable and (Lebesgue) integrable on finite intervals (cf. [10]). Therefore the terms $r_i(t)$ and $p_i(t)$ may have isolated discontinuities but on finite intervals are measurable, (Lebesgue) integrable, and bounded. Thus (4) satisfies a Carathéodory condition and by a solution to (4) we mean a function $x(t)$ satisfying (4) almost everywhere with $x(t)$ and $r_i(t)x'(t)$ absolutely continuous. Therefore (4) does not differ (for our purposes) from a linear second-order equation with continuous coefficients.

Since $R(t)$ and $P(t)$ are continuous, then of course $U(t)$ and $U'(t)$ are continuous. Also $S(t) = R(t)U'(t)U^{-1}(t)$ is differentiable except for any zeros of $\det U(t)$. But $\det U(t) \neq 0$ on (a, ∞) by virtue of the hypothesis that the conjugate point of $t = \infty$ is not on $[a, \infty]$. Thus $S(t)$ satisfies the Riccati matrix differential equation

$$S'(t) = -S(t)R^{-1}(t)S(t) - P(t) \tag{5}$$

on (a, ∞).

The further assumptions of piecewise holomorphy for both $R(t)$ and $P(t)$ then imply that $S(t)$ is piecewise holomorphic. An easy computation shows that the preparedness of $U(t)$ together with the symmetry of both $R(t)$ and $P(t)$ implies that $S(t)$ is symmetric. Thus Lemma 1 applies to $S(t)$ (piecewise). For notational ease, let $\lambda(t)$ be one of the $\lambda_1(t), \ldots, \lambda_n(t)$, say $\lambda_j(t)$, and let $c(t)$ be the corresponding characteristic vector, that is, $c(t) = c_j(t)$.

Differentiate the equality

$$S(t)c(t) = \lambda(t)c(t), \tag{6}$$

multiply the result on the left by $c^*(t)$, use (6) and the fact that $c^*(t)c(t) = 1$,

and obtain

$$\lambda'(t) = c^*(t)S'(t)c(t) \quad \text{(except at isolated points).} \tag{7}$$

Multiplying $c(t)$ on the right of (5) and $c^*(t)$ on the left of (5) and using (6) and (7) yields

$$\lambda(t) = \lambda(b) - \int_b^t \lambda^2(s)r_j^{-1}(s)\,ds - \int_b^t p_j(s)\,ds \tag{8}$$

for all $t \geqslant b$ and any $b > a$, where $r_j^{-1}(t)$ and $p_j(t)$ are defined as in the lemma. Since (8) holds,

$$x_b(t) = \exp \int_b^t r_j^{-1}(s)\lambda(s)\,ds \tag{9}$$

is a solution of (4) that is not zero on $[b, \infty)$. Thus if (4) has the conjugate point of $t = \infty$ at c, $c > a$, then by definition of the conjugate point c, for c_1 close to c and $a < c_1 < c$, there exists a solution of (4) with two zeros on $[c_1, \infty)$. This contradicts $x_b(t)$ having no zeros on $[c_1, \infty)$, if b is taken such that $a < b < c_1$.

LEMMA 3. *Theorem* 1 *is true if $R(t)$ and $P(t)$ are continuous piecewise holomorphic.*

According to Theorem 5.1 of [9], we need to show that the singularity condition holds. It is clear that the singularity condition is satisfied if all characteristic roots of $S(t)$ are bounded away from $-\infty$, where $S(t)$ is the same $S(t)$ defined in Lemma 2. Let us now assume for the moment that the conjugate point of $t = \infty$ of (3) is not on the closed interval $[a, \infty]$. Let $\lambda(t) = \lambda_j(t)$ and $c(t) = c_j(t)$, for some j between 1 and n, be any characteristic root and vector defined as in Lemma 2. Then if $x(t)$ is defined as $x_b(t)$ on $[b, \infty)$ as in (9) and $b \longrightarrow a$, $x(t)$ is defined on (a, ∞). Also, it is readily seen that $x(t)$ can be defined on $[a, \infty)$ as a solution to

$$[r_j(t)x']' + p_j(t)x = 0 \tag{10}$$

and that $x(a) = 0$ since $\lim_{t\to a} \lambda(t) = +\infty$. Since we are assuming momentarily that the conjugate point of $t = \infty$ of (3) is not on $[a, \infty]$, then the same is true of (10) by Lemma 2. Since a is not the first conjugate point of $t = \infty$ and $x(a) = 0$, then $x(t)$ must be an antiprincipal solution of (10).

Also notice that

$$\int_a^t \lambda_1[P(s)]\,ds \leqslant \int_a^t p_j(s)\,ds \leqslant \int_a^t \lambda_n[P(s)]\,ds.$$

Thus (2) indicates that $\int_a^t p_j(s)\,ds$ is bounded. In the proof of Theorem 9.2 in [4], the conditions that (4) does not have a conjugate point of $t = \infty$ on $[a, \infty]$ and $\int_a^t p_j(s)\,ds$ is bounded were shown to imply that $\lambda(t) = r_j(t)x'(t)x^{-1}(t)$ is

bounded if $x(t)$ is antiprincipal. (Actually in the proof of Theorem 9.2 of [4], $r_j(t)$ and $p_j(t)$ are assumed to be continuous; however, there are no difficulties in extending the proof to include the slightly more general conditions that $r_j(t)$ and $p_j(t)$ possess in the above arguments.) Thus the lemma is established in the case that the conjugate point of $t = \infty$ of (3) is not on $[a, \infty]$. Given the assumption in Lemma 3 that the conjugate point of $t = \infty$ of (3) is not on $(a, \infty]$, we have by the definition of the conjugate point of $t = \infty$, that $[c, \infty]$ contains no conjugate point of $t = \infty$ of (3) if $c > a$. Thus according to the above result J possesses an A-minimum limit on $[c, \infty)$ for any $c > a$, and Theorem 5 of [9] shows that J possesses an A-minimum limit on $[a, \infty)$. This completes the proof of Lemma 3.

We now turn to the proof of Theorem 1.

Consider again the original functional with $R(t)$ and $P(t)$ continuous. Assume under the hypothesis of Theorem 1 that J does not possess an A-minimum limit on $[a, \infty)$. There then exists an A-admissible curve $y_0(t)$ and a sequence $t_k \longrightarrow \infty$ such that

$$\lim_{k=\infty} \int_a^{t_k} [y_0^{*\prime}(s)R(s)y_0'(s) - y_0^*(s)P(s)y_0(s)]\, ds = \beta < 0 \tag{11}$$

(β may be equal $-\infty$). Let $t_1 = a$ and let

$$\int_{t_k}^{t_{k+1}} y_0^{*\prime}(s)y_0'(s)\, ds = N_k, \qquad \int_{t_k}^{t_{k+1}} y_0^*(s)y_0(s)\, ds = M_k. \tag{12}$$

Define the matrices $R^{\#}(t)$ and $P^{\#}(t)$ on $[a, \infty)$ to be continuous and equal to $R_k(t)$ and $P_k(t)$ on $[t_k, t_{k+1})$, respectively, where $R_k(t)$ and $P_k(t)$ are piecewise holomorphic matrices, that is, each element in the matrices is continuous on $[t_k, t_{k+1})$ and the interval $[t_k, t_{k+1})$ is divided into a finite number of subintervals on each of which this element is holomorphic. (Actually each element can be a polynomial.) Also $R_k(t)$ and $P_k(t)$ will be subject to further conditions. But first define for any positive semidefinite matrix A, $\|A\| = \lambda_n[A]$. Now let $R_k(t)$ and $P_k(t)$ also satisfy the following conditions:

$$P(t) - P_k(t) > 0 \quad \text{and} \quad R_k(t) - R(t) > 0 \quad \text{for all } t; \tag{13}$$

that is, $P(t) - P_k(t)$ and $R_k(t) - R(t)$ are positive definite;

$$\max(\|R_k(t) - R(t)\| : s \in [t_k, t_{k+1})) \leqslant |\alpha| 2^{-k-2} N_k^{-1}, \tag{14}$$

where $\alpha = \max(\beta, -1)$ (therefore α is finite);

$$\max(\|P(t) - P_k(t)\| : s \in [t_k, t_{k+1})) \leqslant |\alpha| 2^{-k-2} \min[M_k^{-1}, (t_{k+1} - t_k)^{-1}] \tag{15}$$

where $N_k^{-1} = 0$ if $N_k = 0$ and $M_k^{-1} = 0$ if $M_k = 0$. Statements (14) and (15) are true by continuity. That (13) can be made true simultaneously with (14) and (15) is elementary to see. Now define the functional $J^{\#}$ as

$$J^{\#}(z) = \liminf_{t=\infty} \int_a^t [z^{*\prime}(s)R^{\#}(s)z'(s) - z^*(s)P^{\#}(s)z(s)]\, ds$$

for any A-admissible curve z. There must exist a subsequence of $\{t_k\}$ which we continue to call $\{t_k\}$, such that

$$\lim_{k=\infty} \int_a^{t_k} [y_0^{*\prime}(s)R^{\#}(s)y_0'(s) - y_0^*(s)P^{\#}(s)y_0(s)]\, ds = \lim_{k=\infty} J^{\#}(y_0)|_a^{t_k} = L$$

exists, finite or infinite. Let

$$J(y_0)|_a^t = \int_a^t [y_0^{*\prime}(s)R(s)y_0'(s) - y_0^*(s)P(s)y_0(s)]\, ds.$$

Then using the well-known inequalities

$$y_0^{*\prime}[R^{\#} - R]y_0' \leqslant \lambda_n[R^{\#} - R]y_0^{*\prime}y_0' = \|R^{\#} - R\|y_0^{*\prime}y_0'$$

and

$$y_0^*[P - P^{\#}]y_0 \leqslant \lambda_n[P - P^{\#}]y_0^*y_0 = \|P - P^{\#}\|y_0^*y_0$$

together with conditions (12)–(15) show that $\left|J^{\#}(y_0)|_a^{t_k} - J(y_0)|_a^{t_k}\right| \leqslant |\alpha|/2$. This together with (11) then indicates that $L \leqslant \alpha/2 < 0$ and thus $J^{\#}(y_0) \leqslant \alpha/2 < 0$ and $J^{\#}$ does not have an A-minimum limit on $[a, \infty)$. By Lemma 3 there are three possibilities. Either

$$\liminf_{t=\infty} \int_a^t \lambda_1[P^{\#}(s)]\, ds = -\infty, \qquad \limsup_{t=\infty} \int_a^t \lambda_n[P^{\#}(s)]\, ds = +\infty,$$

or the conjugate point, c, of $t = \infty$ of $J^{\#}$ is greater than a. The first one is impossible by virtue of condition (2) of Theorem 1 and the inequality

$$\int_a^t \lambda_1[P(s)]\, ds \leqslant \int_a^t \lambda_1[P^{\#}(s)]\, ds + |\alpha|$$

which follows readily from (15). The second is also impossible by virtue of (2) and the inequality $\lambda_n[P^{\#}(t)] < \lambda_n[P(t)]$. The third possibility assumes that the conjugate point of $t = \infty$, c, of $[R^{\#}(t)X']' + P^{\#}(t)X = 0$ is greater than a. But this is impossible in view of (13) and the Morse comparison theorem. Thus our original assumption that J does not possess an A-minimum limit on $[a, \infty)$ given the hypothesis of our theorem leads to the above contradiction. This then completes the proof of Theorem 1.

We continue with the following theorem.

THEOREM 2. *If*

$$\liminf_{t=\infty} \int_a^t \lambda_1[P(s)]\, ds > -\infty, \tag{16}$$

then necessary and sufficient conditions that $[a, \infty)$ *afford an A-minimum limit to J are that* $(a, \infty]$ *not contain the conjugate point of* $t = \infty$ *and that*

$$\limsup_{t=\infty} \int_a^t \lambda_n[P(s)]\, ds < +\infty. \tag{17}$$

The sufficiency is of course just Theorem 1. For the necessity, using essentially the same proof as found in Theorem 7.1 and Theorem 7.2 in [4], one finds that

$$\limsup_{t=\infty} \int_a^t c^* P(s) c \, ds < +\infty,$$

for every constant vector c. By taking c to be the constant vector with one in the ith position and zeros in all the rest, one finds that

$$\limsup_{t=\infty} \int_a^t d_i(s) \, ds < +\infty,$$

where $d_i(t)$ is the ith diagonal element of $P(t)$. This in turn implies that

$$\limsup_{t=\infty} \int_a^t \operatorname{trace} P(s) \, ds < +\infty.$$

Since trace $P(t) = \Sigma_{i=1}^n \lambda_i[P(t)]$ and condition (16) holds, it follows immediately that (17) holds. This completes the proof of Theorem 2.

We now state another necessary condition. It is not stated here in the most general form possible.

THEOREM 3. *Suppose that $\lambda[P(t)]$ is a continuous characteristic root of $P(t)$ and that the corresponding characteristic vector $c(t)$ is differentiable; then a necessary condition that J is afforded an A-minimum limit on $[a, \infty)$ is that*

$$\limsup_{t=\infty} \int_a^t \lambda[P(s)] \, ds < +\infty. \tag{18}$$

The proof is again similar to Theorem 7.1 in [4]. Notice that the conditions are satisfied if for example $\lambda[P(t)] = \lambda_n[P(t)]$ were a distinct characteristic root and $P(t)$ were differentiable. If $P(t)$ were holomorphic, then $\lambda_n[P(t)]$ and the characteristic vector $c_n(t)$ are holomorphic. This yields a second example.

Variable end point. We now turn to the variable end point problem. In [9] a definition of the kth focal point, $f_k(t_0)$, of the hyperplane $t = t_0$ was defined. It was shown in Lemma 7.1 of [9] that $f_k(t_0)$ is strictly increasing if $P(t_0)$ is positive definite and strictly decreasing if $P(t_0)$ is negative definite. Of course we are not assuming here that $P(t)$ is positive or negative definite. We then define the kth focal point of $t = \infty$ for general $P(t)$ to be $f_k = \limsup_{t=\infty} f_k(t)$. If on some interval (b, ∞), $f_k(t)$ does not exist for $t \in (b, \infty)$, we say that the focal point of $t = \infty$ does not exist. This agrees with [4] for $n = 1$.

We now state a sufficient condition.

THEOREM 4. *If the first focal point of $t = \infty$ is not on $(a, \infty]$ and if $\liminf_{t=\infty} \int_a^t \lambda_1[P(s)] \, ds > -\infty$, then J possesses and F-minimum limit on (a, ∞).*

We first show that no conjugate point of $t = \infty$ is on the interval $(a, \infty]$. Suppose the contrary; there then exists $c \in (a, \infty)$ such that the determinant of the solution matrix $U(t)$ of (3) satisfying $U(c) = 0$, $U'(c) = I$, has a zero, e, on (c, ∞); thus, det $U(t)$ has $(n + 1)$ zeros on $[c, e]$. By the Morse separation theorem, a focal point of any $t_0 > e$ will be on $[c, e]$. Thus $f_1 \geqslant c > a$ which is a contradiction.

We shall now show that if $f_1 < +\infty$, then $\limsup_{t=\infty} \int_a^t \lambda_n [P(s)]\, ds < +\infty$. Define $S_k(t) = R(t)U_k'(t)U_k^{-1}(t)$, where $U_k(t)$ is the solution of (3) such that $U_k(k) = 0$, $U_k'(k) = I$, for any integer $k \geqslant a$. Since there is no conjugate point on $(a, \infty]$, $S_k(t)$ is defined for all $t > k$. Thus

$$S_k(t) = S_k(b) - \int_b^t S_k(s)R(s)S_k(s)\,ds - \int_b^t P(s)\,ds, \qquad t \geqslant b, \tag{19}$$

for any $b > k$. Now assume that $\limsup_{t=\infty} \int_a^t \lambda_n [P(s)]\, ds = +\infty$; then since $\liminf_{t=\infty} \int_a^t \lambda_1 [P(s)]\, ds > -\infty$, $\limsup_{t=\infty} \int_a^t \operatorname{trace} P(s)\, ds = +\infty$. But $\int_a^t \operatorname{trace} P(s)\, ds = \operatorname{trace} \int_a^t P(s)\, ds$. Thus $\limsup_{t=\infty} \lambda_n [\int_a^t P(s)\, ds] = +\infty$. The Courant-Hilbert min-max theorem applied to (19) then implies that for any k one of the characteristic roots of $S_k(t)$ is eventually negative. But $S^{-1}(t)$ is well defined at $t = k$ and since $S_k^{-1}(k) = 0$, $[S_k^{-1}(k)]' = R^{-1}(k)$, a positive definite matrix. Thus near k, $S_k^{-1}(t)$ has all characteristic roots positive. Thus $S_k(t)$ has all roots positive near k. Thus the root that is eventually negative must have been zero at some $t_k > k$, i.e., det $U_k'(t_k) = 0$. But this implies that a focal point of t_k is k, so that $f_1(t_k) \geqslant k$. Thus $\limsup_{t=\infty} f_1(t) = +\infty$, a contradiction.

We now have no conjugate point on $(a, \infty]$ and

$$-\infty < \liminf_{t=\infty} \int_a^t \lambda_1 [P(s)]\, ds \leqslant \limsup_{t=\infty} \int_a^t \lambda_n [P(s)]\, ds < +\infty. \tag{20}$$

If $R(t)$ and $P(t)$ are holomorphic, Lemma 2 applies and (20) shows that $\int_a^t p_i(s)\, ds$ is bounded for $i = 1, \ldots, n$. Reference to the proof of Lemma 3 indicates that the characteristic roots of the singularity matrix defined in Lemma 2 are bounded away from $-\infty$ and the singularity condition is satisfied, yielding by (3.1) an F-minimum limit. (In Theorem 3.1 of [9], $y(t)$ is assumed A-admissible, but (3.1) of [9] is true if y is F-admissible.)

Now an approximation technique, precisely the same as was used in Theorem 1, yields the theorem for $R(t)$, $P(t)$ continuous.

For completeness we state a necessary condition.

THEOREM 5. *If J possesses an F-minimum limit on (a, ∞), then the focal point of $t = \infty$ is not on $(a, \infty]$.*

The proof is the same as in [4] for $n = 1$.

Fixed end point problem and focal points. In [3], Leighton discovered a most curious connection between the fixed end point problem and infinite focal points. We generalize this theorem to $n = n$ in the case $P(t)$ is positive definite for large t.

THEOREM 6. *If $P(t)$ is positive definite for large t and if n focal points of $t = \infty$ are infinite, then J does not possess an A-minimum limit.*

We may restrict ourselves to the case that the conjugate point of $t = \infty$ is not infinite. (Example 4.1 in [3] illustrates this possibility.) Let b be such that $[b, \infty]$ does not contain the conjugate point of $t = \infty$, $P(t)$ is positive definite on $[b, \infty)$, and $f_1(t^*) \leqslant b$ for some $t^* > b$.

Define the solution matrix of (3) by $U(b) = 0$, $U'(b) = I$. Then Lemma 6.4 of [9] indicates that $U(t)$ is an antiprincipal prepared solution of (3). Then Hartman [1] showed that

$$W(t) = U(t) \int_t^\infty [U^*(s)R(s)U(s)]^{-1}\, ds, \qquad t > b,$$

is a principal prepared solution. Now consider the solution matrix of (3)

$$Z(t, t_0) = W(t)U^{*\prime}(t_0)R(t_0) - U(t)W^{*\prime}(t_0)R(t_0).$$

A computation, using the preparedness of $U(t)$, shows that

$$Z_t(t_0, t_0) = W'(t_0)U^{*\prime}(t_0)R(t_0) - U'(t_0)W^{*\prime}(t_0)R(t_0) = 0.$$

Thus if $f_k(t)$ is the kth focal point of t, then

$$\det Z(f_k(t_0), t_0) = 0. \tag{21}$$

Now if we define $S(t) = R(t)U'(t)U^{-1}(t)$ for $t > b$, then as in the proof of Theorem 4, $S(t)$ is positive definite for $t > b$ and t sufficiently close to b. Also $P(t)$ positive definite implies that $S'(t)$ is negative definite and therefore the characteristic roots of $S(t)$ are monotone decreasing. This then implies that det $S(t)$, and hence det $U'(t)$, has only a finite number of zeros. We now assume that t_0 is sufficiently large to insure that $f_n(t_0) > b$ and that det $U'(t) \neq 0$ on $[t_0, \infty)$. Thus (21) is equivalent to

$$\det [U^{-1}(f_k(t_0))W(f_k(t_0)) - W^{*\prime}(t_0)U^{*\prime -1}(t_0)] = 0. \tag{22}$$

Using the preparedness of U, one readily sees that $W^{*\prime}(t_0)U^{*\prime -1}(t_0) = U'^{-1}(t_0)W'(t_0)$. Thus (22) is equivalent to

$$\det [U^{-1}(f_k(t_0))W(f_k(t_0)) - U'^{-1}(t_0)W'(t_0)] = 0. \tag{23}$$

But

$$\begin{aligned} U^{-1}(f_k(t_0))W(f_k(t_0)) - U'^{-1}(t_0)W'(t_0) \\ = \int_{f_k(t_0)}^{\infty} (U^*RU)^{-1}\,ds - \int_{t_0}^{\infty} (U^*RU)^{-1}\,ds + [U^*(t_0)R(t_0)U'(t_0)]^{-1} \\ = \int_{f_k(t_0)}^{t_0} (U^*RU)^{-1}\,ds + [U^*(t_0)R(t_0)U'(t_0)]^{-1}, \end{aligned}$$

where the term $(U^*RU)^{-1}$ in the integrands is meant to be $[U^*(s)R(s)U(s)]^{-1}$. This together with (23) then implies that

$$\det\left[\int_{f_k(t_0)}^{t_0} (U^*RU)^{-1}\,ds + [U^*(t_0)R(t_0)U'(t_0)]^{-1}\right] = 0. \tag{24}$$

Now $\int_{f_k(t_0)}^{t_0} (U^*RU)^{-1}\,ds$ is positive definite and

$$\lim_{t_0=\infty} \int_{f_k(t_0)}^{t_0} (U^*RU)^{-1}\,ds = 0, \quad \text{for } k \leqslant n,$$

since $\lim f_k(t_0) = \infty$ for $k \leqslant n$ and since $\int^t (U^*RU)^{-1}\,ds$ is bounded by virtue of the fact that U is antiprincipal. This together with (24) indicates that one characteristic root of the symmetric matrix $[U^*(t_0)R(t_0)U'(t_0)]^{-1}$ must go to zero through negative values as t_0 goes to infinity. (Actually, U^*RU' is negative definite since $S = RU'U^{-1}$ is negative definite and $U^*SU = U^*RU'$.) This implies that $U^*(t)R(t)U'(t)$ is unbounded. From the equation

$$S^{-1}(t) = S^{-1}(c) + \int_c^t S^{-1}(s)P(s)S^{-1}(s)\,ds + \int_c^t R^{-1}(s)\,ds, \qquad t \geqslant c > t_0,$$

and the fact that $S(t)$ is negative definite and $P(t)$ is positive definite, we see that $\int_c^t R^{-1}(s)\,ds$ must be bounded. If we now assume that $\int_a^t P(s)\,ds$ is bounded, then all solutions of (3) are bounded. (See for example the proof of Theorem 6.4 of [9].) Now $U(t)$ bounded and $\int_a^t P(s)\,ds$ bounded imply by Lemma 6.3 [9] that $R(t)U'(t)$ is bounded. Thus $U^*(t)R(t)U'(t)$ is bounded which is a contradiction. Thus $\int_a^t P(s)\,ds$ must be unbounded and Theorem 2 indicates that J does not possess an A-minimum limit. This completes the proof of Theorem 6.

It is an open question whether or not Theorem 6 is valid if one only assumes the first $n - 1$ focal points are infinite.

Oscillation and focal points. We saw in the last section how focal points, which play a critical role in the variable end point problem, curiously also play a role in the fixed end point problem. In this section we shall see how focal points also play a role in oscillation.

We continue with the following theorem.

THEOREM 7. *Suppose that $P(t)$ is positive definite on $[a, \infty)$ and that p characteristic roots of $\int_a^t P(s)\,ds$ are infinite and that exactly k focal points of* (3) *are infinite. Then if $p > k$,* (3) *is oscillatory.*

We assume that (3) is not oscillatory, that is, that the conjugate point of $t = \infty$ is not infinite. Then for any b sufficiently large, the matrix solution $U(t)$

of (3) defined by $U(b) = 0$, $U'(b) = I$, has the property that $\det U(t) \neq 0$ on (b, ∞). If $S(t) = R(t)U'(t)U^{-1}(t)$, then $S(t)$ is well defined on (b, ∞). Furthermore, just as in the proof of Theorem 6, $S(t)$ is positive definite to the right of b and sufficiently close to b. (Notice that this fact does not depend on $P(t)$ being positive definite.) Since at least one characteristic root of $\int_a^t P(s)\,ds$ is positively unbounded, the Riccati equation

$$S(t) = S(b) - \int_b^t S(s)R^{-1}(s)S(s)\,ds - \int_b^t P(s)\,ds$$

indicates that at least one characteristic root of $S(t)$ must become negative. (The preparedness of $U(t)$ implies $S(t) = S^*(t)$ and thus $\int_b^t SR^{-1}S$ is positive definite.) Therefore there is a first $b_1 > b$ such that $\det S(b_1) = 0$, and thus $\det U'(b_1) = 0$. This implies that $f_1(b_1) \geqslant b$. If a second characteristic root of $\int_a^t P(s)\,ds$ is positively unbounded, then there exists a smallest $b_2 \geqslant b_1$ such that $\det S(b_2) = 0$ and thus $\det U'(b_2) = 0$. (If $b_2 = b_1$, then $\det U'(b_1)$ vanishes with multiplicity at least two.) If $b_2 = b_1$, then $f_2(b_2) \geqslant b$. If $b_2 > b_1$, then $f_j(b_2) = b$ for some j. Lemma 7.1 of [9] shows that the focal points are strictly increasing since $P(t)$ is positive definite. In the case that $b_2 > b_1$, this implies that $f_1(b_2) > f_1(b_1) \geqslant b$ and this rules out the possibility that $f_1(b_2) = b$. Thus if $b_2 > b_1$, $f_j(b_2) = b$ for some $j \geqslant 2$. This in turn implies that $f_2(b_2) \geqslant b$. Since there are in fact p characteristic roots of $\int_a^t P(s)\,ds$ that become positively infinite, then in similar fashion there are $b_1, b_2, \ldots, b_p$, all greater than b and such that $f_j(b_j) \geqslant b$. But b was any arbitrary, sufficiently large number; thus, $f_j = +\infty$ for $j = 1, 2, \ldots, p$. But this contradicts the fact that at most k focal points are infinite and $k < p$. This contradiction then establishes the theorem.

The following oscillation theorem does not require $P(t)$ to be definite. The proof is similar to Theorem 7 and will not be included, except to note that $S(t)$ as defined in Theorem 7 is positive definite just to the right of b whether or not $P(t)$ is positive definite. (This is true since $[S^{-1}(t)]'$ at $t = b$ is just $R^{-1}(b)$ as in Theorem 4.)

THEOREM 8. *Suppose that* (3) *has no infinite focal points and that* $\limsup_{t=\infty} \lambda_n[\int_a^t P(s)\,ds] = +\infty$; *then* (3) *is oscillatory.*

REFERENCES

1. Philip Hartman, *Self-adjoint, non-oscillatory systems of ordinary, second order, linear differential equations*, Duke Math. J. 24 (1957), 25–35. MR 18, 576.

2. T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New York, 1966. MR 34 #3324.

3. Walter Leighton, *Principal quadratic functionals*, Trans. Amer. Math. Soc. 68 (1949), 253–274. MR 11, 603.

4. Walter Leighton and A. D. Martin, *Quadratic functionals with a singular end point*, Trans. Amer. Math. Soc. 78 (1955), 98–128. MR 16, 598.

5. Marston Morse and Walter Leighton, *Singular quadratic functionals*, Trans. Amer. Math. Soc. 40 (1936), 252–286.

6. Marston Morse, *The calculus of variations in the large*, 2nd ed., Amer. Math. Soc. Colloq. Publ., vol. 18, Amer. Math. Soc., Providence, R. I., 1960.

7. ———, *A generalization of the Sturm separation and comparison theorems to n-space*, Math. Ann. 103 (1930), 52–69.

8. ———, *Singular quadratic functionals*, Math. Ann. 201 (1973), 315–340. MR 49 #7894.

9. E. C. Tomastik, *Singular quadratic functionals of n dependent variables*, Trans. Amer. Math. Soc. 124 (1966), 60–76. MR 33 #4743.

10. William T. Reid, *Some elementary properties of proper values and proper vectors of matrix functions*, SIAM J. Appl. Math. 18 (1970), 259–266. MR 41 #8623.

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