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ASYMPTOTIC EQUIPARTITION OF ENERGY FOR DIFFERENTIAL EQUATIONS IN HILBERT SPACE

BY

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ABSTRACT. Of concern are second order differential equations of the form $(d/dt - if_1(A))(d/dt - if_2(A))u = 0$. Here A is a selfadjoint operator and f_1, f_2 are real-valued Borel functions on the spectrum of A. The Cauchy problem for this equation is governed by a certain one parameter group of unitary operators. This group allows one to define the energy of a solution; this energy depends on the initial data but not on the time t. The energy is broken into two parts, kinetic energy $K(t)$ and potential energy $P(t)$, and conditions on A, f_1, f_2 are given to insure asymptotic equipartition of energy: $\lim_{t\to\pm\infty} K(t) = \lim_{t\to\pm\infty} P(t)$ for all choices of initial data. These results generalize the corresponding results of Goldstein for the abstract wave equation $d^2u/dt^2 + A^2u = 0$. (In this case, $f_1(\lambda) \equiv \lambda$, $f_2(\lambda) \equiv -\lambda$.)

1. Introduction. Let A be a selfadjoint operator on a complex Hilbert space $\mathcal{H}$. The Cauchy problem

$$u''(t) + A^2u(t) = 0 \quad (t \in \mathbf{R} = (-\infty, \infty)), \tag{1.1}$$

$$u(0) = f_1 \in \mathcal{D}(A^2), \quad u'(0) = f_2 \in \mathcal{D}(A) \tag{1.2}$$

is well posed; here $' = d/dt$ and $\mathcal{D}(A)$ denotes the domain of A. The *energy*

$$E_f = \|u'(t)\|^2 + \|Au(t)\|^2$$

depends on the data $f = (f_1, f_2)$ but not on t. Let

$$K(t) = \|u'(t)\|^2, \quad P(t) = \|Au(t)\|^2$$

denote the *kinetic* and *potential energy* at time t. Goldstein [6] showed that the energy is *asymptotically equipartitioned*:

$$\lim_{t\to\pm\infty} K(t) = \lim_{t\to\pm\infty} P(t) = E_f/2$$

for all choices of initial data f if and only if $\lim_{t\to\pm\infty} \int_{-\infty}^{\infty} e^{it\lambda}\, d(\|\Pi_\lambda h\|^2) = 0$ for all $h \in \mathcal{H}$, where $\{\Pi_\lambda : \lambda \in \mathbf{R}\}$ is the resolution of the identity associated with A, i.e., $A = \int_{-\infty}^{\infty} \lambda\, d\Pi_\lambda$ is the spectral integral representation of A.

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Now consider the more general problem

$$v'(t) = iBv(t) \quad (t \in \mathbf{R}), \quad v(0) = g \in \mathcal{D}(B) \tag{1.3}$$

in a complex Hilbert space, where B is selfadjoint. (1.1), (1.2) is a special case of (1.3) (cf. [6]). The solution of (1.3) is given by $v(t) = U(t)g$, where $\{U(t) = \exp(itB)\colon t \in \mathbf{R}\}$ is a one parameter group of unitary operators. Interpret $E_g = \|v(t)\|^2$ to be the energy at time t; E_g depends on the data g but not on t. The *question* which motivated the present paper is: *With which Cauchy problems* (1.3) *can one decompose the energy into different types of energy and prove an asymptotic equipartition of energy theorem*?

This seems to be a very difficult question, even if one specializes the context to symmetric hyperbolic systems (Friedrichs [4]) or higher order hyperbolic equations (Mizohata [8]). Mochizuki [9], [10], generalizing [6], proved an asymptotic equipartition of energy theorem for a special class of higher order hyperbolic equations. In this paper, we generalize [6], [7] in a different way, proving asymptotic equipartition of energy for a class of second order equations of the form

$$u''(t) + iSu'(t) + Tu(t) = 0 \qquad (t \in \mathbf{R})$$

where S and T are certain commuting selfadjoint operators.

Other papers dealing with equipartition of energy are Bobisud and Calvert [1], Brodsky [2], Duffin [3], Glassey [5], and Shinbrot [14].

After some preliminaries in §2, the main results are presented in §§3 and 4. §5 is devoted to an example.

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2. **Preliminaries.** Let H be a complex Hilbert space, and let B_1, B_2 be (not necessarily bounded) selfadjoint operators on H. We say that B_1 and B_2 *commute* if and only if $(\lambda iI - B_1)^{-1}$ and $(\mu iI - B_2)^{-1}$ commute for all $\lambda, \mu \in \mathbf{R}\backslash\{0\}$ (if and only if $\exp(isB_1)$ and $\exp(itB_2)$ commute for all $s, t \in \mathbf{R}$). Also, if B_1, B_2 commute, there is a selfadjoint operator A on H and real-valued Borel functions g_1, g_2 defined on the spectrum of A, $\sigma(A) \subset \mathbf{R}$, such that $B_j = g_j(A)$ for $j = 1, 2$ (cf. [12, p. 358]).

Let B_1, B_2 be commuting selfadjoint operators with $B_j = g_j(A)$ as above. The equation

$$u'' + iB_1u' + B_2u = 0 \tag{2.1}$$

can be factored as

$$(d/dt - if_1(A))(d/dt - if_2(A))u = 0 \tag{2.2}$$

where f_1, f_2 are Borel functions on $\sigma(A)$ such that $g_1 = f_1 + f_2$, $g_2 = -f_1 f_2$. There are real-valued solutions f_1, f_2 of these equations as long as $g_1^2 + 4g_2 \geqslant 0$ on $\sigma(A)$.

Our asymptotic equipartition of energy theorem will be for the Cauchy problem (2.2). As indicated above, a great many of the equations (2.1) can be written in this form. The advantage of working with (2.2) is that one may exploit the techniques developed by Sandefur [13] to deal with the factored equation $\Pi_{j=1}^m (d/dt - C_j)u = 0$.

3. **The main theorem.** Let A be a selfadjoint operator on a complex Hilbert space H. Let f_1, f_2 be real-valued Borel functions on the spectrum $\sigma(A)$ of A. Let $A_j = if_j(A)$ for $j = 1, 2$ and let

$$\mathcal{D}_2 = \bigcap \{\mathcal{D}(A_j A_k) \colon 1 \leqslant j, k \leqslant 2\},$$

$$\mathcal{D}_3 = \bigcap \{\mathcal{D}(A_j A_k A_l) \colon 1 \leqslant j, k, l \leqslant 2\};$$

$\mathcal{D}_2$ and $\mathcal{D}_3$ are dense in H. Define

$$B = -2^{-1}(A_1 + A_2), \quad C = 2^{-1}(-A_1 + A_2)$$

with domain $\mathcal{D}(B) = \mathcal{D}(C) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2)$. Note that iB and iC are essentially selfadjoint operators, i.e. their closures are selfadjoint.

We consider the initial value problem

$$(d/dt - A_1)(d/dt - A_2)u(t) = 0 \quad (t \in \mathbf{R}), \tag{3.1}$$

$$u(0) = \phi_1 \in \mathcal{D}_3, \quad u'(0) = \phi_2 \in \mathcal{D}_2. \tag{3.2}$$

THEOREM 3.1. *Let A, A_1, A_2 be as above. Then for any $\phi_1 \in \mathcal{D}_3$, $\phi_2 \in \mathcal{D}_2$, the problem* (3.1), (3.2) *has a unique twice continuously differentiable solution u. Let*

$$K(t) \equiv \|u'(t) + Bu(t)\|, \quad P(t) \equiv \|Cu(t)\|^2, \quad t \in \mathbf{R}.$$

Then

$$E_\phi \equiv K(t) + P(t)$$

depends on ϕ but not on t. Let $\{\Pi_\lambda \colon \lambda \in \mathbf{R}\}$ be the resolution of the identity associated with A. Then

$$\lim_{t \to \pm\infty} K(t) = \lim_{t \to \pm\infty} P(t) = E_\phi/2 \tag{3.3}$$

for all choices of initial data ϕ as in (3.2) *if and only if*

$$\lim_{t \to \pm\infty} \int_{-\infty}^{\infty} \exp\{it(f_1(\lambda) - f_2(\lambda))\}\, d_\lambda(\|\Pi_\lambda x\|^2) = 0 \tag{3.4}$$

for all $x \in H$.

When $f_1(t) \equiv -f_2(t) \equiv t$, (3.1) reduces to (1.1), and Theorem 3.1 contains Goldstein's result [6] as a special case. {Strictly speaking more is demanded of the initial data (cf. (3.2)) than in [6], but there is no loss of generality as long as one works with a dense set of initial data. For example, it suffices to prove the theorem in [6] for initial data $\phi_1, \phi_2 \in \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$; the general case follows by a simple density argument.}

With the aid of the Riemann-Lebesgue lemma it is easy to write down sufficient conditions for (3.4) to hold. For example it is sufficient to suppose A is spectrally absolutely continuous and that for each $x \in H$, $g_x(h')^{-1} \in L^1(h(\sigma(A)))$ where $g_x(\lambda) = (d/d\lambda)(\|\Pi_\lambda x\|^2)$, and where $h = f_1 - f_2$ is increasing and absolutely continuous on $\sigma(A)$. We omit the simple verification (based on the change of variables $\mu = h(\lambda)$).

PROOF OF THEOREM 3.1. Consider the initial value problem

$$\frac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad (t \in \mathbf{R}) \tag{3.5}$$

in $H \oplus H$ with

$$\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{D}\left(\begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix}\right). \tag{3.6}$$

Since A_j is skew-adjoint it generates a unitary group $\{T_j(t) = \exp(tA_j): t \in \mathbf{R}\}$ for $j = 1, 2$; therefore

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

generates the unitary group

$$T(t) = \left\{\begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix} : t \in \mathbf{R}\right\}$$

on $H \oplus H$. Since the operator $\left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$ is bounded,

$$\begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

generates a group $\{S(t): t \in \mathbf{R}\}$ given by

$$S(t)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} T_1(t)\psi_1 + \int_0^t T_1(t-s)T_2(s)\psi_2\, ds \\ T_2(t)\psi_2 \end{pmatrix} \tag{3.7}$$

according to the Phillips' perturbation theorem [11] and a short calculation. For

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$S(t)\psi$ is the unique twice strongly continuously differentiable solution to (3.5) and (3.6). (One can, of course, easily directly check that for $S(t)\psi$ defined by (3.7),

$$(d/dt)S(t)\psi = \begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix} S(t)\psi,$$

$S(0)\psi = \psi$.)

If we choose $\psi_1 = \phi_1$, $\psi_2 = \phi_2 - A_1\phi_1$ and set

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = S(t)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad t \in \mathbf{R},$$

it follows that u_1 is a solution of (3.1), (3.2). Also by (3.2) and the definition of $\mathcal{D}_2$ and $\mathcal{D}_3$, we have

$$\psi_1, \psi_2 \in \mathcal{D}\left(\begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix}^2\right)$$

and so u_1 and $u_2 = u_1' - A_1 u_1$ are twice strongly continuously differentiable.

If v is any twice strongly continuously differentiable solution to (3.1), (3.2), then

$$\begin{pmatrix} v \\ v' - A_1 v \end{pmatrix}$$

is a solution to (3.5), (3.6). But by uniqueness for (3.5), (3.6), it follows that $v = u_1$, and so the solution to (3.1), (3.4) is unique.

Note that

$$\|u_1'(t) - A_1 u(t)\|^2 = \|u_2(t)\|^2 = \|T_2(t)(\phi_2 - A_1\phi_1)\|^2 = \|\phi_2 - A_1\phi_1\|^2 \tag{3.8}$$

since $T_2(t)$ is unitary. We have the companion equality

$$\|u_1'(t) - A_2 u_1(t)\|^2 = \|\phi_2 - A_2\phi_1\|^2 \tag{3.9}$$

for all $t \in \mathbf{R}$. That we can interchange the subscripts 1 and 2 in ϕ and A follows from that fact that $(d/dt - A_1)(d/dt - A_2)u_1 = 0$ if and only if $(d/dt - A_2) \cdot (d/dt - A_1)u_1 = 0$ by the closedness and commutativity of A_1 and A_2; thus the proof of (3.8) also yields (3.9). From now on let u $(= u_1)$ denote the unique

solution of (3.1), (3.2). By (3.8), (3.9) and the parallelogram law,

$$\begin{aligned} K(t) + P(t) &= \|u'(t) + Bu(t)\|^2 + \|Cu(t)\|^2 \\ &= 2^{-1}\{\|u'(t) + Bu(t) - Cu(t)\|^2 + \|u'(t) + Bu(t) + Cu(t)\|^2\} \\ &= 2^{-1}\{\|u'(t) - A_2u(t)\|^2 + \|u'(t) - A_1u(t)\|^2\} \\ &= 2^{-1}\{\|\phi_2 - A_2\phi_1\|^2 + \|\phi_2 - A_1\phi_1\|^2\} \\ &= \|\phi_2 + B\phi_1\|^2 + \|C\phi_1\|^2 = E_\phi \end{aligned}$$

for all $t \in \mathbf{R}$. Thus the first two assertions of the theorem are proved. Next, using (3.8), (3.9) again,

$$\begin{aligned} P(t) &= \|Cu(t)\|^2 = 4^{-1}\|(u'(t) - A_1u(t)) - (u'(t) - A_2u(t))\|^2 \\ &= 4^{-1}\{\|u'(t) - A_1u(t)\|^2 + \|u'(t) - A_2u(t)\|^2 \\ &\qquad - 2\,\mathrm{Re}\langle T_1(t)(\phi_2 - A_2\phi_1), T_2(t)(\phi_2 - A_1\phi_1)\rangle\} \\ &= 4^{-1}\{\|\phi_2 - A_1\phi_1\|^2 + \|\phi_2 - A_2\phi_1\|^2 \\ &\qquad - 2\,\mathrm{Re}\langle T_2(-t)T_1(t)(\phi_2 - A_2\phi_1), \phi_2 - A_1\phi_1\rangle\} \\ &= 2^{-1}E_\phi - \alpha(t, \phi) \end{aligned} \tag{3.10}$$

where

$$\alpha(t, \phi) = 2^{-1}\,\mathrm{Re}\langle T_2(-t)T_1(t)(\phi_2 - A_2\phi_1), \phi_2 - A_1\phi_1\rangle. \tag{3.11}$$

Thus $\lim_{t\to\pm\infty} P(t) = E_\phi/2$ for all ϕ as in (3.2) if and only if $\lim_{t\to\pm\infty} \alpha(t, \phi) = 0$ for all such ϕ. Taking $\phi_1 = 0$, we see that (3.3) implies

$$\lim_{t\to\pm\infty} \mathrm{Re}\langle T_2(-t)T_1(t)x, x\rangle = 0 \tag{3.12}$$

for all $x \in \mathcal{D}_2$, whence for all $x \in H$. But (3.12) implies

$$\lim_{t\to\pm\infty} \langle T_2(-t)T_1(t)x, y\rangle = 0 \tag{3.13}$$

for all $x, y \in H$ by polarization. Thus (3.3) implies

$$\lim_{t\to\pm\infty} \alpha(t, \phi) = 0; \tag{3.14}$$

simply take $x = \phi_2 - A_2\phi_1$, $y = \phi_2 - A_1\phi_1$ in (3.13). Conversely, (3.12) implies (3.13) implies (3.14) implies (3.3); thus (3.3) and (3.12) are equivalent.

According to the spectral theorem and the associated operational calculus,

$$\langle T_2(-t)T_1(t)x, x\rangle = \int_{-\infty}^{\infty} \exp\{it(f_1(\lambda) - f_2(\lambda))\}\, d_\lambda\langle \Pi_\lambda x, x\rangle,$$

whence (3.12) is equivalent to (3.4) and the theorem follows. □

4. Asymptotic equipartition in the Cesàro sense.

THEOREM 4.1. *Let A, A_1, A_2 be as in Theorem* 3.1. *Then*

$$\lim_{\tau\to\pm\infty} \tau^{-1}\int_0^\tau K(t)\,dt = \lim_{\tau\to\pm\infty} \tau^{-1}\int_0^\tau P(t)\,dt = E_\phi/2 \tag{4.1}$$

for all initial data $\phi = (\phi_1, \phi_2) \in \mathcal{D}_3 \times \mathcal{D}_2$ if and only if

$$\lim_{\tau\to\pm\infty} \tau^{-1}\int_0^\tau \alpha(t, \phi)\,dt = 0 \tag{4.2}$$

for all such ϕ where α is defined by (3.11).

In particular, (4.1) *holds if* 0 *is not an eigenvalue of A (i.e. A is one-to-one) and for each $\delta > 0$,*

$$\inf\{\,|f_1(t) - f_2(t)|: t \in \sigma(A),\ |t| \geqslant \delta\} > 0. \tag{4.3}$$

This generalizes a theorem in [7] in the same way that Theorem 3.1 generalized [6].

PROOF OF THEOREM 4.1. According to (3.10), $P(t) = 2^{-1}E_\phi - \alpha(t, \phi)$ for all initial data ϕ. The equivalence of (4.1) and (4.2) follows easily from this and from $E_\phi = K(t) + P(t)$. By Theorem 3.1 it also follows that (4.2) holds if and only if

$$\lim_{\tau\to\pm\infty} \tau^{-1}\int_0^\tau \langle T_2(-t)T_1(t)x, x\rangle\,dt = 0$$

for all $x \in H$. If the hypotheses of the second paragraph of the theorem hold, we have

$$\begin{aligned}\tau^{-1}\int_0^\tau \langle T_2(-t)T_1(t)\lambda, x\rangle\,dt &= J_1 + J_2\\ &\equiv \tau^{-1}\int_0^\tau \int_{|\lambda|\leqslant\delta} \exp\{it(f_1(\lambda) - f_2(\lambda))\}\,d(\|\Pi_\lambda x\|^2)\\ &\quad + \tau^{-1}\int_0^\tau \int_{|\lambda|>\delta} \exp\{it(f_1(\lambda) - f_2(\lambda))\}\,d(\|\Pi_\lambda x\|^2);\end{aligned}$$

$$|J_1| \leqslant \tau^{-1}\int_0^\tau \int_{|\lambda|\leqslant\delta} d(\|\Pi_\lambda x\|^2) \leqslant \|\Pi_{\delta+}x - \Pi_{\delta-}x\|^2.$$

Given $\epsilon > 0$, choose and fix $\delta > 0$ such that $|J_1| < \epsilon/2$; this can be done since A is one-to-one. Next, if $b > 0$ represents the left-hand side of (4.3) and if $h = f_1 - f_2$,

$$\begin{aligned}|J_2| &= \left|\int_{|\lambda|>\delta} \tau^{-1}\int_0^\tau \exp\{ith(\lambda)\}\,dt\,d(\|\Pi_\lambda x\|^2)\right|\\ &= \left|\int_{|\lambda|>\delta} [\exp\{ith(\lambda)\} - 1]\,[i\tau h(\lambda)]^{-1}\,d(\|\Pi_\lambda x\|^2)\right|\\ &\leqslant 2\|x\|^2|\tau b|^{-1} < \epsilon/2\end{aligned}$$

if $|\tau| > 4\|x\|^2/\epsilon b \equiv \tau(\epsilon, x)$. Consequently for each $x \in H$ and $\epsilon > 0$, $|\tau| > \tau(\epsilon, x)$ implies

$$\left|\tau^{-1} \int_0^\tau \langle T_2(-t)T_1(t)x, x\rangle \, dt\right| < \epsilon,$$

and so (4.1) holds. □

If $Ah = 0$ for some $h \neq 0$ and if $f_1(0) = f_2(0)$, then $T_1(t)h = T_2(t)h = h$ for all real t, whence

$$\tau^{-1} \int_0^\tau \langle T_2(-t)T_1(t)h, h\rangle \, dt = \|h\|^2$$

does not tend to zero as $\tau \longrightarrow \pm\infty$, hence (4.2) and (4.1) fail to hold in this case.

5. **An example.** The maximal operator A for id/dx on the complex Hilbert space $H = L^2(\mathbf{R})$ is selfadjoint. Consider the initial value problem for

$$u''(t) + P(A)u'(t) + Q(A)u(t) = 0 \qquad (t \in \mathbf{R}) \tag{5.1}$$

where

$$P(\xi) = \sum_{j=0}^{n} a_j \xi^j, \qquad Q(\xi) = \sum_{j=0}^{m} b_j \xi^j/4$$

for $\xi \in \mathbf{R}$, where $a_j, b_j \in \mathbf{C}$, $a_n \neq 0$, $b_m \neq 0$. (Condition (3.2) in this case becomes $u(0) \in \mathcal{D}(A^{3N})$, $u'(0) \in \mathcal{D}(A^{2N})$ where $N = \max(n, m)$.) (5.1) can be rewritten in the form

$$[d/dt - 2^{-1}(-P(A) + R(A))]\,[d/dt - 2^{-1}(-P(A) - R(A))]\,u = 0 \tag{5.2}$$

where

$$R(\xi) = \{P(\xi)^2 - 4Q(\xi)\}^{1/2}, \qquad \xi \in \mathbf{R}.$$

Thus Theorem 3.1 (and Theorem 4.1) can be applied if $\mathrm{Re}(-P(\xi) \pm R(\xi)) = 0$ for each $\xi \in \mathbf{R}$. This is equivalent to:

(i) For $j = 0, 1, \ldots, n$, $a_j = ic_j$ where $c_j \in \mathbf{R}$;

(ii) $-(\Sigma_{j=0}^n c_j\xi^j)^2 - \Sigma_{j=0}^m b_j\xi^j$ is real and nonpositive for all real ξ.

In particular, each b_j must be real. We now suppose $c_j = -ia_j$ and b_k are real for each j, k.

Case I. Suppose $2n > m$. The left side of (ii) is bounded above. Consequently (ii) automatically holds if we increase b_0 enough.

Case II. Suppose $2n = m$. If $-c_n^2 - b_{2n} < 0$, the left side of (ii) is bounded above, so (ii) holds if we increase b_0 enough.

Case III. Suppose $2n < m$. In order for the left-hand side of (ii) to be bounded above, m must be even. If this is so and if $b_m > 0$, then the conclusion of Cases I and II hold in this case as well. Thus we assume one of the following conditions:

(α) $2n > m$, or

(β) $2n = m$, $c_n^2 + b_{2n} > 0$, or

(γ) $2n < m$, m is even, $b_m > 0$,

and if necessary we enlarge b_0 so that (i) and (ii) hold.

Let $\{\Pi_\lambda : \lambda \in \mathbf{R}\}$ be the resolution of the identity associated with A and let $\{\Gamma_\mu : \mu \geqslant 0\}$ be the resolution of the identity associated with the positive operator $S(A) = \{4Q(A) - P(A)^2\}^{1/2}$. Both A and $S(A)$ are spectrally absolutely continuous. Let

$$\mathcal{D} = \{x \in H : \Pi_\lambda x - \Pi_{-\lambda} x = \lambda \text{ for some } \lambda > 0\}.$$

$\mathcal{D}$ is dense in H. To simplify the calculation we suppose that $S(\xi) = \{4Q(\xi) - P(\xi)^2\}^{1/2}$ is monotone decreasing on $(-\infty, L)$ and monotone increasing on (L, ∞). (Without this assumption the following computations become more complicated, but the results remain valid.) For $x \in \mathcal{D}$, choose M so that $(\Pi_M - \Pi_{-M})x = x$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp(it(f_1(\lambda) - f_2(\lambda)))\, d(\|\Pi_\lambda x\|^2) &= \int_{-\infty}^{\infty} e^{itS(\lambda)}\, d(\|\Pi_\lambda x\|^2) \\
&= \int_{-M}^{L} e^{itS(\lambda)}\, d(\|\Pi_\lambda(\Pi_L x)\|^2) + \int_{L}^{M} e^{itS(\lambda)}\, d(\|\Pi_\lambda(x - \Pi_L x)\|^2) \\
&= \int_{S(L)}^{S(-M)} e^{-it\mu} \frac{d}{d\lambda}(\|\Pi_\lambda(\Pi_L x)\|^2) \frac{d}{d\mu}(S^{-1}(\mu))\, d\mu \\
&\quad + \int_{S(L)}^{S(M)} e^{it\mu} \frac{d}{d\lambda}(\|\Pi_\lambda(x - \Pi_L x)\|^2) \frac{d}{d\mu}(S^{-1}(\mu))\, d\mu \\
&\longrightarrow 0 \quad \text{as } t \longrightarrow \pm\infty
\end{aligned}$$

by the Riemann-Lebesgue lemma. Since $\mathcal{D}$ is dense, a simple density argument shows that (3.4) holds for all $x \in H$. Thus Theorem 3.1 gives asymptotic equipartition of energy for the equation

$$(\partial^2/\partial t^2 + P(i\partial/\partial x)(\partial/\partial t) + Q(i\partial/\partial x))u(t, x) = 0$$

where

$$P\left(i\frac{\partial}{\partial x}\right) = \sum_{j=0}^{n} i^{j+1} c_j \left(\frac{\partial}{\partial x}\right)^j, \quad Q\left(i\frac{\partial}{\partial x}\right) = 4^{-1} \cdot \sum_{j=0}^{m} i^j b_j \left(\frac{\partial}{\partial x}\right)^j,$$

c_j, $b_k \in \mathbf{R}$, $c_n \neq 0$, $b_m \neq 0$, and (α) or (β) or (γ) hold (and b_0 is large enough). □

CONCLUDING REMARK. It would be of interest to find an equipartition of energy theorem subsuming both the result of this paper and the result of Mochizuki [9].

ADDED IN PROOF. David G. Costa has informed us that he has obtained a result similar to Theorem 3.1.

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