TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 220, 1976

HYPERSURFACES OF ORDER TWO

BY

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ABSTRACT. A hypersurface S^{n-1} of order two in the real projective n-space is met by every straight line in maximally two points; cf. [1, p. 391]. We develop a synthetic theory of these hypersurfaces inductively, basing it upon a concept of differentiability. We define the index and the degree of degeneracy of an S^{n-1} and classify the S^{n-1} in terms of these two quantities. Our main results are (i) the reduction of the theory of the S^{n-1} to the nondegenerate case; (ii) the Theorem (A.5.11) that a nondegenerate S^{n-1} of positive index must be a quadric and (iii) a comparison of our theory with Marchaud's discussion of "linearly connected" sets; cf. [3].

Preface. The theory of plane curves is of major importance in our study of the hypersurfaces of order two. This theory is the first step in our induction and the means by which we define tangents. The introduction of these curves follows the approach of P. Scherk in [5] and R. Park in [4].

A. Marchaud introduced in [2] the "surfaces of order two" in the real projective three-space. Our theory is based on that paper.

We compare our hypersurfaces with the quadrics by direct construction (see Appendix) and also by showing that these hypersurfaces are identical with the common boundaries of certain pairs of linearly connected sets; cf. [3].

This paper has developed out my doctoral thesis of the same title, written at the University of Toronto, under the supervision of Professor Peter Scherk. My thanks are due to him for his continued help in the preparation of this paper. I would also like to thank Professor O. Haupt for bringing to my attention the theory of linearly connected sets.

1. Curves in P^2.

The theory of hypersurfaces is based on that of curves in the real projective plane. In this chapter, we give a precise definition of the curves of order two and introduce tangents.

1.1 *Introduction.* Let P^n be a real projective space of n dimensions; $n \geq 1$. We define a topology of P^n in the usual manner. Thus, P^n is compact and connected.

Received by the editors April 7, 1975.

AMS (MOS) subject classifications (1970). Primary 53C75; Secondary 53C75.

Key words and phrases. Index, degree of degeneracy, tangent hyperplanes, boundaries of linearly connected sets, ruled quadrics.

Let $P^k, Q^k, \ldots$ denote k-flats in P^n; $-1 \leqslant k \leqslant n-1$. The $(n-1)$- and 0-flats in P^n are the *hyperplanes* and the *points* of P^n respectively. The unique (-1)-flat is denoted by $\varnothing$.

For a collection of flats $P^k, P^m, \ldots, P^r$, $[P^k, P^m, \ldots, P^r]$ denotes the flat in P^n spanned by them.

1.2. *Plane curves.*

1.2.1. A *parameter curve* C is a continuous map from $P^1 = \{t, t', \ldots\}$ into P^2. A line T is the *tangent* of a parameter curve C at a point $t \in P^1$ if $T = \lim_{t' \to t; t' \neq t} [C(t), C(t')]$.

1.2.2. The parameter curve C is *differentiable* if the tangent of C at every $t \in P^1$ exists. C is *degenerate* if C is injective and $C(P^1)$ is a line. Finally, C is *totally degenerate* if $C(P^1)$ is a point.

If C is degenerate, then $C(P^1)$ is the tangent of C at t for each $t \in P^1$.

1.2.3. A (*plane*) *curve* Γ is the union of a finite collection of sets $C_\sigma(P^1)$ where the C_σ's are parameter curves.

A line T is a *tangent* of Γ at p if T is the tangent of some C_σ at t where $p = C_\sigma(t) \in C_\sigma(P^1) \subset \Gamma$.

1.3. *Order.*

1.3.1. A differentiable parameter curve C is of *order* 2 if 2 is the maximum of the number of points of P^1 mapped into collinear points by C and if a line meets $C(P^1)$ at exactly one point if and only if it is the tangent of C at that point.

As a parameter curve is a closed curve, this implies that a differentiable parameter curve C of order 2 is injective and there is a unique tangent at each point of $C(P^1)$.

1.3.2. Let Γ be a plane curve. Then Γ is of *order* 1 if $\Gamma = C(P^1)$ where C is a degenerate parameter curve. Γ is a *nondegenerate curve of order* 2 if $\Gamma = C(P^1)$ where C is a differentiable parameter curve of order 2. Finally, Γ is a *degenerate curve of order* 2 if either $\Gamma = C(P^1)$ where C is a totally degenerate parameter curve or $\Gamma = C_1(P^1) \cup C_2(P^1)$ where C_1 and C_2 are distinct degenerate parameter curves.

We refer to the plane curves in 1.3.2 as the curves of order $\leqslant 2$. We shall denote a nondegenerate plane curve of order 2 by S^1. We quote without proof:

1.3.3. LEMMA. *Let $S^1 \subset P^2$ be a curve of order* 2.

(1) *A line T is the tangent of S^1 at p if and only if $T \cap S^1 = \{p\}$.*

(2) *There is a line $L \subset P^2$ such that $L \cap S^1 = \varnothing$.*

2. **Hypersurfaces of order two.** We shall study the (differentiable) hypersurfaces S^{n-1} of order two by constructing their tangent hyperplanes and introducing two invariants: the index and the degree of degeneracy of an S^{n-1}.

2.1. *Introduction.*

2.1.1. A *hypersurface of order two* in P^2 is an S^1. We wish to define hypersurfaces S^{n-1} of order two in P^n; $n \geqslant 3$.

2.1.2. A set $S \subset P^n$ ($n \geqslant 3$) is a *hypersurface in* P^n if it is compact and if every point $p \in S$ has a neighbourhood in S which is the continuous image of the union of a finite number of open $(n-1)$-balls such that outside an $(n-2)$-dimensional subset of that union, the mapping is locally homeomorphic.

In view of 2.1.1, we may assume that S^{m-1}, a hypersurface of order 2 in P^m, is already defined; $2 \leqslant m \leqslant n-1$.

2.1.3. Let $n \geqslant 3$. Let M be a set in P^n; $P^k \subset P^n$, $2 \leqslant k \leqslant n-1$. The *k-section* $P^k \cap M$ is

(1) *nondegenerate* if $P^k \cap M$ is an S^{k-1};

(2) *degenerate* if $P^k \cap M$ is either an m-flat or a pair of distinct $(k-1)$-flats; $-1 \leqslant m \leqslant k-1$.

2.1.4. A hypersurface $S^{n-1} \subset P^n$ ($n \geqslant 3$) is of *order* 2 if every intersection of S^{n-1} with a hyperplane is either degenerate or nondegenerate and there is a hyperplane P_0^{n-1} such that $P_0^{n-1} \cap S^{n-1}$ is an S^{n-2}.

Henceforth, S^{n-1} will be a hypersurface of order 2; $n \geqslant 3$.

2.1.5. Lemma. *Let $P^k \subset P^n$, $1 \leqslant k \leqslant n-2$. Then $P^k \cap S^{n-1}$ is a flat or a pair of $(k-1)$-flats or an S^{k-1}.*

Proof. Let P^{n-1} be a hyperplane through P^k. By 2.1.4, $P^{n-1} \cap S^{n-1}$ is either degenerate or nondegenerate.

If $P^{n-1} \cap S^{n-1}$ is degenerate, then our assertion is trivial.

Suppose $P^{n-1} \cap S^{n-1}$ is an S^{n-2}. Let $n = 3$. Then $P^2 \cap S^2$ is an S^1 and the lemma follows from §1. Assume that the lemma is true for $P^k \subset P^m$; $3 \leqslant m \leqslant n-1$.

Since $P^{n-1} \cap S^{n-1}$ is an S^{n-2}, we have $P^k \cap S^{n-1} = P^k \cap S^{n-2}$. If $k = n-2$, then $P^{n-2} \cap S^{n-2}$ is either degenerate or nondegenerate by 2.1.4. If $k < n-2$, the lemma follows by the induction hypothesis.

Corollary. (1) *The plane section $P^2 \cap S^{n-1}$ is either a flat or a curve of order* 2.

(2) *Any line, not lying in S^{n-1}, meets S^{n-1} at most twice.*

2.1.6. Let H^{n-1} be a hypersurface in P^n such that $P^{n-1} \cap H^{n-1}$ is either a flat or a pair of $(n-2)$-flats for all $P^{n-1} \subset P^n$. It is immediate that H^{n-1} is either a hyperplane or a pair of distinct hyperplanes.

2.2. *Differentiability.* Let $p \in P^k \cap S^{n-1}$, $2 \leqslant k \leqslant n-1$. If there is a P^2 such that $p \in P^2 \subset P^k$, $P^2 \not\subset S^{n-1}$ and $P^2 \cap S^{n-1} \neq \{p\}$, then $P^k \cap S^{n-1}$ is *proper at p.*

If $P^2 \cap S^{n-1}$ is proper at p, then there is a tangent of $P^2 \cap S^{n-1}$ at p by 2.1.5, Corollary and 1.2.3.

2.2.1. A line is a *tangent* of S^{n-1} at p if it is a tangent of $P^2 \cap S^{n-1}$ at p for some P^2 through p.

2.2.2. LEMMA. *S^{n-1} has a tangent at p for all p.*

PROOF. Let p and q be distinct points of S^{n-1}. By 2.1.2, there is a point $r \in P^n \backslash S^{n-1}$. If $[p, q, r] \subset P^2$, then $P^2 \cap S^{n-1}$ is proper at p.

2.2.3. A point $p \in S^{n-1}$ is *differentiable* if there is a hyperplane π containing all the tangents of S^{n-1} at p. Otherwise, p is a *singular* point of S^{n-1}.

2.2.4. LEMMA. *Let $p \neq q$ in S^{n-1}. Then $[p, q]$ is a tangent of S^{n-1} at p if and only if $[p, q] \subset S^{n-1}$.*

PROOF. Let $[p, q]$ be a tangent of S^{n-1} at p. Then $[p, q]$ is a tangent of $P^2 \cap S^{n-1}$ at p for some P^2 through $[p, q]$. If $P^2 \cap S^{n-1}$ is a curve of order 1, then $[p, q] = P^2 \cap S^{n-1}$. If $P^2 \cap S^{n-1}$ is a curve of order 2, then $P^2 \cap S^{n-1}$ is degenerate by 1.3.3. Thus, $P^2 \cap S^{n-1}$ is a pair of lines, one of which is $[p, q]$.

Conversely, let $[p, q] \subset S^{n-1}$. Choose a point $r \in P^n \backslash S^{n-1}$. Then $[p, q]$ is a tangent of $[p, q, r] \cap S^{n-1}$ at p.

2.2.5. THEOREM. *Let $p \in S^{n-1}$ be differentiable. Then every line through p in π is a tangent of S^{n-1} at p; cf.* 2.2.3. *In particular, the tangent hyperplane $\pi = \pi(p)$ of S^{n-1} at p is unique.*

PROOF. Let L be a line; $p \in L \subset \pi$. Choose a point $q \in S^{n-1} \backslash \pi$. By 2.2.4, $[p, q] \not\subset S^{n-1}$. Then $[L, q] \cap S^{n-1}$ is proper at p with a tangent T at p. As $T \subset \pi$, we have $T = \pi \cap [L, q] = L$.

2.2.6. S^{n-1} is *differentiable* if each point contained in any $S^1 \subset S^{n-1}$ is differentiable.

We shall prove that the hypersurfaces of order 2 are in fact all differentiable; cf. 2.2.9.

2.2.7. LEMMA. *Let $\{p, q\} \subset S^{n-1}$; $[p, q] \not\subset S^{n-1}$. Let T be a line such that $T \cap S^{n-1} = \{p\}$. Then T is a tangent of S^{n-1} at p.*

PROOF. Let $P^2 = [T, q]$. By 2.1.5, $P^2 \cap S^{n-1}$ is a flat or a pair of lines or an S^1. Since $T \cap S^{n-1} = \{p\}$, the first two instances imply that $[p, q] \subset S^{n-1}$. Thus, $P^2 \cap S^{n-1}$ is an S^1 and by 1.3.3, T is a tangent of S^1 at p.

2.2.8. LEMMA. *Let $\{p, q\} \subset S^{n-1}$; $[p, q] \not\subset S^{n-1}$. Then $\tau(p) = \{r \in P^n \mid r$ lies on a tangent of S^{n-1} at $p\}$ is a flat.*

PROOF. By 2.2.2, $\tau(p) \neq \emptyset$. Let $r_1 \neq r_2$ be points in $\tau(p)$. We may assume that $p \notin [r_1, r_2]$. Thus, $P^2 = [r_1, r_2, p]$ is a plane and $T_i = [p, r_i]$ are distinct tangents of S^{n-1} at p; $i = 1, 2$.

If $T_1 \cup T_2 \subset S^{n-1}$, then either $P^2 \subset S^{n-1}$ or $P^2 \cap S^{n-1} = T_1 \cup T_2$. If say $T_1 \subset S^{n-1}$ and $T_2 \not\subset S^{n-1}$, then either $P^2 \cap S^{n-1} = T_1$ or $P^2 \cap S^{n-1}$ is a pair of lines. But $T_2 \not\subset S^{n-1}$ implies that $T_2 \cap S^{n-1} = \{p\}$ by 2.2.4. Thus if $P^2 \cap S^{n-1}$ is a pair of lines, then both lines pass through p.

If $T_i \cap S^{n-1} = \{p\}$; $i = 1, 2$, then $T_1 \neq T_2$ implies that $P^2 \cap S^{n-1}$ is not an S^1 by 1.3.3. Hence $P^2 \cap S^{n-1}$ is the point p or a line or a pair of lines, all through p.

Let T be a line; $p \in T \subset P^2$. By the preceding, either $T \subset S^{n-1}$ or $T \cap S^{n-1} = \{p\}$. Then $T \subset \tau(p)$ by 2.2.4 and 2.2.7 respectively. Hence, $[r_1, r_2] \subset P^2 \subset \tau(p)$ and $\tau(p)$ is a flat.

2.2.9. THEOREM. *S^{n-1} is differentiable*; $n \geqslant 3$.

PROOF. Let P^2 be a plane such that $P^2 \cap S^{n-1}$ is an S^1. Let $p \in S^1$. Then $[p, q] \not\subset S^{n-1}$ for each $q \in S^1 \backslash \{p\}$ and thus, $q \notin \tau(p)$ by 2.2.4. Hence, $\dim \tau(p) \leqslant n - 1$ and there is a hyperplane π containing $\tau(p)$.

2.2.10. THEOREM. *A point $v \in S^{n-1}$ is singular if and only if* $[v, p] \subset S^{n-1}$ *for all $p \in S^{n-1}$.*

PROOF. From 2.1.4, it is immediate that there is a plane P_0^2 such that $P_0^2 \cap S^{n-1}$ is an S^1. By 1.3.3, there is a line $L \subset P_0^2$ such that $L \cap S^1 = L \cap S^{n-1} = \emptyset$.

Let $v \in S^{n-1}$ be singular. By 2.2.6 and 2.2.9, $v \notin S^1$. Then by 2.1.5 Corollary, $[L, v] \cap S^{n-1} = \{v\}$.

Let $p \in S^{n-1} \backslash \{v\}$. Then clearly, $P^3 = [L, v, p]$ is a 3-flat. Let $P^2 \subset P^3$ be a plane through $[v, p]$. By 2.2.6 and 2.2.9, $P^2 \cap S^{n-1}$ is either a line or a pair of lines. As the line $P^2 \cap [L, v]$ meets S^{n-1} only at v, every line in $P^2 \cap S^{n-1}$ passes through v. Thus, $[v, p] \subset S^{n-1}$.

Since S^{n-1} is not contained in any hyperplane, the converse follows by 2.2.4 and 2.2.3.

COROLLARY. *The set V of all the singular points of S^{n-1} is a flat; moreover, $V \subset \pi(p)$ for all $p \in S^{n-1} \backslash V$.*

2.2.11. S^{n-1} is *d-(times) degenerate* if $\dim V = d - 1$. Obviously, $0 \leqslant d \leqslant n - 2$. For brevity, S^{n-1} is [*non*]*degenerate* if $[d = 0]$ $d > 0$.

2.3. *The index of S^{n-1}.*

2.3.1. S^{n-1} has the *index i* if $i = \operatorname{ind} S^{n-1}$ is the maximum dimension of any flat of P^n contained in S^{n-1}.

2.3.2. REMARK. $\dim V < \text{ind}\, S^{n-1} \leqslant n-2$.

Let M be a set in P^n. Then $[M]$ is the flat spanned by the points of M.

2.3.3. LEMMA. *Let* $\text{ind}\, S^{n-1} = i$. *Then every point of* S^{n-1} *lies in an* i*-flat contained in* S^{n-1}.

PROOF. Since $\text{ind}\, S^{n-1} = i$, there is an i-flat $P^i \subset S^{n-1}$.

Let $v \in V$. By 2.2.10, $[P^i, v]$ is a flat in S^{n-1} and thus, $i \leqslant \dim[P^i, v] \leqslant \text{ind}\, S^{n-1} = i$. Hence, $[P^i, v] = P^i$ and $V \subset P^i$.

Let $p \in S^{n-1}\backslash P^i$. Then p is differentiable and $p \notin \pi(p) \cap P^i$. By 2.2.4, $[p, \pi(p) \cap P^i] \subset S^{n-1}$. Clearly, $\dim(\pi(p) \cap P^i) = i-1$ and $\dim[p, \pi(p) \cap P^i] = i$.

COROLLARY 1. *Let* $\text{ind}\, S^{n-1} = i$; $p \in S^{n-1}\backslash V$. *Then*

(1) $P^i \subset \pi(p) \cap S^{n-1}$ *if and only if* $p \in P^i \subset S^{n-1}$;

(2) $V = \bigcap_{P^i \subset S^{n-1}} P^i$.

PROOF. The proof of 2.3.3 implies (1) and $V \subset \bigcap_{P^i \subset S^{n-1}} P^i = W$.

Let $q \in W$ and let $p \in S^{n-1}$. By 2.3.3, p lies in an i-flat $P_0^i \subset S^{n-1}$. Thus, $[p, q] \subset P_0^i \subset S^{n-1}$ and by 2.2.10, $q \in V$.

COROLLARY 2. *Let* $\pi(p) \cap S^{n-1} = S^{n-2}$ *for some* $p \in S^{n-1}\backslash V$. *Then* $\text{ind}\, S^{n-1} \leqslant n-3$.

PROOF. By 2.1.4, there are no $(n-2)$-flats in S^{n-2}. Thus p does not lie in an $(n-2)$-flat. By 2.3.3, $\text{ind}\, S^{n-1} \leqslant n-3$.

2.3.4. LEMMA. *Let* $\{p, q\} \subset S^{n-1}$; $[p, q] \not\subset S^{n-1}$. *Then* $\dim[\pi(p) \cap S^{n-1}] = \dim[\pi(q) \cap S^{n-1}]$.

PROOF. Let $k = \dim[\pi(p) \cap S^{n-1}]$. By the symmetry in p and q, it is sufficient to prove that $k \leqslant \dim[\pi(q) \cap S^{n-1}]$. Thus, we may assume $k > 0$.

Choose points $p_\sigma \in \pi(p) \cap S^{n-1}$ such that $[\pi(p) \cap S^{n-1}] = [p, p_1, \ldots, p_k]$. By 2.2.4, $L_\sigma = [p, p_\sigma] \subset S^{n-1}$ and $q \notin \pi(p)$. Thus, $[L_\sigma, q] \cap S^{n-1}$ contains the line $[q, q_\sigma]$ for some $q_\sigma \in L_\sigma$. Then $q_\sigma \in \pi(q)$ and $[q, q_1, \ldots, q_k] \subset [\pi(q) \cap S^{n-1}]$. Since $[p, q] \not\subset S^{n-1}$, we have $q \neq q_\sigma \neq p$ for $\sigma = 1, \ldots, k$. Thus, $[\pi(p) \cap S^{n-1}] = [L_1, L_2, \ldots, L_k] = [p, q_1, \ldots, q_k]$ and $P^{k-1} = [q_1, \ldots, q_k]$ is a $(k-1)$-flat. Obviously, $q \notin P^{k-1}$ and thus the k-flat $[q, P^{k-1}] \subset \pi(q) \cap S^{n-1}$. Hence, $\dim[\pi(q) \cap S^{n-1}] \geqslant k$.

2.3.5. REMARK. Let $p \in S^{n-1}\backslash V$. Then

(1) $\pi(p) \cap S^{n-1}$ is a k-flat if and only if $\dim[\pi(p) \cap S^{n-1}] = k$, $0 \leqslant k \leqslant n-2$;

(2) $\pi(p) \cap S^{n-1}$ is an S^{n-2} if and only if $\dim[\pi(p) \cap S^{n-1}] = n-1$ and $\text{ind}\, S^{n-1} < n-2$.

2.3.6. THEOREM. *Let* $\text{ind}\, S^{n-1} = i$; $0 \leqslant i \leqslant n-2$. *Then we have* *precisely one of the following three cases*:

(1) $\pi(p) \cap S^{n-1}$ *is an i-flat for all* $p \in S^{n-1}\backslash V$;

(2) $\pi(p) \cap S^{n-1}$ *is a pair of distinct* $(n-2)$*-flats for all* $p \in S^{n-1}\backslash V$, $i = n-2$;

(3) $\pi(p) \cap S^{n-1}$ *is an* S^{n-2} *for all* $p \in S^{n-1}\backslash V$, $0 < i < n-2$.

PROOF. Let $p \in S^{n-1}\backslash V$. By 2.3.3, there is a P^i through p in S^{n-1}. By 2.2.4, $P^i \subset \pi(p)$ and thus, $k(p) = \dim[\pi(p) \cap S^{n-1}] \geqslant i$. By 2.3.5, either $k(p) = i$ or $k(p) = n-1$. It remains to be shown that $k(p)$ is independent of p.

Let $\{p_1, p_2\} \subset S^{n-1}\backslash V; p_1 \neq p_2$. If $[p_1, p_2] \not\subset S^{n-1}$, then 2.3.4 implies that $k(p_1) = k(p_2)$. Let $[p_1, p_2] \subset S^{n-1}$. Since $S^{n-1}\backslash V \not\subset \pi(p_1) \cup \pi(p_2)$, there is a $p_3 \in S^{n-1}\backslash V$ such that $p_3 \notin \pi(p_1) \cup \pi(p_2)$. Therefore, $k(p_1) = k(p_3) = k(p_2)$.

2.4. *Nondegenerate* S^{n-1}. In this and the following section, we examine the properties of the nondegenerate and the degenerate S^{n-1} and we consider a relationship between them.

2.4.1. LEMMA. *Let* S^{n-1} *be nondegenerate*; ind $S^{n-1} = i$. *Then either* $i = 0$ *or* $\pi(p) = [\pi(p) \cap S^{n-1}]$ *for* $p \in S^{n-1}$.

PROOF. Let $\{p_0, p_1\} \subset S^{n-1}$; $[p_0, p_1] \not\subset S^{n-1}$. By 2.3.3, there is a $P^i_\sigma, p_\sigma \in P^i_\sigma \subset S^{n-1}$; $\sigma = 0, 1$. Clearly, $P^i_0 \neq P^i_1$.

Suppose a $q \in P^i_0 \cap P^i_1$. Then $P^i_0 \cup P^i_1 \subset \pi(q) \cap S^{n-1}$ and by 2.3.6, $[\pi(q) \cap S^{n-1}] = \pi(q)$. Assume that $[\pi(p) \cap S^{n-1}] \neq \pi(p)$ for some and hence for all $p \in S^{n-1}$ (2.3.6). Then by the preceding, $P^i_0 \cap P^i_1 = \varnothing$. By 2.3.6, $\pi(p_0) \cap S^{n-1} = P^i_0$ and thus, $\pi(p_0) \cap P^i_1 = \varnothing$. Hence, $P^i_1 = \{p_1\}$ and $i = 0$.

2.4.2. LEMMA. *Let* $\{p_0, p_1\} \subset S^{n-1}\backslash V; P^1 = [p_0, p_1] \subset S^{n-1}$. *Then* $\pi(p_0) \cap \pi(p_1) \subsetneq \pi(p)$ *for all* $p \in P^1\backslash V$.

PROOF. Let $u \in (\pi(p_0) \cap \pi(p_1))\backslash P^1$.

If $u \in S^{n-1}$, then $[P^1, u] \cap S^{n-1}$ contains the lines P^1, $[p_0, u]$ and $[p_1, u]$ by 2.2.4. Thus, $[P^1, u] \subset S^{n-1}$ by 2.1.5.

If $u \notin S^{n-1}$, then $[p_\sigma, u] \cap S^{n-1} = \{p_\sigma\}$ by 2.2.4. Hence, 2.1.5 implies that $[P^1, u] \cap S^{n-1} = P^1$. Thus, $[p, u] \cap S^{n-1} = \{p\}$ for all $p \in P^1$ and the lemma follows by 2.2.7.

2.4.3. LEMMA. *Let* S^{n-1} *be nondegenerate. Let* $P^k = [p_0, \ldots, p_k] \subset S^{n-1}$. *Then* $M = \bigcap_{\sigma=0}^{k}\pi(p_\sigma)$ *is an* $(n-k-1)$*-flat.*

PROOF. Our statement is trivial for $k = 0$.

Assume that the lemma is true for any $P^{k-1} \subset S^{n-1}$. Since $[p_0, \ldots, p_{k-1}] \subset S^{n-1}$, we have that $P^{n-k}_1 = \bigcap_{j=0}^{k-1}\pi(p_j)$ is an $(n-k)$-flat. Then $M = \pi(p_k) \cap P^{n-k}_1$ and $n-k-1 \leqslant \dim M \leqslant n-k$.

Let $q \in S^{n-1} \setminus \bigcup_{j=0}^{k} \pi(p_j)$. Then $[p_k, p_\sigma, q]$ is not in S^{n-1} and there is a line $L_\sigma \subset [p_k, p_\sigma, q] \cap S^{n-1}$ such that $L_\sigma = [q, q_\sigma]$ for some $q_\sigma \in [p_k, p_\sigma]$; $\sigma = 0, 1, \ldots, k-1$. By 2.2.4, $p_k \neq q_\sigma \neq p_\sigma$ and $q \in \pi(q_\sigma)$.

By our construction, $P^k = [p_0, p_1, \ldots, p_k] = [q_0, q_1, \ldots, q_{k-1}, p_k]$. Thus, $[q_0, \ldots, q_{k-1}] \subset S^{n-1}$ is a $(k-1)$-flat and by the induction hypothesis, $\bigcap_{\sigma=0}^{k-1} \pi(q_\sigma) = P_2^{n-k}$ is an $(n-k)$-flat.

By 2.4.2, $q_\sigma \in [p_k, p_\sigma] \subset S^{n-1}$ implies that $\pi(p_k) \cap \pi(p_\sigma) \subseteq \pi(q_\sigma)$; $\sigma = 0, 1, \ldots, k-1$. Thus, $M = \pi(p_k) \cap P_1^{n-k} \subseteq P_2^{n-k}$ and $q \in P_2^{n-k} \setminus M$.

Since $P^k \subset M$ in 2.4.3, we have

2.4.4. THEOREM. *Let S^{n-1} be nondegenerate. Then* ind $S^{n-1} \leqslant \frac{1}{2}(n-1)$.

COROLLARY 1. *If $n = 3$, then* ind $S^2 = 0$ *or* 1 *and $\pi(p) \cap S^2$ is the point p or a pair of distinct lines through p respectively.*

COROLLARY 2. *If $n \geqslant 4$ and* ind $S^{n-1} > 0$, *then $\pi(p) \cap S^{n-1}$ is an S^{n-2}.*

It should be noted that our results concerning the surfaces of order 2 in P^3 coincide with the theory in [2]. Of particular importance is the following result of Marchaud. While his proof is incomplete, it is easy to verify.

2.4.5. THEOREM (MARCHAUD [2]). *A nondegenerate S^2 with the index 1 is a quadric.*

2.5. *Degenerate S^{n-1}.* Let S^{n-1} be d-degenerate; ind $S^{n-1} = i$. Thus, $\dim V = d - 1$ and $0 \leqslant d \leqslant i \leqslant n-2$. Put $m = n - d$ and let P_v^m be an m-flat such that $P_v^m \cap V = \varnothing$. Then $P^n = [V, P_v^m]$.

2.5.1. LEMMA. $S^{n-1} = \bigcup [V, p], p \in P_v^m \cap S^{n-1}$.

PROOF. Let $q \in S^{n-1} \setminus V$. Then $[V, q] \cap P_v^m$ is a point p and $[V, q] = [V, p] \subset S^{n-1}$.

2.5.3. LEMMA. *$P_v^m \cap S^{n-1}$ is a nondegenerate S^{m-1}.*

PROOF. Let $p \in P_v^m \cap S^{n-1}$ (2.5.1). By 2.2.10, $p \notin V$ implies that there is a $p' \in S^{n-1}$ such that $[p, p'] \not\subset S^{n-1}$. Hence $p' \notin V$. By 2.5.1, $p' \in [v, q]$ where $v \in V$ and $q \in P_v^m \cap S^{n-1}$. Thus, $[v, p, p']$ is a plane and by 2.1.5 and 2.2.10, $[v, p, p'] \cap S^{n-1} = [v, p] \cup [v, q, p']$. Thus, $[p, q] \cap P_v^m \cap S^{n-1} = \{p, q\}$ and $P_v^m \cap S^{n-1}$ is not a flat.

Let $P^k \subset P_v^m \cap S^{n-1}$. Then 2.2.10 and $P^k \cap V = \varnothing$ imply that the $(k+d)$-flat $[V, P^k]$ lies in S^{n-1}. Thus, $k + d \leqslant i \leqslant n-2$ and $k \leqslant i - d \leqslant n - d - 2 = m - 2$.

By 2.1.5, $P_v^m \cap S^{n-1}$ is an S^{m-1}. By 2.2.10 and the preceding, S^{m-1} is nondegenerate.

2.5.3. LEMMA. ind $S^{m-1} = i - d$.

PROOF. Since ind $S^{n-1} = i$, there is a $P^i \subset S^{n-1}$. Then $V \subset P^i$ and $P^n = [P^i, P_v^m]$. Thus, $P^i \cap P_v^m$ is an $(i - d)$-flat contained in S^{m-1}.
From the proof of 2.5.2, ind $S^{m-1} \leqslant i - d$.
In summary, we have

2.5.4. THEOREM. *Let S^{n-1} be d-degenerate;* ind $S^{n-1} = i$. *Let $P^{n-d} \cap V = \emptyset$. Then $P^{n-d} \cap S^{n-1}$ is a nondegenerate S^{n-d-1} with the index $i - d$ and $S^{n-1} = \bigcup_{p \in S^{n-d-1}} [V, p]$.*

This theorem reduces the study of a degenerate S^{n-1} to that of a suitable nondegenerate S^{n-d-1}. But in 2.4, we have already classified the latter according to the index.

2.5.5. THEOREM. *Let S^{n-1} be d-degenerate*; ind $S^{n-1} = i$. *Let $p \in S^{n-1} \backslash V$ and let $P^{n-d} \cap V = \emptyset$; $p \in P^{n-d}$. Then there are exactly the following three cases*:

(1) $\pi(p) \cap S^{n-d-1} = \{p\}$. *Then* $\pi(p) \cap S^{n-1} = [V, p]$ *and* $i = d$.

(2) $\pi(p) \cap S^{n-d-1} = P_1^1 \cup P_2^1$, $P_1^1 \neq P_2^1$; $n - d = 3$. *Then* $\pi(p) \cap S^{n-1} = P_1^{n-2} \cup P_2^{n-2}$ *where* $P_\sigma^{n-2} = [P_\sigma^1, V]$; $\sigma = 1, 2$.

(3) $\pi(p) \cap S^{n-d-1} = S^{n-d-2}$; $n - d \geqslant 4$. *Then* $\pi(p) \cap S^{n-1}$ *is an* S^{n-2} *and* $d < i \leqslant \frac{1}{2}(n + d - 1)$.

PROOF. Let $\pi'(p)$ be the tangent hyperplane of S^{n-d-1} at p. Then $\pi'(p) = \pi(p) \cap P^{n-d}$ and $\pi'(p) \cap S^{n-d-1} = \pi(p) \cap S^{n-d-1}$. As S^{n-d-1} is nondegenerate, we obtain the intersections $\pi'(p) \cap S^{n-1}$ by 2.4.1 and 2.4.4.

(1) If $\pi(p) \cap S^{n-d-1} = \{p\}$, then ind $S^{n-d-1} = i - d = 0$. Since dim $V = d - 1$, $[V, p] \subset \pi(p) \cap S^{n-1}$ is a d-flat. Clearly, ind $S^{n-1} = d$ implies that $\pi(p) \cap S^{n-1} = [V, p]$.

(2) If $\pi(p) \cap S^{n-d-1} = P_1^1 \cup P_2^1$, then ind $S^{n-d-1} = i - d = 1$ and $n - d = 3$. Thus, $i = n - 2$ and the result follows by 2.3.6.

(3) If $\pi(p) \cap S^{n-d-1} = S^{n-d-2}$, then $S^{n-d-2} \subset \pi(p) \cap S^{n-1}$, $n - d \geqslant 4$ and $0 < i - d \leqslant \frac{1}{2}(n - d - 1)$ by 2.4.4. Obviously, $\pi(p) \cap S^{n-1}$ is an S^{n-2}.

2.6. *Decomposition.* To facilitate the study of nondegenerate S^{n-1}'s, we shall decompose them, whenever possible, into nondegenerate hypersurfaces of smaller dimension.

Let S^{n-1} be nondegenerate; ind $S^{n-1} = i \geqslant 1$ and $n \geqslant 4$.

2.6.1. LEMMA. *For any $p \in S^{n-1}$, $\pi(p) \cap S^{n-1}$ is a* 1*-degenerate S^{n-2} with* ind $S^{n-2} = i$ *and the singular point p.*

PROOF. By 2.5.5, $\pi(p) \cap S^{n-1}$ is an S^{n-2}. Since ind $S^{n-1} = i$, there is a $P^i \subset \pi(p) \cap S^{n-1}$ and thus, ind $S^{n-2} = i$.

Let $q \in S^{n-2}$. By 2.2.4, $[p, q] \subset \pi(p) \cap S^{n-1} = S^{n-2}$ and by 2.2.10, p is a singular point of S^{n-2}.

If $p \neq q$, then $\pi(p) \neq \pi(q)$ by 2.4.3. Hence by 2.4.1,

$$[\pi(q) \cap S^{n-1}] = \pi(q) \neq \pi(p) = [\pi(p) \cap S^{n-1}] = [S^{n-2}].$$

Thus, there is a point $u \in S^{n-2}$ such that $[u, q] \not\subset S^{n-1}$. In particular, $[u, q] \not\subset S^{n-2}$ and S^{n-2} is differentiable at q by 2.2.10. Hence, p is the only singular point of S^{n-2}.

2.6.2. Lemma. *Let* $\{p, q\} \subset S^{n-1}$; $[p, q] \not\subset S^{n-1}$. *Then* $\pi(p) \cap \pi(q) \cap S^{n-1}$ *is a nondegenerate* S^{n-3} *with* ind $S^{n-3} = i - 1$.

Proof. Since $[p, q] \not\subset S^{n-1}$, $P^{n-2} = \pi(p) \cap \pi(q)$ is an $(n-2)$-flat. Note that $P^{n-2} \cap S^{n-1} \subset \pi(p) \cap S^{n-1} \subset S^{n-1}$.

By 2.6.1, $\pi(p) \cap S^{n-1}$ is 1-degenerate with ind $S^{n-1} = i$ and with the singular point p. Since $\pi(p) = [P^{n-2}, p]$, 2.5.4 yields that $P^{n-2} \cap (\pi(p) \cap S^{n-1}) = P^{n-2} \cap S^{n-2}$ is a nondegenerate S^{n-3}; ind $S^{n-3} = i - 1$.

Corollary. $P^n = [P^{n-2}, p, q]$.

2.6.3. Theorem. *There is a sequence of points* $p_0, q_0, \ldots, p_i, q_i$ *in* S^{n-1} *such that, for* $0 \leqslant k \leqslant i$,

(1) $\bigcap_{\sigma=0}^{k-1}(\pi(p_\sigma) \cap \pi(q_\sigma)) = R^{n-2k}$ *is an* $(n-2k)$*-flat,*

(2) $\{p_k, q_k\} = [p_k, q_k] \cap (R^{n-2k} \cap S^{n-1})$, *and*

(3) *if* $n - 2k > 1$, *then* $R^{n-2k} \cap S^{n-1}$ *is a nondegenerate* S^{n-2k-1}; ind $S^{n-2k-1} = i - k$.

Proof. Let $p_0 \in S^{n-1}$. Since S^{n-1} is differentiable, there is a $q_0 \in S^{n-1}$ such that $[p_0, q_0] \not\subset S^{n-1}$ by 2.2.10. Then $R^{n-2} = \pi(p_0) \cap \pi(p_0)$ is an $(n-2)$-flat and by 2.6.2, $R^{n-2} \cap S^{n-1}$ is a nondegenerate S^{n-3}; ind $S^{n-3} = i - 1$. We now choose points $\{p_1, q_1\} \subset S^{n-3}$ such that $[p_1, q_1] \not\subset S^{n-3}$. The tangent hyperplane of S^{n-3} is the $(n-3)$-flat $\pi(p) \cap R^{n-2}$; thus

$$R^{n-4} = \pi(p_1) \cap \pi(q_1) \cap R^{n-2} = \bigcap_{\sigma=0}^{1} (\pi(p_\sigma) \cap \pi(q_\sigma))$$

is an $(n-4)$-flat.

If $n \geqslant 6$ and $i \geqslant 2$, we can repeat this construction. By 2.6.2, $R^{n-4} \cap S^{n-3} = R^{n-4} \cap S^{n-1}$ is a nondegenerate S^{n-5}; ind $S^{n-5} = i - 2$. Obviously, we can choose $\{p_2, q_2\} \subset S^{n-5}$ such that $[p_2, q_2] \not\subset S^{n-5}$.

Thus, as long as $n - 2(k-1) \geqslant 4$ and $i \geqslant k$, we can repeat our construction obtaining a sequence of points $p_0, q_0, \ldots, p_k, q_k$ which satisfy the conditions (1)–(3).

By 2.4.4, $n \geqslant 2i + 1$. If $n > 2i + 1$, then $i \geqslant k$ implies that $n - 2(k-1)$

$\geqslant 4$. Thus, the construction yields $p_0, q_0, \ldots, p_i, q_i$.

If $n = 2i + 1$, then $i - 1 \geqslant k$ implies that $n - 2(k - 1) \geqslant 5$. Now the construction yields $p_0, q_0, \ldots, p_{i-1}, q_{i-1}$. In particular, $R^{n-2(i-1)} = R^3$ and $R^3 \cap S^{n-2}$ is a nondegenerate S^2; ind $S^2 = 1$.

By 2.2.4 Corollary 1, $\pi(p) \cap S^2$ is a pair of lines through p for $p \in S^2$. Since $[p_{i-1}, q_{i-1}] \not\subset S^2$, the line $R^1 = \pi(p_{i-1}) \cap \pi(q_{i-1}) \cap R^3$ does not pass through p_{i-1} or q_{i-1}. Thus, R^1 meets S^2 at exactly two distinct points, say p_i and q_i. Then $[p_i, q_i] \not\subset S^2$ and $\{p_i, q_i\} = R^1 \cap S^2 = \bigcap_{\sigma=0}^{i-1} [(\pi(p_\sigma) \cap \pi(q_\sigma))] \cap S^{n-1}$.

COROLLARY. $\dim [p_0, \ldots, p_i, q_0, \ldots, q_i] = 2i + 1$.

PROOF. By 2.6.2 Corollary, we obtain

$$P^n = [R^{n-2}, p_0, q_0] = [R^{n-4}, p_1, q_1, p_0, q_0]$$
$$= [R^{n-2i}, p_{i-1}, q_{i-1}, \ldots, p_0, q_0].$$

If $n = 2i + 1$, then $R^{n-2i} = R^1 = [p_i, q_i]$. If $n > 2i + 1$, then ind $S^{n-2i-1} = 0$ and $R^{n-2i} = [\pi(p_i) \cap \pi(q_i) \cap R^{n-2i}, p_i, q_i]$. Since $(\pi(p_i) \cap \pi(q_i)) \cap [p_i, q_i] = \varnothing$, the result follows.

2.6.4. Let $r_0, r_1, \ldots, r_k$ be a sequence of points in P^n. We shall denote by $[r_0, \ldots, \hat{r}_\alpha, \ldots, r_k]$, the flat of P^n spanned by the points r_σ; $\sigma = 0, 1, \ldots, k$ and $\sigma \neq \alpha$.

Let $I = \{0, 1, \ldots, i\}$. From 2.6.3, we observe that

(1) $P^n = [p_0, \ldots, p_i, R^s, q_0, \ldots, q_i]$ where

(2) $R^s = \bigcap_{\sigma=0}^{i} (\pi(p_\sigma) \cap \pi(q_\sigma))$; $s = n - (2i + 2) \geqslant -1$. Thus for $\alpha \in I$,

(3) $\pi(p_\alpha) = [p_0, \ldots, p_i, R^s, q_0, \ldots, \hat{q}_\alpha, \ldots, q_i]$ and

(4) $\pi(q_\alpha) = [q_0, \ldots, q_i, R^s, p_0, \ldots, \hat{p}_\alpha, \ldots, p_i]$.

We put

(5) $\widetilde{P}^i = [p_0, \ldots, p_i]$,

(6) $\widetilde{Q}^i = [q_0, \ldots, q_i]$,

(7) $P^i_\alpha = [p_0, \ldots, \hat{p}_\alpha, \ldots, p_i, q_\alpha]$, $\alpha \in I$, and

(8) $Q^i_\alpha = [q_0, \ldots, \hat{q}_\alpha, \ldots, q_i, p_\alpha]$, $\alpha \in I$.

Thus, $P^n = [\widetilde{P}^i, R^s, \widetilde{Q}^i]$.

2.6.5. LEMMA. *The i-flats $\widetilde{P}^i$, $\widetilde{Q}^i$, P^i_α and Q^i_α lie in S^{n-1}*; $\alpha \in I$.

PROOF. We prove that $\widetilde{P}^i \subset S^{n-1}$.

By 2.1.5, $\widetilde{P}^i \cap S^{n-1}$ is a flat or a pair of $(i - 1)$-flats or an S^{i-1}. By 2.6.4(3), $p_\alpha \in \pi(p_\beta)$ and thus, $[p_\alpha, p_\beta] \subset S^{n-1}$ for $\{\alpha, \beta\} \subset I$ by 2.2.4. Then $\widetilde{P}^i = [p_0, \ldots, p_i]$ implies that either $\widetilde{P}^i \subset S^{n-1}$ or $\widetilde{P}^i \cap S^{n-1}$ is an S^{i-1}.

Suppose $\widetilde{P}^i \cap S^{n-1}$ is an S^{i-1}. Since $[p_\alpha, p_\beta] \subset S^{n-1}$ for $\{\alpha, \beta\} \subset I$,

p_α must be a singular point of S^{i-1} for $\alpha \in I$. Thus, $\widetilde{P}^i$ is the singular flat of S^{i-1}; a contradiction. Thus $\widetilde{P}^i \subset S^{n-1}$.

By similar arguments, the other i-flats lie in S^{n-1}.

COROLLARY. $R^s \cap S^{n-1} = \varnothing$.

PROOF. $R^s \subset \bigcap_{\sigma=0}^{i}\pi(p_\sigma)$ implies that $\widetilde{P}^i \subset \pi(p) \cap S^{n-1}$ for any $p \in R^s \cap S^{n-1}$. But $R^s \cap \widetilde{P}^i = \varnothing$ and 2.3.3 Corollary 1 imply that this is not possible. Thus $R^s \cap S^{n-1} = \varnothing$.

Let $n \geqslant 2i + 2$. Thus there is a point $r \in R^s$. By 2.6.4(1), $P^{2k+2} = [p_0, \ldots, p_k, r, q_0, \ldots, q_k]$ is a $(2k+2)$-flat; $0 \leqslant k \leqslant i$.

2.6.6. LEMMA. *$P^{2k+2} \cap S^{n-1}$ is a nondegenerate S^{2k+1} with the index k*; $0 \leqslant k \leqslant i$.

PROOF. Obviously, $P^2 = [p_0, q_0, r] \subset P^{2k+2}$. Since $[p_0, q_0] \not\subset S^{n-1}$ and $r \in \pi(p_0) \cap \pi(q_0)$, $r \notin S^{n-1}$ implies that $P^2 \cap S^{n-1}$ is an S^1 by 2.1.5. Hence, $P^{2k+2} \cap S^{n-1}$ is an S^{2k+1} with the tangent hyperplane $\pi(p) \cap P^{2k+2}$ at $p \in S^{2k+1}$. Furthermore, $[p_\alpha, q_\alpha] \not\subset S^{2k+1}$ implies that S^{2k+1} is differentiable at each of the points p_α, q_α; $\alpha = 0, \ldots, k$.

By 2.6.4(3) and 2.6.4(4),

$$\pi(p_\alpha) \cap P^{2k+2} = [p_0, \ldots, p_k, r, q_0, \ldots, \hat{q}_\alpha, \ldots, q_k]$$

and

$$\pi(q_\alpha) \cap P^{2k+2} = [q_0, \ldots, q_k, r, p_0, \ldots, \hat{p}_\alpha, \ldots, p_k]; \quad \alpha = 0, \ldots, k.$$

Therefore, $\bigcap_{\alpha=0}^{k}(\pi(p_\alpha) \cap \pi(q_\alpha) \cap P^{2k+2}) = \{r\}$. Since $r \notin S^{n-1}$, S^{2k+1} is nondegenerate by 2.2.10 Corollary.

By 2.4.4, ind $S^{2k+1} \leqslant k$. As $[p_0, \ldots, p_k] \subset P^{2k+2} \cap \widetilde{P}^i \subseteq P^{2k+2} \cap S^{n-1} = S^{2k+1}$, we obtain ind $S^{2k+1} = k$.

The preceding lemma is readily extended to the case $n = 2i + 1$ if we assume that $0 \leqslant k \leqslant i - 1$. Then by 2.6.3, $R^1 = \bigcap_{\sigma=0}^{i-1}(\pi(p_\sigma) \cap \pi(q_\sigma))$ and $R^1 \cap S^{n-1} = \{p_i, q_i\}$. We then choose $r \in R^1 \backslash S^{n-1}$ and our result is valid for $P^{2k+2} = [p_0, \ldots, p_k, r, q_0, \ldots, q_k]$; $0 \leqslant k \leqslant i - 1$.

In summary, we have

2.6.7. THEOREM. *Let S^{n-1} be nondegenerate*; ind $S^{n-1} = i \geqslant 1$ *and* $n \geqslant 4$. *Let $0 \leqslant k \leqslant i - 1$ $[0 \leqslant k \leqslant i - 2]$ and let $r \in R^s$ $[r \in R^1 \backslash S^{n-1}]$ when $n > 2i + 1$ $[n = 2i + 1]$. Then $P^n = [R^{n-2(k+1)}, P^{2(k+1)}]$ where*

(1) $R^{n-2(k+1)} = \bigcap_{\sigma=0}^{k}(\pi(p_\sigma) \cap \pi(q_\sigma))$,

(2) $P^{2(k+1)} = [p_0, \ldots, p_k, r, q_0, \ldots, q_k]$,

(3) $R^{n-2(k+1)} \cap P^{2(k+1)} = \{r\}$,

(4) $R^{n-2(k+1)} \cap S^{n-1}$ *is a nondegenerate S^{n-2k-3} of index $i - (k+1)$*,

and

(5) $P^{2(k+1)} \cap S^{n-1}$ *is a nondegenerate* S^{2k+1} *of index* k.

3. Linearly connected sets. In this chapter, we shall prove that a nondegenerate $S^{n-1} \subset P^n$ with a positive index is the boundary of a certain type of linearly connected set as introduced in [3]. Marchaud's results then imply that such an S^{n-1} is a quadric.

3.1. *Introduction.*

3.1.1. A set M in P^n is *linearly connected* if $P^1 \cap M$ is connected for all $P^1 \subset P^n$.

Thus M is connected and $P^n \backslash M$ is linearly connected.

3.1.2. Let M_σ be sets in P^k; $k \geqslant 2$; $\sigma = 1, 2$. M_1 and M_2 are a *linearly connected pair in* P^k if M_1 and M_2 are nonvoid, open, disjoint, linearly connected sets such that $P^k \backslash (M_1 \cup M_2) = \overline{M}_1 \cap \overline{M}_2$.

Let $n \geqslant 3$. Let A and B be a linearly connected pair in P^n. Let $F = \overline{A} \cap \overline{B}$. Then $P^n = A \cup B \cup F$, $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, $\overline{B} = B \cup F$ and $\overline{A} = A \cup F$.

We collect some definitions and results regarding such linearly connected sets; cf. [3].

3.1.3. Let $P^k \subset P^n$ be a k-flat. P^k is a *secant* if $P^k \cap A \neq \emptyset \neq P^k \cap B$. P^k *supports* $A[B]$ if $P^k \subset \overline{B}[\overline{A}]$, $P^k \cap F \neq \emptyset$ and $P^k \cap B \neq \emptyset$ $[P^k \cap A \neq \emptyset]$.

3.1.4. Let $P^k \subset P^n$ be a secant. Then $P^k \cap A$ and $P^k \cap B$ are a linearly connected pair in P^k with the common boundary $P^k \cap F$. In particular, a line is a secant if and only if it meets F in exactly two distinct points.

3.1.5. A point p in F is *regular* if there exists a secant line through p. Otherwise, p is an *irregular* point of F.

3.1.6. The *index* of $A[B]$ is the maximum dimension of any flat of P^n contained in $A[B]$. We shall denote the indices by i_mA and i_mB respectively.

3.1.7. A set O in P^2 is an *oval* if O is an injective continuous image of P^1 into P^2 and O is the common boundary of a linearly connected pair in P^2.

3.1.8. A oval O has a paratingent at each point p; i.e. a line of accumulation of lines through two distinct points of O which tend simultaneously to p.

Let F be regular; that is, every $p \in F$ is regular.

3.1.9. Let $P^2 \subset P^n$ be a secant. Then $P^2 \cap F$ is either a pair of lines or an oval. Thus, any line which meets F in three distinct points is contained in F.

3.1.10. Let $c \in A$ $[B]$. Then there is a k-flat through c in A $[B]$ where $k = i_mA$ $[i_mB]$.

3.1.11. Let $\min\{i_mA, i_mB\} > 0$. Then a line is a paratingent of an oval if and only if it meets the oval at exactly one point.

3.1.12. Let F be regular. If $\min\{i_mA, i_mB\} > 0$, then F is a nondegenerate ruled quadric.

3.2. *Linearly connected pairs.* Let A and B be a linearly connected pair in P^n such that

$$(1)_n \qquad (1)\ F = P^n \backslash (A \cup B) = \overline{A} \cap \overline{B}, \quad (2)\ F \text{ is regular}, \quad (3)\ \min\{i_m A, i_m B\} \geq 1.$$

We shall show that F is a hypersurface of order 2 in P^n; $n \geq 3$.

3.2.1. LEMMA. *Let F satisfy $(1)_n$. Then F is a hypersurface in P^n.*

PROOF. Clearly, F is closed.

Let $p \in F$. Since p is regular, there exists a secant $L = [p, q]$ such that $L \cap F = \{p, q\}$. Thus, there is a point $a_0 \in L \cap A$ and a point $b_0 \in L \cap B$. Choose a hyperplane P^{n-1} through a_0 such that $L \cap P^{n-1} = \{a_0\}$. Thus, there is an open $(n-1)$-ball N_0 about a_0 in $P^{n-1} \cap A$.

If $a \in N_0$, the secant $L_a = [b_0, a]$ meets F in exactly two points. Let $\{a_\sigma\}$ be a sequence of points in N_0 with the limit point a_0, $L_{a_\sigma} = [b_0, a_\sigma]$. Then $\lim L_{a_\sigma} = [b_0, a_0] = [p, q]$.

Let U be an open neighbourhood of p in F such that $q \notin \overline{U}$. For a_σ sufficiently close to a_0, L_{a_σ} meets U in exactly one point. Thus, there exists an $(n-1)$-ball N_1 about a_0 in N_0 such that $L_a \cap U$ is a point for all $a \in N_1$.

Let $U_1 = \{u \in U | \{u\} = L_a \cap U \text{ where } a \in N_1\}$.

Since $a_0 \in N_1$, $p \in U_1$. Suppose that $\{u\} = L_{a_1} \cap U = L_{a_2} \cap U$ for $u \in U_1$ and $a_1 \neq a_2$ in N_1. Then $\{a_1, a_2\} \subset N_1 \subset N_0 \subset P^{n-1}$ implies that $b_0 \in [u, a_1] = [a_1, a_2] \subset P^{n-1}$; a contradiction. Hence, the correspondence between $a \in N_1$ and $L_a \cap U \in U_1$ is a bijection. Obviously, it is a homeomorphism.

Since p is arbitrary, the lemma follows.

COROLLARY. *Let $P^k \subset P^n$ be a secant; $3 \leq k \leq n-1$. Then $F_k = P^k \cap F$ is a hypersurface in P^k.*

PROOF. If F_k is regular, then the result follows by 3.2.1.

Let $v_1 \neq v_2$ be points in V, the set of irregular points of F_k. The line $[v_1, v_2]$ meets F_k at a third point by 3.1.4 and 3.1.5. Since $F_k \subset F$, 3.1.9 implies that $[v_1, v_2] \subset F_k$. Hence V is a flat and clearly, $\dim V \leq k-2$.

From the proof of 3.2.1, $F_k \backslash V$ is locally homeomorphic to the union of a finite number of open $(k-1)$-balls.

3.2.2. LEMMA. *Let $F \subset P^3$ satisfy $(1)_3$. Then every $P^2 \subset P^3$ is a secant.*

PROOF. By $(1)_3$, there is a line $L_A \subset A$ and a line $L_B \subset B$. Hence, $P^2 \cap L_A \neq \emptyset \neq P^2 \cap L_B$ for all $P^2 \subset P^3$.

3.2.3. LEMMA. *Under the hypothesis of 3.2.2, let $P^2 \cap F$ be an oval O. Then O is an S^1.*

PROOF. Since F is regular, every line meets O in at most two distinct points. Therefore, O is an S^1 if O has a unique tangent at each point.

Let $p \in O$. By 3.1.8 and 3.1.11, there is a paratingent T of O at p and $T \cap O = \{p\}$. Thus T is not a secant and by 3.1.3, T supports A or B. We may assume $T \subset \overline{B}$; thus, $T \cap A = \emptyset$.

Let $q \in O\backslash\{p\}$. Since $[p, q] \cap O = \{p, q\}$, $L = [p, q]$ is a secant and $L = [a, b]$ where $a \in L \cap A$ and $b \in L \cap B$. By 3.1.10, there is a line $L_a \subset A$ through a. Since $T \cap A = \emptyset$, we have $[L_a, L] \neq [T, L] = P^2$.

Obviously, $[L_a, L]$ is a secant. By 3.1.9, $L_a \cap F = \emptyset$ implies that $[L_a, L] \cap F$ is an oval O'. Then $\{p, q\} \subset O'$ and O' has a paratingent T' at p. Since $T' \cap O' = \{p\}$ and $T' \cap L_a \in A$, $T' \subset \overline{A}$ by 3.1.3. As $T \subset \overline{B}$, this implies that $P_0^2 = [T, T']$ is a plane.

Again, P_0^2 is a secant through p. It is immediate that $i_m(P^2 \cap A) = i_m(P^2 \cap B) = 0$. Thus, $P_0^2 \cap F$ is a pair of lines L_1 and L_2 through p by 3.1.9.

Since $p \in O$ is arbitrary, there are two distinct lines of F through each point of O. Since every plane is a secant, there are exactly two such lines through each point of O by 3.1.9.

Let $T'' \subset P^2$ be a paratingent of O at p. Since $T \subset \overline{B}$, $T'' \subset \overline{B}$ as well. From the preceding, $[T'', T'] \cap F$ is a pair of distinct lines through p. Since these must be L_1 and L_2, we have

$$T'' = P^2 \cap [T'', T'] = P^2 \cap [L_1, L_2] = P^2 \cap [T, T'] = T.$$

Therefore, O has a unique tangent at each point p.

3.2.4. THEOREM. *Let $F \subset P^3$ satisfy $(1)_3$. Then F is a nondegenerate S^2 with index* 1.

PROOF. By 3.2.2, 3.2.3 and 3.1.9, $P^2 \cap F$ is either a pair of lines or an S^1 for all $P^2 \subset P^3$. As there is a line $L \subset A$, $P^2 \cap F$ is an S^1 for all P^2 through L. By 3.2.1 and 2.1.4, F is an S^2.

Obviously, F is nondegenerate and from the proof of 3.2.3, F has the index 1.

Let $F \subset P^n$ satisfy $(1)_n$; $n \geqslant 4$.

3.2.5. LEMMA. *If P^k is not a secant, then $P^k \cap F$ is a flat*; $-1 \leqslant k \leqslant n-1$.

PROOF. We may assume that $\dim(P^k \cap F) > 0$. By 3.1.3, we may assume that $P^k \cap A = \emptyset$ say.

Let $p \neq q$ in $P^k \cap F$. Since $[p, q] \cap A = \emptyset$, $[p, q]$ is not a secant. By 3.1.4 and 3.1.9, $[p, q] \subset F$.

3.2.6. LEMMA. *Let $P^2 \subset P^n$. Then $P^2 \cap F$ is a flat or a pair of lines or an S^1.*

PROOF. By 3.2.5 and 3.1.9, we may assume that $P^2 \cap F$ is an oval O. Obviously, there is a line L contained in either $P^2 \cap A$ or $P^2 \cap B$. Let $L \subset P^2 \cap B$. Since P^2 is a secant, there is a point $a \in P^2 \cap A$. As $i_m A \geqslant 1$, 3.1.10 implies that there is a line $L' \subset A$ through a. Since $A \cap B = \varnothing$, $P^3 = [L, L']$ is a 3-flat. Clearly, P^3 is a secant and $P^2 \subset P^3$.

By 3.1.4, $A_3 = P^3 \cap A$ and $B_3 = P^3 \cap B$ are a linearly connected pair in P^3 with $F_3 = P^3 \cap F = \bar{A}_3 \cap \bar{B}_3$. As $L \subset B_3$ and $L' \subset A_3$, we have $\min\{i_m A_3, i_m B_3\} > 0$. Then every plane in P^3 is a secant and it is immediate that F_3 is regular. Thus, F_3 is a nondegenerate S^2 with the index 1 by 3.2.4. In particular, $P^2 \cap S^2 = P^2 \cap F = O$ is an S^1 by 2.1.5.

3.2.7. LEMMA (MARCHAUD [3]). *Let $P^k \subset P^n$ be a secant such that $i_m(P^k \cap A) = i_m(P^k \cap B) = 0$; $1 \leqslant k \leqslant n-1$. Then $P^k \cap F$ is a pair of $(k-1)$-flats.*

3.2.8. LEMMA. *Let $P^k \subseteq P^n$; $2 \leqslant k \leqslant n$. Then $P^k \cap F$ is a flat or a pair of $(k-1)$-flats or an S^{k-1}.*

PROOF. By 3.2.6, the assertion is true for $k = 2$. Suppose it has been proven up to $k-1$. By 3.2.5 and 3.2.7, we may assume that P^k is a secant with $i_m(P^k \cap B) > 0$. Thus, there is a line $L \subset P^k \cap B$ and a point $a \in P^k \cap A$. Clearly, we wish to show that $P^k \cap F$ satisfies 2.1.4.

By the induction hypothesis, $P^{k-1} \cap F$ is a flat or a pair of $(k-2)$-flats or an S^{k-2} for all $P^{k-1} \subset P^k$. As $L \subset P^k \cap B$ and $L \cap F = \varnothing$, this implies that every $(k-1)$-section of $P^k \cap F$ is either degenerate or nondegenerate; cf. 2.1.3.

Obviously, the plane $P^2 = [L, a]$ is a secant and $P^2 \cap F$ is an S^1 by 3.1.9 and 3.2.6. Let $P^{k-1} \subset P^k$ contain P^2. Then $S^1 \subset P^{k-1} \cap F$ implies that $P^{k-1} \cap F$ is an S^{k-2}. By 3.2.1 Corollary and 2.1.4, $P^k \cap F$ is an S^{k-1}.

3.2.9. THEOREM. *Let $n \geqslant 3$. If F satisfies $(1)_n$, then F is a nondegenerate S^{n-1} with* ind $S^{n-1} \geqslant 1$.

PROOF. As $\min\{i_m A, i_m B\} \geqslant 1$, F is an S^{n-1} by 3.2.8. As in 3.2.6, we construct a P^2 such that $P^2 \cap F$ is an S^1 and P^3 through P^2 such that $P^3 \cap F$ is a nondegenerate S^2 with ind $S^2 = 1$. Then ind $F \geqslant$ ind $S^2 = 1$. As F is regular, F is nondegenerate by 2.2.10.

3.3. *Nondegenerate S^{n-1}*. Let S^{n-1} be nondegenerate with ind $S^{n-1} = i > 0$; $n \geqslant 3$. We shall prove that $F = S^{n-1}$ satisfies $(1)_n$; cf. 3.2.

3.3.1. Let $\{c, c'\} \subset P^n \backslash S^{n-1}$. We define $c \sim c'$ if $[c, c'] \cap S^{n-1}$ is void or a point or a pair of distinct points p_1 and p_2 such that c and c' lie on the same line segment bounded by p_1 and p_2.

3.3.2. Clearly, $\sim$ is both reflexive and symmetric. It is easy to verify that it is, in fact, an equivalence relation on $P^n \backslash S^{n-1}$.

Let A be an equivalence class [$\sim$-class] of $P^n \backslash S^{n-1}$. Since there is a line meeting S^{n-1} in exactly two distinct points, there exists a second $\sim$-class of $P^n \backslash S^{n-1}$, say B, by 3.3.1. For any $P^2 \subset P^n$, it is immediate that

$$P^2 = (P^2 \cap A) \cup (P^2 \cap S^{n-1}) \cup (P^2 \cap B)$$

and thus, A and B are the only two such classes. Hence, $A \cup B = P^n \backslash S^{n-1}$ and $A \cap B = \varnothing$.

3.3.3. Let $P^1 \subset P^n$. By 3.3.1, $P^1 \cap A$ [$P^1 \cap B$] is a flat or an open segment. Hence, A and B are linearly connected sets by 3.1.1.

Obviously, $\overline{A} = A \cup S^{n-1} = P^n \backslash B$ and $\overline{B} = B \cup S^{n-1} = P^n \backslash A$. Thus $A \cap B = \varnothing$ implies that $S^{n-1} = \overline{A} \cap \overline{B}$. By 2.2.10, S^{n-1} is regular.

In conclusion, we have

3.3.4. THEOREM. *Let S^{n-1} be nondegenerate with* ind $S^{n-1} > 0$; $n \geqslant 3$. *Then $S^{n-1} = P^n \backslash (A \cup B) = \overline{A} \cap \overline{B}$ where A and B are a linearly connected pair in P^n. Moreover, S^{n-1} is regular.*

It remains to show that $\min\{i_m A, i_m B\} \geqslant 1$. This is true when $n = 3$, as a nondegenerate S^2 with ind $S^2 = 1$ is a quadric by 2.4.5.

3.3.5. LEMMA. *Under the hypothesis of* 3.3.4, *there is a* 3*-flat P^3 such that $P^3 \cap S^{n-1}$ is a nondegenerate S^2 with* ind $S^2 = 1$.

PROOF. We may assume that $n \geqslant 4$. Since S^{n-1} is nondegenerate, there is a P^2 such that $P^2 \cap S^{n-1}$ is an S^1. Let $\{p_0, p_1\} \subset S^1$; $p_0 \neq p_1$. Let P_k^1 be the tangent of S^1 at p_k, $k = 0, 1$. By 1.3.3, $P_k^1 \cap S^1 = \{p_k\}$ and $P_0^1 \cap P_1^1$ is a point $r \notin S^{n-1}$.

By 2.6.2, $R^{n-2} = \pi(p_0) \cap \pi(p_1)$ is an $(n-2)$-flat and $R^{n-2} \cap S^{n-1}$ is a nondegenerate S^{n-3} with ind $S^{n-3} = i - 1 \geqslant 0$. Obviously, $r \in R^{n-2} \backslash S^{n-3}$. Let $p \in S^{n-3}$.

Since S^{n-3} is differentiable at p, there is a point $p' \in S^{n-1}$ such that $[p, p'] \not\subset S^{n-3}$. Whether $[r, p, p']$ is a plane or a line, there is a P^1 through r intersecting S^{n-3} at distinct points u_1 and u_2. Then $P^1 = [r, u_1, u_2] \not\subset S^{n-1}$ and $P^1 \subset R^{n-2}$.

By 2.2.4, $u_\sigma \in S^{n-3}$ implies that $[u_\sigma, p_0] \subset S^{n-1}$, $\sigma = 1, 2$. Thus $P^1 \cap P^2 = \{r\}$ and $P^3 = [P^2, P^1]$ is a 3-flat. Since $S^1 \subset P^3 \cap S^{n-1}$; by 2.1.5, $P^3 \cap S^{n-1}$ is an S^2. Hence, $[u_\sigma, p_0] \subset S^2$ and ind $S^2 = 1$.

Since $p_0 \in S^1 \subset S^2$, p_0 is a differentiable point of S^2 with the tangent plane $\tilde{\pi}(p_0) = \pi(p_0) \cap P^3$. Since $P^1 = [u_1, u_2] \not\subset S^2$, $[p_0, u_1] \neq [p_0, u_2]$ and thus, $\tilde{\pi}(p_0) \cap S^2 = [p_0, u_1] \cup [p_0, u_2]$ by 2.3.6. Hence, S^2 is nondegenerate by 2.3.3 Corollary 1.

3.3.6. LEMMA. *Under the hypothesis of* 3.3.4, $\min\{i_m A, i_m B\} \geqslant 1$.

PROOF. By 3.3.5, there is a P^3 such that $P^3 \cap S^{n-1}$ is a nondegenerate S^2 with ind $S^2 = 1$. By 3.3.4,

$$(1) \qquad S^2 = P^3 \backslash (A_3 \cup B_3) = \overline{A}_3 \cap \overline{B}_3,$$

where A_3 and B_3 are a linearly connected pair in P^3. As S^2 is a quadric, this yields that $i_m A_3 = i_m B_3 = 1$.

By 3.3.4, $P^n = A \cup S^{n-1} \cup B$ and hence,

$$(2) \qquad P^3 = (P^3 \cap A) \cup (P^3 \cap S^{n-1}) \cup (P^3 \cap B),$$

where $P^3 \cap A$ and $P^3 \cap B$ are open, disjoint, linearly connected and thus, connected. Since S^2 is a quadric, $P^3 \backslash S^2$ is the union of precisely two nonvoid connected sets. Thus, $P^3 \cap A \neq \varnothing \neq P^3 \cap B$ and (1) and (2) imply, for example, $P^3 \cap A = A_3$ and $P^3 \cap B = B_3$. Since $A_3 \subset A$ and $B_3 \subset B$, we have $i_m A \geqslant i_m A_3$ and $i_m B \geqslant i_m B_3$.

Thus we obtain by 3.1.12,

3.3.7. THEOREM. *Let* $n \geqslant 3$. *An* $S^{n-1} \subset P^n$ *is nondegenerate with* ind $S^{n-1} \geqslant 1$ *if and only if* $F = S^{n-1}$ *satisfies* $(1)_n$ *in* 3.2; *moreover, such an* S^{n-1} *is a quadric.*

Appendix: Quadrics. We shall now prove, independently of the concept of linear connectedness, that a nondegenerate $S^{n-1} \subset P^n$ with a positive index is a quadric.

Let $u_\sigma \equiv (\delta_{\sigma 0}, \delta_{\sigma 1}, \ldots, \delta_{\sigma n})$ be the base points of a (homogeneous) coordinate system of P^n; $\sigma = 0, 1, \ldots, n$. Let $\mathbf{R}$ be the set of real numbers.

A quadric $\mathbf{Q}^{n-1}$ P^n is given by an equation

$$(1) \qquad \sum_{\sigma,\mu=0}^{n} a_{\sigma\mu} x_\sigma x_\mu = 0$$

where $P^0 \equiv (x_0, \ldots, x_n) \in P^n$ and $a_{\sigma\mu} = a_{\mu\sigma}$; $\sigma, \mu = 0, 1, \ldots, n$.

A $\mathbf{Q}^{n-1}$ is *nondegenerate* if $\det(a_{\sigma\mu}) \neq 0$ where $(a_{\sigma\mu})$ is the matrix of coefficients in (1). Finally, if $\mathbf{Q}^{n-1}$ is nondegenerate, then the tangent hyperplane $\omega(p)$ of $\mathbf{Q}^{n-1}$ exists at each point p in $\mathbf{Q}^{n-1}$.

A.1. Preparatory lemmas.

A.1.1. LEMMA. *Let* S^{n-1} *be nondegenerate*; $n \geqslant 4$, ind $S^{n-1} = i \geqslant 1$. *Let* $P^k \cap S^{n-1} = S^{k-1}$ *have the index* 0; $2 \leqslant k \leqslant \min\{n-2, n-i+1\}$. *Then there is a* P^{k+1} *through* P^k *such that* $P^{k+1} \cap S^{n-1}$ *is a nondegenerate* S^k *with the index* 1.

PROOF. Since ind $S^{k-1} = 0$ and $k \geqslant 2$, there is a $P^2 \subset P^k$ such that

$P^2 \cap S^{k-1}$ is an S^1. Then there is a P^3 through P^2 such that $P^3 \cap S^{n-1}$ is a nondegenerate S^2 with the index 1; cf. the proof of 3.3.5.

Clearly, $P^3 \cap P^k = P^2$ and $P^{k+1} = [P^k, P^3]$ is a $(k+1)$-flat. By 2.1.5, $P^{k+1} \cap S^{n-1}$ is necessarily an S^k. As ind $S^2 = 1$ [ind $S^{k-1} = 0$], we have ind $S^k \geqslant 1$ [ind $S^k \leqslant 1$]. Hence, ind $S^k = 1$. Now S^k is clearly nondegenerate.

The following assertions are obvious.

A.1.2. LEMMA. *Let* ind $S^{n-1} = 0$; $n \geqslant 3$. *Choose* $p_\sigma \in S^{n-1}$ *such that* $P^n = [p_0, \ldots, p_n]$. *If* S^{n-1} *is a quadric, then* $\bigcap_{\sigma=0}^{n} \pi(p_\sigma) = \varnothing$.

COROLLARY. $\bigcap_{\sigma=0;\sigma\neq k}^{n} \pi(p_\sigma) = P(k) \not\subset [p_0, \ldots, \hat{p}_k, \ldots, p_n]$; $k = 0, 1, \ldots, n$.

A.2. **Nondegenerate S^{n-1} with index 1.** In order to prove our theorem, we shall construct a set in S^{n-1} and coordinates in P^n for every i. Then we show that there is a unique nondegenerate quadric containing this set and that this quadric is identical with S^{n-1}.

In the following sections, we deal with special values of i.

A.2.1. By 2.4.5, a nondegenerate $S^2 \subset P^3$ with the index 1 is a quadric. Assume that every $S^{k-1} \subset P^k$ with ind $S^{k-1} = 1$ is a quadric; $3 \leqslant k \leqslant n-1$. Let $S^{n-1} \subset P^n$ be nondegenerate with ind $S^{n-1} = 1$; $n \geqslant 4$.

By 2.6.3, there are points p_0, q_0 in S^{n-1}; $[p_0, q_0] \not\subset S^{n-1}$. Then $R^{n-2} = \pi(p_0) \cap \pi(q_0)$ is an $(n-2)$-flat and $R^{n-2} \cap S^{n-1}$ is an S^{n-3} of index 0.

Choose $n-2$ points $r_\sigma \in S^{n-3}$ such that $R^{n-3} = [r_0, \ldots, r_{n-3}] \subset R^{n-2}$ is an $(n-3)$-flat. We assume $\{p_1, q_1\} \subset \{r_0, \ldots, r_{n-3}\}$; cf. 2.6.3. By A.1.1 and the induction hypothesis, S^{n-3} is a quadric. Thus $\bigcap_{\sigma=0}^{n-3}(\pi(r_\sigma) \cap R^{n-2})$ is a point $r_{n-2} \notin R^{n-3}$, by A.1.2 Corollary. Clearly, $R^{n-2} = [R^{n-3}, r_{n-2}]$ and $r_{n-2} \in R^s$; cf. 2.6.4. Hence $r_{n-2} \notin S^{n-1}$.

From 2.6.7 with $r = r_{n-2}$ and $k = 0$, we obtain $P^n = [R^{n-2}, P^2]$ where $P^2 = [p_0, q_0, r_{n-2}]$, $R^{n-2} \cap P^2 = \{r_{n-2}\}$ and $P^2 \cap S^{n-1}$ is an S^1. Thus, $P^n = [r_0, \ldots, r_{n-2}, r_{n-1}, r_n]$ where $r_{n-1} = p_0$, $r_n = q_0$. By A.1.1 and 2.4.5, S^1 is a conic.

Finally, we observe

A.2.2. LEMMA. (1) $P^2 \subset \pi(r_\sigma)$; $\sigma = 0, 1, \ldots, n-3$; cf. 2.2.4.
(2) $R^{n-3} \subset \pi(p)$ *for each* $p \in S^1$.

A.2.3. Let r_σ be the base points (of a coordinate system) of P^n; $\sigma = 0, \ldots, n$. Let $Q^{n-1} \subset P^n$ be given by

$$(1) \qquad x_{n-2}^2 + 2a_{n-1,n}x_{n-1}x_n + \sum_{\sigma,\mu=0;\sigma\neq\mu}^{n-3} a_{\sigma\mu}x_\sigma x_\mu = 0; \quad \det(a_{\sigma\mu}) \neq 0.$$

Then $r_{n-2} \equiv (0, \ldots, 0, 1, 0, 0) \notin Q^{n-1}$ and the tangent hyperplanes

$\omega(r_{n-1})$ and $\omega(r_n)$ are given by $x_n = 0$ and $x_{n-1} = 0$ respectively. Thus

$$\pi(r_{n-1}) \cap \pi(r_n) = [r_0, \dots, r_{n-2}] = R^{n-2} = \omega(r_{n-1}) \cap \omega(r_n),$$
$$\pi(r_\sigma) = [R^{n-2}, r_\sigma] = \omega(r_\sigma); \quad \sigma = n-1, n. \tag{2}$$

Since S^{n-3} and S^1 are a quadric and conic, respectively, we choose $a_{\sigma\mu}$ satisfying (1) so that

$$S^{n-3} = R^{n-2} \cap Q^{n-1} \quad \text{and} \quad S^1 = P^2 \cap Q^{n-1}. \tag{3}$$

Then Q^{n-1} is uniquely determined.

As $\pi(p) \cap R^{n-2} = \omega(p) \cap R^{n-2}$ is the tangent hyperplane in R^{n-2} at a point p, (2) implies that

$$\pi(p) = [\pi(p) \cap R^{n-2}, r_{n-1}, r_n] = \omega(p) \quad \text{for } p \in S^{n-3}. \tag{4}$$

Similarly, $\pi(p) \cap P^2 = \omega(p) \cap P^2$ for $p \in S^1$ and thus

$$\pi(p) = [R^{n-3}, \pi(p) \cap S^{n-1}] = \omega(p) \quad \text{for all } p \in S^1. \tag{5}$$

Therefore

$$R^{n-3} = \bigcap_{p \in S^1} \pi(p). \tag{6}$$

By (2) and (3), $\pi(r_\sigma) \cap S^{n-1}$ and $\omega(r_\sigma) \cap Q^{n-1}$ are cones with the same vertex r_σ and the same $(n-2)$-section S^{n-3}. Thus,

$$\pi(r_\sigma) \cap S^{n-1} = \omega(r_\sigma) \cap Q^{n-1}; \quad \sigma = n-1, n. \tag{7}$$

Similarly,

$$[p, R^{n-3}] \cap S^{n-1} = [p, R^{n-3}] \cap Q^{n-1} \quad \text{for each } p \in S^1. \tag{8}$$

A.2.4. Lemma. *$R^{n-3} = \pi(p) \cap \pi(q) \cap \pi(r)$ for any three mutually distinct points p, q and r in S^1.*

Proof. Since S^1 is of order 2, we have $P^2 = [p, q, r]$ and $\pi(p) \cap \pi(q) \cap \pi(r) \cap S^1 = \emptyset$. Thus, $\dim(\pi(p) \cap \pi(q) \cap \pi(r)) \leqslant n-3$ and the result follows by A.2.3(6).

A.2.5. Lemma. *Let $p \in S^1$. Then $\pi(p) \cap S^{n-1} = \pi(p) \cap Q^{n-1}$.*

Proof. By 2.6.1, $\pi(p) \cap S^{n-1}$ is a 1-degenerate S^{n-2} with the singular point p; ind $S^{n-2} = 1$. Thus any line $L \subset S^{n-2}$ passes through p. By A.2.3(7), we may assume $p \neq r_{n-1}, r_n$. Then $p \notin \pi(r_\sigma)$ and thus, $\pi(r_\sigma) \cap L$ is a point u_σ; $\sigma = n-1, n$. By A.2.3(7), $u_\sigma \in Q^{n-1}$.

If $u_{n-1} \neq u_n$, then L meets Q^{n-1} at three mutually distinct points and thus, $L \subset Q^{n-1}$; cf. A.2.3(3). If $u_{n-1} = u_n = u$, then by A.2.4, $u \in \pi(r_{n-1}) \cap$

$\pi(r_n) \cap \pi(p) = R^{n-3}$. Hence, $L = [p, u] \subset [p, R^{n-3}] \cap S^{n-1} \subset Q^{n-1}$ by A.2.3(8). Thus, $\pi(p) \cap S^{n-1} \subseteq \pi(p) \cap Q^{n-1}$.

The preceding argument is symmetric in S^{n-1} and Q^{n-1}.

A.2.6. Lemma. *Let $p \in S^{n-3} \backslash R^{n-3}$. Then $\pi(p) \cap S^{n-1} = \pi(p) \cap Q^{n-1}$.*

Proof. As in A.2.5, we apply 2.6.1 and obtain $\pi(p) \cap S^{n-1} = S^{n-2}$. Let $L \subset S^{n-2}$ be a line. Thus $p \in L$. Since $S^{n-3} = R^{n-2} \cap S^{n-1}$ has the index 0, this yields $L \cap R^{n-2} = \{p\}$. As $p \notin R^{n-3}$, we obtain $L \cap R^{n-3} = \emptyset$.

If L meets P^2 at a point q, then $q \in S^1$ and by A.2.5, $L \subset \pi(q) \cap S^{n-1} \subset Q^{n-1}$. Let $L \cap P^2 = \emptyset$. Then for $q \in S^1$, $u(q) = \pi(q) \cap L$ is a point in Q^{n-1}. But A.2.4 and $L \cap R^{n-3} = \emptyset$ imply that $u = u(q)$ has at most two solutions in S^1 for any u. Hence, L meets Q^{n-1} in three mutually distinct points and thus, $L \subset Q^{n-1}$. The lemma now follows; cf. the proof of A.2.5.

A.2.7. Theorem. *Under the hypotheses of* A.2.1 *and* A.2.3, $S^{n-1} = Q^{n-1}$.

Proof. We first prove $Q^{n-1} \subseteq S^{n-1}$.

Let $L \subset Q^{n-1}$ be a line. By A.2.5, we assume that $L \cap P^2 = \emptyset$. Thus, $u(q) = \pi(q) \cap L$ is a point in S^{n-1} for each $q \in S^1$. Let $U = \{u(q) | q \in S^1\}$.

If $|U| \geqslant 3$, then $L \subset S^{n-1}$ by 2.1.5. If $U = \{u_1, u_2\}$, then say $u_1 \in R^{n-3}$ by

A.2.4. Let $u_2 = u(q)$. Then $L = [u_1, u_2] \subset [R^{n-3}, \pi(q) \cap L] \subset \pi(q)$ by A.2.3(5); a contradiction by assumption.

Let $U = \{u\}$. Then $u \in R^{n-3} \cap S^{n-1} \subset S^{n-3}$ by A.2.4. Observe that $u \in L \subset Q^{n-1}$ implies that $L \subset \omega(u) = \pi(u)$ by A.2.3(4).

Choose a point $p \in S^{n-3} \backslash R^{n-3}$. By A.2.6, $\pi(p) \cap L \subset S^{n-1}$. Since ind $S^{n-3} = 0$ and $\{u, p\} \subset S^{n-3}$, $u \notin \pi(p)$ and thus, $L = [u, \pi(p) \cap L]$. By 2.2.4, $L \subset \pi(u)$ implies that $L \subset S^{n-1}$. Thus $Q^{n-1} \subseteq S^{n-1}$.

The preceding argument is symmetric in S^{n-1} and Q^{n-1}.

A.3. Nondegenerate S^{n-1} with ind $S^{n-1} = [\frac{1}{2}(n-1)]$; n even.

A.3.1. In this section, we prove that a nondegenerate $S^{2i+1} \subset P^{2i+2}$ of index i is a quadric for $i > 0$. By A.2, this assertion is true for $i = 1$. We assume that it has been proven up to $i - 1$.

Put $n = 2i + 2$. Let $\{p_\sigma, q_\sigma | \sigma \in I = \{0, 1, \ldots, i\}\} \subset S^{n-1}$ satisfy 2.6.3. From 2.6.4(2), $R^s = \bigcap_{\sigma=0}^{i}(\pi(p_\sigma) \cap \pi(q_\sigma))$ is an s-flat; $s = n - 2(i+1) = 0$. By 2.6.5 Corollary, R^s is a point $r \notin S^{n-1}$.

We introduce the i-flats $\widetilde{P}^i$, $\widetilde{Q}^i$, P^i_α and Q^i_α as in 2.6.4; $\alpha \in I$. By 2.6.5, all of these i-flats are contained in S^{n-1}; moreover,

(1) $P^n = [p_0, \ldots, p_i, r, q_0, \ldots, q_i] = [\widetilde{P}^i, r, \widetilde{Q}^i]$,

(2) $\pi(p_\alpha) = [\widetilde{P}^i, r, q_0, \ldots, \hat{q}_\alpha, \ldots, q_i]$, $\alpha \in I$, and

(3) $\pi(q_\alpha) = [\widetilde{Q}^i, r, p_0, \ldots, \hat{p}_\alpha, \ldots, p_i]$, $\alpha \in I$.

We apply 2.6.7 in the case $k = 0$ and $R^s = \{r\}$. Then $P^n = [R^{n-2}, P^2]$ where $R^{n-2} = \pi(p_0) \cap \pi(q_0)$, $P^2 = [p_0, q_0, r]$, $R^{n-2} \cap P^2 = \{r\}$, $P^2 \cap S^{n-1}$ is an S^1 and $R^{n-2} \cap S^{n-1}$ is a nondegenerate S^{n-3}; ind $S^{n-3} = i - 1$. By the preceding,

$$R^{n-2} = \pi(p_0) \cap \pi(q_0) = [p_1, \ldots, p_i, r, q_1, \ldots, q_i].$$

A.3.2. Choose the following base points u_k of P^n:

$$u_k = \begin{cases} p_k, & k = 0, 1, \ldots, i, \\ r, & k = i+1, \\ q_{n-k}, & k = i+2, i+3, \ldots, n. \end{cases}$$

Let $Q^{n-1} \subset P^n$ be given by

$$x_{i+1}^2 + 2\sum_{\sigma=0}^{i} a_{\sigma,n-\sigma} x_\sigma x_{n-\sigma} = 0; \quad \det(a_{\sigma\mu}) \neq 0. \tag{1}$$

Clearly, Q^{n-1} contains $\widetilde{P}^i$, $\widetilde{Q}^i$, P^i_α and Q^i_α, $\alpha \in I$, and

$$r = u_{i+1} \equiv (0, \ldots, 0, x_{i+1}, 0, \ldots, 0) \notin Q^{n-1} \qquad (x_{i+1} = 1).$$

For $\sigma = 0, \ldots, n$, $\sigma \neq i+1$, (1) implies that $\omega(u_\sigma)$ is given by $x_{n-\sigma} = 0$. Thus

$$\omega(u_\sigma) = [u_0, \ldots, \hat{u}_{n-\sigma}, \ldots, u_n] = \pi(u_\sigma) \equiv x_{n-\sigma} = 0. \tag{2}$$

By the induction hypothesis, $S^{n-3} = R^{n-2} \cap S^{n-1}$ is a quadric. Obviously, $S^1 = P^2 \cap S^{n-1}$ is a conic. Thus, we determine Q^{n-1} uniquely by choosing $a_{\sigma\mu}$ in (1) so that

$$S^{n-3} = R^{n-2} \cap Q^{n-1} \quad \text{and} \quad S^1 = P^2 \cap Q^{n-1}. \tag{3}$$

A.3.3. LEMMA. $\pi(p_\sigma) \cap \pi(q_\sigma) \cap S^{n-1} = \pi(p_\sigma) \cap \pi(q_\sigma) \cap Q^{n-1}$; $\sigma \in I$.

PROOF. By A.3.2(3), we may assume $\sigma \neq 0$; e.g. $\sigma = i$. Let $j = i - 1$.

Clearly, $P^{n-2} = \pi(p_i) \cap \pi(q_i)$ is an $(n-2)$-flat and by 2.6.2, $\widetilde{S}^{n-3} = P^{n-2} \cap S^{n-1}$ is a nondegenerate hypersurface of order 2; ind $\widetilde{S}^{n-3} = i - 1$.

Using the coordinate system of A.3.2, P^{n-2} is given by $x_i = x_{i+2} = 0$ by A.3.2(2). Observe that $P^{n-2} \cap \widetilde{P}^i$, $P^{n-2} \cap \widetilde{Q}^i$, $P^{n-2} \cap P^i_\alpha$ and $P^{n-2} \cap Q^i_\alpha$ are j-flats in $\widetilde{S}^{n-3}$; $\alpha \in J = I\backslash\{i\}$. Finally, by the induction hypothesis, $\widetilde{S}^{n-3} \subset P^{n-2}$ is a quadric, say

$$\widetilde{S}^{n-3} = Q^{n-3} \equiv x_{i+1}^2 + 2\sum_{\sigma=0}^{j} c_{\sigma,n-\sigma} x_\sigma x_{n-\sigma} = 0, \quad x_i = x_{i+2} = 0. \tag{1}$$

Let $\mu \in J$ and consider the plane $P^2_\mu = [p_\mu, q_\mu, r]$. Clearly, $P^2_\mu \cap S^{n-1} = S^1_\mu$ is a nondegenerate curve of order 2 by 2.1.5. By A.1.2 and 2.4.5, S^1_μ is a conic.

Observe that $P^2_0 = P^2$ (A.3.1) and thus, $S^1_0 = S^1 \subset \widetilde{S}^{n-3}$. For $\mu \neq 0$, $S^1_\mu \subset S^{n-3} \cap \widetilde{S}^{n-3}$. As $S^1 \cup S^{n-3} \subset \mathbf{Q}^{n-1}$ and $\widetilde{S}^{n-3} = \mathbf{Q}^{n-3}$, this implies that $S^1_\mu \subset \mathbf{Q}^{n-1} \cap \mathbf{Q}^{n-3}$ and in particular,

$$(2) \qquad S^1_\mu = P^2_\mu \cap \mathbf{Q}^{n-1} = P^2_\mu \cap \mathbf{Q}^{n-3}; \qquad \mu \in J.$$

Then A.3.2(1), (1) and (2) imply that $c_{\mu,n-\mu} = a_{\mu,\ n-\mu}$ for $\mu \in J$. Thus, $\mathbf{Q}^{n-3} \subset \mathbf{Q}^{n-1}$ and $P^{n-2} \cap S^{n-1} = \widetilde{S}^{n-3} = \mathbf{Q}^{n-3} = P^{n-2} \cap \mathbf{Q}^{n-1}$.

COROLLARY. $\pi(u_\sigma) \cap S^{n-1} = \pi(u_\sigma) \cap \mathbf{Q}^{n-1}$; $\sigma = 0, \ldots, n$, $\sigma \neq i+1$.

PROOF. Both $\pi(u_\sigma) \cap S^{n-1}$ and $\pi(u_\sigma) \cap \mathbf{Q}^{n-1}$ are cones with the vertex u_σ and the $(n-2)$-section $\pi(u_\sigma) \cap \pi(u_{n-\sigma}) \cap S^{n-1}$.

A.3.4. THEOREM. *Under the hypotheses of* A.3.1 *and* A.3.2, $S^{n-1} = \mathbf{Q}^{n-1}$.

PROOF. We first show $\mathbf{Q}^{n-1} \subsetneq S^{n-1}$. Let $Q^i \subset \mathbf{Q}^{n-1}$ be an i-flat. By A.3.3 Corollary, we may assume $\pi(u_\sigma) \cap Q^i$ is an $(i-1)$-flat in S^{n-1}; $\sigma = 0, \ldots, n$, $\sigma \neq i+1$. Then dim $Q^i = i \geqslant 2$ and A.3.2(2) imply that there are at least three mutually distinct $\pi(u_\sigma) \cap Q^i$'s. By 2.1.5, $Q^i \subset S^{n-1}$ and thus $\mathbf{Q}^{n-1} \subsetneq S^{n-1}$.

The preceding argument is symmetric in $\mathbf{Q}^{n-1}$ and S^{n-1}.

A.4. Nondegenerate S^{n-1} with ind $S^{n-1} = i$; $n = 2i+1 \geqslant 5$.

A.4.1. Put $n = 2i+1$. We wish to prove that a nondegenerate $S^{2i} \subset P^{2i+1}$ of index i is a quadric. By 2.4.5, this assertion is true for $i = 1$. We assume that it has been proven up to $i-1$.

Let $\{p_\sigma, q_\sigma | \sigma \in I\} \subset S^{n-1}$ satisfy 2.6.3. Then $R^s = \varnothing$ (2.6.4(2)) and by 2.6.5, the i-flats $\widetilde{P}^i$, $\widetilde{Q}^i$, P^i_α, and Q^i_α are contained in S^{n-1}; $\alpha \in I$. Finally, 2.6.4 and $R^s = \varnothing$ yield

(1) $P^n = [p_0, \ldots, p_i, q_0, \ldots, q_i] = [\widetilde{P}^i, \widetilde{Q}^i]$,

(2) $\pi(p_\alpha) = [\widetilde{P}^i, q_0, \ldots, \hat{q}_\alpha, \ldots, q_i]$, $\alpha \in I$, and

(3) $\pi(q_\alpha) = [\widetilde{Q}^i, p_0, \ldots, \hat{p}_\alpha, \ldots, p_i]$, $\alpha \in I$.

As $\widetilde{P}^i \cap \widetilde{Q}^i = \varnothing$, this implies that for $\alpha \neq \beta$ in I

(4) $\widetilde{P}^i \cap P^i_\alpha = [p_0, \ldots, \hat{p}_\alpha, \ldots, p_i]$ is an $(i-1)$-flat, and

(5) $P^i_\alpha \cap P^i_\beta = [p_0, \ldots, \hat{p}_\alpha, \ldots, \hat{p}_\beta, \ldots, p_i]$ is an $(i-2)$-flat.

Similarly, $\dim(\widetilde{Q}^i \cap Q^i_\alpha) = i-1$ and $\dim(Q^i_\alpha \cap Q^i_\beta) = i-2$.

A.4.2. Let $q_{0\beta} \in [q_0, q_\beta] \backslash \{q_0, q_\beta\}$; $\beta \in I \backslash \{0\}$. Then $q_{0\beta} \notin \widetilde{P}^i$. As $P^i_\alpha = [p_0, \ldots, \hat{p}_\alpha, \ldots, p_i, q_\alpha]$, this implies that for $\{\alpha, \beta\} \subset I$, $q_{0\beta} \in [q_0, q_\beta] \subset$

$\bigcap_{\sigma=1;\sigma\neq\beta}^{i}\pi(p_\sigma)$. Thus by 2.2.4, $P_0^i \cap P_\beta^i = [p_1, \ldots, \hat{p}_\beta, \ldots, p_i] \subset \pi(q_{0\beta}) \cap S^{n-1}$ and $[q_{0\beta}, P_0^i \cap P_\beta^i] \subset S^{n-1}$.

By 2.1.5, the $(i+1)$-section $[\widetilde{P}^i, q_{0\beta}] \cap S^{n-1}$ is a pair of i-flats, $\widetilde{P}^i$ and say $P_{0\beta}^i$. Clearly $q_{0\beta} \in P_{0\beta}^i$ and thus, $[q_{0\beta}, P_0^i \cap P_\beta^i] \subset P_{0\beta}^i$. Since $\dim(\widetilde{P}^i \cap P_{0\beta}^i) = i-1$ and $[p_0, p_\beta] \cap (P_0^i \cap P_\beta^i) = \varnothing$, we obtain that $P_{0\beta}^i \cap [p_0, p_\beta]$ consists of a point, say $p_{0\beta}$. Thus, $P_{0\beta}^i = [q_{0\beta}, p_{0\beta}, P_0^i \cap P_\beta^i]$.

Since $[q_0, q_\beta] \cap \pi(p_\beta) = \{q_0\}$, we have $q_{0\beta} \notin \pi(p_\beta)$. Then $P_{0\beta}^i \not\subset \pi(p_\beta)$ and equivalently, $p_\beta \notin P_{0\beta}^i$. Since $p_{0\beta} \in P_{0\beta}^i$, we have $p_{0\beta} \neq p_\beta$. By a symmetric argument, $p_{0\beta} \neq p_0$ as well.

In summary, $P_{0\beta}^i = [q_{0\beta}, p_{0\beta}, P_0^i \cap P_\beta^i]$ where $q_{0\beta} \in [q_0, q_\beta] \setminus \{q_0, q_\beta\}$ and $p_{0\beta} \in [p_0, p_\beta] \setminus \{p_0, p_\beta\}$; $\beta \in I \setminus \{0\}$. Clearly, $\pi(p_{0\beta}) = [\widetilde{P}^i, q_{0\beta}, q_1, \ldots, \hat{q}_\beta, \ldots, q_i]$.

A.4.3. LEMMA. *There is a unique nondegenerate $\mathbf{Q}^{n-1} \subset P^n$ containing the i-flats $\widetilde{P}^i$, $\widetilde{Q}^i$, P_α^i and $P_{0\beta}^i$, $\{\alpha, \beta\} \subset I$, $\beta \neq 0$; $n = 2i+1 \geqslant 5$.*

PROOF. Choose the base points u_k of P^n as follows:

$$u_k = \begin{cases} p_k, & k = 0, 1, \ldots, i, \\ q_{n-k}, & k = i+1, \ldots, n. \end{cases}$$

Let $\beta \in I \setminus \{0\}$. Since $q_{0\beta} \in [q_0, q_\beta] \setminus \{q_0, q_\beta\}$, let $q_{0\beta} \equiv (0, \ldots, 0, x_{n-\beta}, 0, \ldots, 0, 1)$, $x_{n-\beta} = d_\beta \neq 0$. Then $p_{0\beta} \in [p_0, p_\beta] \setminus \{p_0, p_\beta\}$ is determined and hence, $p_{0\beta} \equiv (1, 0, \ldots, 0, x_\beta, 0, \ldots, 0)$, $x_\beta = c_\beta \neq 0$. Thus,

$$P_{0\beta}^i \equiv \{(x_0, \ldots, x_i, 0, \ldots, 0, x_{n-\beta}, 0, \ldots, 0, x_n) \mid x_\beta = x_0 c_\beta \text{ and } x_{n-\beta} = x_n d_\beta\}.$$

It is easy to verify that the quadric $\mathbf{Q}^{n-1}$ given by

$$\sum_{\sigma=1}^{i} (c_\sigma d_\sigma)^{-1} x_\sigma x_{n-\sigma} - x_0 x_n = 0 \tag{1}$$

satisfies A.4.3.

COROLLARY. $\omega(u_\sigma) = \pi(u_\sigma)$; $\sigma = 0, 1, \ldots, n$.

PROOF. By A.4.3(1), $\omega(u_\sigma) \equiv x_{n-\sigma} = 0$; $\sigma = 0, 1, \ldots, n$. From A.4.1,

$$\pi(u_\sigma) = [u_0, \ldots, \hat{u}_{n-\sigma}, \ldots, u_n] \equiv x_{n-\sigma} = 0; \qquad \sigma = 0, 1, \ldots, n.$$

A.4.4. LEMMA. $\pi(p_\sigma) \cap \pi(q_\sigma) \cap S^{n-1} = \pi(p_\sigma) \cap \pi(q_\sigma) \cap \mathbf{Q}^{n-1}$; $\sigma \in I \setminus \{0\}$.

PROOF. We may assume e.g. $\sigma = i$. From A.4.1, $P^{n-2} = \pi(p_i) \cap \pi(q_i) =$

$[p_0, \ldots, p_{i-1}, q_0, \ldots, q_{i-1}]$ is an $(n-2)$-flat and by 2.6.2, $P^{n-2} \cap S^{n-1}$ is a nondegenerate S^{n-3}; ind $S^{n-3} = i - 1 = \frac{1}{2}(n-3)$. As $n - 2 \geq 3$, $S^{n-3} \subset P^{n-2}$ is a quadric by the induction hypothesis.

The following $(i-1)$-flats are contained in S^{n-3}: $P^{n-2} \cap \widetilde{P}^i$, $P^{n-2} \cap \widetilde{Q}^i$, $P^{n-2} \cap P^i_\alpha$, $\alpha \in \Lambda\backslash\{i\}$ and $P^{n-2} \cap P^i_{0\beta} = [q_{0\beta}, p_{0\beta}, (P^{n-2} \cap P^i_0) \cap (P^{n-2} \cap P^i_\beta)]$, $\beta \in \Lambda\backslash\{0, i\}$.

Using the coordinate system in A.4.3, P^{n-2} is given by $x_i = x_{i+1} = 0$. But then (cf. A.4.3) S^{n-3} is defined by

$$\sum_{\sigma=1}^{i-1} (c_\sigma d_\sigma)^{-1} x_\sigma x_{n-\sigma} - x_0 x_n = 0, \quad x_i = x_{i+1} = 0.$$

By A.4.3(1), $S^{n-3} = P^{n-2} \cap \mathrm{Q}^{n-1}$ and the lemma follows.

A.4.5. Lemma. $\pi(u_\sigma) \cap S^{n-1} = \pi(u_\sigma) \cap \mathrm{Q}^{n-1}$; $\sigma = 0, 1, \ldots, n$.

Proof. Recall that

$$\pi(u_\sigma) = \omega(u_\sigma) \equiv x_{n-\sigma} = 0; \qquad \sigma = 0, 1, \ldots, n. \tag{1}$$

For $0 \neq \sigma \neq n$, A.4.4 and (1) imply that $\pi(u_\sigma) \cap S^{n-1}$ and $\pi(u_\sigma) \cap \mathrm{Q}^{n-1}$ are cones with the vertex u_σ and the $(n-2)$-section $\pi(u_\sigma) \cap \pi(u_{n-\sigma}) \cap S^{n-1}$ and thus equal.

Let $\sigma = 0$. We first prove that $\pi(u_0) \cap \mathrm{Q}^{n-1} \subseteq \pi(u_0) \cap S^{n-1}$. Let $Q^i \subset \pi(u_0) \cap \mathrm{Q}^{n-1}$ be an i-flat. By the preceding, we may assume that $\pi(u_\sigma) \cap Q^i$ is an $(i-1)$-flat in S^{n-1}; $\sigma = 1, \ldots, n-1$. By 2.1.5, $Q^i \subset S^{n-1}$ if

$$|\{\pi(u_\sigma) \cap Q^i | \sigma = 1, \ldots, n-1\}| \geq 3. \tag{2}$$

Since $Q^i \subset \pi(u_0)$ is an i-flat and since $\pi(u_0)$ is given by $x_n = 0$, (1) implies that there are at least $i+1$ mutually distinct $\pi(u_\sigma) \cap Q^i$'s for $\sigma = 1, 2, \ldots, n$. But then $u_n \notin Q^i$ and $i \geq 2$ imply (2). Therefore, $\pi(u_0) \cap \mathrm{Q}^{n-1} \subseteq \pi(u_0) \cap S^{n-1}$.

The preceding argument is symmetric in Q^{n-1} and S^{n-1}. Similarly, $\pi(u_n) \cap \mathrm{Q}^{n-1} = \pi(u_n) \cap S^{n-1}$.

A.4.6. Theorem. *Under the hypotheses of* A.4.1 *and* A.4.3, $S^{n-1} = \mathrm{Q}^{n-1}$.

Proof. Cf. the proof of A.3.4.

A.5. Nondegenerate S^{n-1}; $1 < \text{ind } S^{n-1} < [\frac{1}{2}(n-1)]$.

A.5.1. Let $S^{n-1} \subset P^n$ be nondegenerate; $1 < i = \text{ind } S^{n-1} < [\frac{1}{2}(n-1)]$; thus $n \geq 7$. Let $\{p_\sigma, q_\sigma | \sigma \in \Gamma\} \subset S^{n-1}$ satisfy 2.6.3. Let $m = n - 2i$. Then $R^m = \bigcap_{\sigma=0}^{i-1}(\pi(p_\sigma) \cap \pi(q_\sigma))$ is an m-flat and $R^m \cap S^{n-1}$ is a nondegenerate S^{m-1}; ind $S^{m-1} = 0$. By A.1.1, there is a P^{m+1} through R^m such that

$P^{m+1} \cap S^{n-1}$ is a nondegenerate S^m; ind $S^m = 1$. From A.2, S^m and thus S^{m-1} are quadrics.

Choose m points $r_\gamma \in S^{m-1}$ such that $R^{m-1} = [r_1, \ldots, r_m] \subset R^m$ is an $(m-1)$-flat. As S^{m-1} is a quadric, A.1.2 Corollary implies that $\bigcap_{\gamma=1}^m (\pi(r_\gamma) \cap R^m)$ is a point $r_0 \notin R^{m-1}$. We assume $\{p_i, q_i\} \subset \{r_1, \ldots, r_m\}$ and thus, $r_0 \notin S^{n-1}$ as

$$\{r_0\} = \bigcap_{\gamma=1}^{m} (\pi(r_\gamma) \cap R^m) \subseteq \bigcap_{\sigma=0}^{i} (\pi(p_\sigma) \cap \pi(q_\sigma)) = R^s; \quad \text{cf. 2.6.5.}$$

We apply 2.6.7 in the case $k = i - 1$ and $r = r_0$. Then $P^n = [R^m, P^{2i}]$ where $R^m = [r_0, \ldots, r_m]$, $P^{2i} = [p_0, \ldots, p_{i-1}, r_0, q_0, \ldots, q_{i-1}]$, $R^m \cap P^{2i} = \{r_0\}$ and $P^{2i} \cap S^{n-1}$ is a nondegenerate S^{2i-1} with ind $S^{2i-1} = i - 1$.

We introduce the i-flats $\widetilde{P}^i$, $\widetilde{Q}^i$, P^i_α and Q^i_α in S^{n-1} as in 2.6.4; $\alpha \in I$. Let $j = i - 1$ and $J = I \backslash \{i\}$. Then the j-flats $\widetilde{P}^j = P^{2i} \cap \widetilde{P}^i$, $\widetilde{Q}^j = P^{2i} \cap \widetilde{Q}^i$, $P^j_\alpha = P^{2i} \cap P^i_\alpha$ and $Q^j_\alpha = P^{2i} \cap Q^i_\alpha$ are contained in S^{2i-1}; $\alpha \in J$. Finally, $\{p_i, q_i\} \subset \{r_1, \ldots, r_m\}$ implies $R^m = [R^s, p_i, q_i]$; (cf. the proof of 2.6.3 Corollary). Then

$$\begin{aligned} \pi(p_\alpha) &= [p_0, \ldots, p_i, R^s, q_0, \ldots, \hat{q}_\alpha, \ldots, q_i] \\ &= [\widetilde{P}^j, R^m, q_0, \ldots, \hat{q}_\alpha, \ldots, q_j], \\ \pi(q_\alpha) &= [\widetilde{Q}^j, R^m, p_0, \ldots, \hat{p}_\alpha, \ldots, p_j], \; \alpha \in I; \quad \text{cf. 2.6.4.} \end{aligned} \tag{1}$$

A.5.2. Lemma. (1) $P^{2i} \subset \pi(r_\gamma)$; $\gamma = 1, 2, \ldots, m$.
(2) $R^{m-1} \subset \pi(p)$ *for each* $p \in S^{2i-1}$.

A.5.3. Let u_k be the base points of P^n where

$$u_k = p_k; \quad k = 0, 1, \ldots, i-1,$$
$$u_{i+k} = r_k; \quad k = 0, 1, \ldots, n-2i,$$
$$u_{n-k} = q_k; \quad k = 0, 1, \ldots, i-1.$$

Let $\mathrm{Q}^{n-1} \subset P^n$ be given by

$$x_i^2 + 2\sum_{\sigma=0}^{i-1} a_{\sigma,n-\sigma} x_\sigma x_{n-\sigma} + 2 \sum_{\sigma,\mu=i+1;\sigma\neq\mu}^{n-i} a_{\sigma\mu} x_\sigma x_\mu = 0; a_{\sigma\beta} \neq 0. \tag{1}$$

Then $r_0 \notin \mathrm{Q}^{n-1}$ and Q^{n-1} contains $\widetilde{P}^j$, $\widetilde{Q}^j$, P^j_α and Q^j_α; $\alpha \in J$.

We have observed that S^{m-1} is a quadric. As ind $S^{2i-1} = [\frac{1}{2}(2i-1)]$, S^{2i-1} is a quadric from A.3. Thus we can choose $u_{\sigma\beta}$ in (1) so that $S^{m-1} = R^m \cap \mathrm{Q}^{n-1}$ and $S^{2i-1} = P^{2i} \cap \mathrm{Q}^{n-1}$. This determines Q^{n-1} uniquely.

Let $b \equiv (b_0, \ldots, b_n) \in S^{2i-1}$. Then $b_{i+1} = \cdots = b_{n-i} = 0$ and $\omega(b) \equiv b_i x_i + \Sigma_{\sigma=0}^{i-1} a_{\sigma,n-\sigma}(b_\sigma x_{n-\sigma} + b_{n-\sigma} x_\sigma) = 0$. Also for $\sigma - 1, \ldots, m$, $\omega(r_\sigma) = \omega(u_{i+\sigma}) \equiv \Sigma_{\mu=i+1;\mu\neq\sigma}^{n-i} a_{\sigma\mu} x_\mu = 0$.

Clearly, $P^{2i} \subset \omega(r_\sigma)$ for $\sigma = 1, \ldots, m$ and $R^{m-1} \subset \omega(p)$ for each

$p \in S^{2i-1}$. Finally, $\omega(p_\sigma)$ is the hyperplane given by $x_{n-\sigma} = 0$. Thus,

$$\omega(p_\sigma) = [p_0, \ldots, p_{i-1}, r_0, \ldots, r_m, q_0, \ldots, \hat{q}_\sigma, \ldots, q_{i-1}] = \pi(p_\sigma).$$

Similarly, $\omega(q_\sigma) = \pi(q_\sigma)$ is given by $x_\sigma = 0$; $\sigma \in J$. Hence, $\{u_i\} = \{r_0\} \subsetneq \bigcap_{\sigma=0;\sigma\neq i}^{n} \omega(u_\sigma)$.

A.5.4. We have shown that a nondegenerate $S^{n-1} \subset P^n$ with ind $S^{n-1} = 1$ or $[\frac{1}{2}(n-1)]$ is a quadric; $n \geqslant 3$. In particular, every nondegenerate S^{n-1} with ind $S^{n-1} > 0$ is a quadric when $n = 3, 4, 5$ or 6. Since $n \geqslant 7$, we assume every nondegenerate $S^{k-1} \subset P^k$, with ind $S^{k-1} > 0$ and $k < n$ is a quadric.

A.5.5. LEMMA. $\pi(p_\sigma) \cap \pi(q_\sigma) \cap S^{n-1} = \pi(p_\sigma) \cap \pi(q_\sigma) \cap Q^{n-1}$; $\sigma \in J$.

PROOF. By the symmetry in $\sigma \in J$, we may assume $\sigma = i - 1$. Now

$$P^{n-2} = \pi(p_{i-1}) \cap \pi(q_{i-1}) = [p_0, \ldots, p_{i-2}, R^m, q_0, \ldots, q_{i-2}]$$

is an $(n-2)$-flat. By 2.6.2, $P^{n-2} \cap S^{n-1}$ is a nondegenerate S^{n-3}; ind $S^{n-3} = i - 1$. By A.5.4, S^{n-3} is a quadric Q^{n-3}.

Clearly, $S^{m-1} \subset S^{n-3}$ and S^{n-3} contains the $(i-2)$-flats $P^{n-2} \cap \widetilde{P}^j$, $P^{n-2} \cap \widetilde{Q}^j$, $P^{n-2} \cap P_\alpha^j$ and $P^{n-2} \cap Q_\alpha^j$; $\alpha = 0, 1, \ldots, i-2$.

Using the base points in A.5.3, P^{n-2} is given by $x_{i-1} = x_{n-i+1} = 0$. Let Q^{n-3} be given by

$$x_i^2 + 2\sum_{\sigma=0}^{i-2} d_{\sigma,n-\sigma} x_\sigma x_{n-\sigma} + 2\sum_{\sigma,\mu=i+1;\sigma\neq\mu}^{n-i} d_{\sigma\mu} x_\sigma x_\mu = 0, \quad x_{i-1} = x_{n-i+1} = 0.$$

Then $S^{m-1} = R^m \cap Q^{n-1} = R^m \cap Q^{n-3}$ implies that $a_{\sigma\mu} = d_{\sigma\mu}$; $\sigma \neq \mu$, $\sigma, \mu = i+1, \ldots, n-i$. Similarly (cf. the proof of A.3.3), $S_\mu^1 = P_\mu^2 \cap Q^{n-1} = P_\mu^2 \cap Q^{n-3}$ implies that $a_{\mu,n-\mu} = d_{\mu,n-\mu}$; $\mu = 0, 1, \ldots, i-2$. Thus $Q^{n-3} \subset Q^{n-1}$ and $P^{n-2} \cap S^{n-1} = S^{n-3} = Q^{n-3} = P^{n-2} \cap Q^{n-1}$.

COROLLARY. $\pi(u_\sigma) \cap S^{n-1} = \pi(u_\sigma) \cap Q^{n-1}$; $\sigma = 0, \ldots, i-1, n-i+1, \ldots, n$.

A.5.6. LEMMA. *Let* $p \in S^{2i-1} \cup S^{m-1}$. *Then* $\pi(p) = \omega(p)$.

PROOF. Let $\pi'(p)[\widetilde{\pi}(p)]$ be the tangent hyperplane of $S^{2i-1}[S^{m-1}]$ at a point p. Then dim $\pi'(p) = 2i - 1$ and dim $\widetilde{\pi}(p) = m - 1$.

If $p \in S^{2i-1}$, then clearly $\pi(p) = [\pi'(p), R^{m-1}]$. Since $S^{2i-1} = P^{2i} \cap Q^{n-1}$, we have $\pi'(p) \subsetneq \omega(p)$ and thus, $\omega(p) = [\pi'(p), R^{m-1}]$.

If $p \in S^{m-1} = R^m \cap Q^{n-1}$, then $\widetilde{\pi}(p) \subsetneq \pi(p) \cap \omega(p)$. From the construction, $[\widetilde{P}^{i-1}, \widetilde{Q}^{i-1}] \subsetneq \pi(p) \cap \omega(p)$ and thus, $[\widetilde{\pi}(p), \widetilde{P}^{i-1}, \widetilde{Q}^{i-1}] \subsetneq \pi(p) \cap \omega(p)$. But $\widetilde{\pi}(p) \subset R^m$ and $R^m \cap [\widetilde{P}^{i-1}, \widetilde{Q}^{i-1}] = \emptyset$ imply

$$\dim[\widetilde{\pi}(p), \widetilde{P}^{i-1}, \widetilde{Q}^{i-1}] = n - 1.$$

A.5.7. LEMMA. *Let* $p \in S^{2i-1} \cup S^{m-1}$. *Then* $\pi(p) \cap S^{n-1} = \pi(p) \cap Q^{n-1}$.

PROOF. By A.5.6, it is sufficient to find a $P^{n-2} \subset \pi(p)$ such that $\pi(p) = [P^{n-2}, p]$ and $P^{n-2} \cap S^{n-1} = P^{n-2} \cap Q^{n-1}$; for then, $\pi(p) \cap S^{n-1}$ and $\pi(p) \cap Q^{n-1}$ are cones with the vertex p and the same $(n-2)$-section.

Let $p \in S^{2i-1}$. Since $P^{2i} \cap R^m = \{r_0\} \not\subset S^{n-1}$, we have $p \notin R^m$ and thus, $p \notin \pi(p_0)$ say. Take $P^{n-2} = \pi(p) \cap \pi(p_0)$. Then by A.5.5 Corollary,

$$P^{n-2} \cap S^{n-1} = P^{n-2} \cap (\pi(p_0) \cap S^{n-1}) = P^{n-2} \cap (\pi(p_0) \cap Q^{n-1})$$
$$= P^{n-2} \cap Q^{n-1}.$$

Let $p \in S^{m-1}$ and let $P^2 = [p_0, q_0, r_0]$. Obviously, $P^2 \cap S^{n-1}$ is an $S^1 \subset S^{2i-1}$. Now $r_0 \in \pi(p_0) \cap \pi(q_0)$ implies that $r_0 \notin \pi(q)$ for each $q \in S^1 \backslash \{p_0, q_0\}$. By A.5.2, $R^{m-1} = \pi(q) \cap R^m$ for each $q \in S^1\backslash\{p_0, q_0\}$.

If $p \in S^{m-1}\backslash R^{m-1}$, then $p \notin \pi(q)$ for some $q \in S^1 \subset S^{2i-1}$. By the preceding, $\pi(p) \cap (\pi(q) \cap S^{n-1}) = \pi(p) \cap (\pi(q) \cap Q^{n-1})$.

If $p \in R^{m-1} \cap S^{m-1}$, then ind $S^{m-1} = 0$ implies that $p \notin \pi(p')$ for any $p' \in S^{m-1}\backslash R^{m-1}$. The lemma now follows as above.

A.5.8. Let $\mathcal{M} = \{u_0, \ldots, u_{i-1}, u_{n-i+1}, \ldots, u_n\}$. From A.5.3, $\pi(u_\sigma) = \omega(u_\sigma)$ is given by $x_{n-\sigma} = 0$ for each $u_\sigma \in \mathcal{M}$. Thus for $\{u_{\sigma_1}, \ldots, u_{\sigma_k}\} \subset \mathcal{M}$, $P^{n-k} = \bigcap_{j=1}^k \pi(u_{\sigma_j})$ is an $(n-k)$-flat.

Let M^i be an i-flat; $M^i \cap R^m = \emptyset$. Assume that $M_\sigma^{i-1} = \pi(u_\sigma) \cap M^i$ is an $(i-1)$-flat for each $u_\sigma \in \mathcal{M}$. Let $U = \{M_\sigma^{i-1} | u_\sigma \in \mathcal{M}\}$.

A.5.9. LEMMA. *$M^{i-1} = M_\sigma^{i-1}$ has at most i solutions u_σ in $\mathcal{M}$.*

PROOF. Let $\{u_{\sigma_1}, \ldots, u_{\sigma_k}\} \subset \mathcal{M}$ be the set of solutions of $M^{i-1} = M_\sigma^{i-1}$. Then $M^{i-1} = \bigcap_{j=1}^k M_{\sigma_j}^{i-1} = (\bigcap_{j=1}^k \pi(u_{\sigma_j})) \cap M^i = P^{n-k} \cap M^i$. As $R^m = \bigcap_{u_\sigma \in \mathcal{M}} \pi(u_\sigma)$, this implies that $[R^m, M^{i-1}] \subseteq P^{n-k}$. But $R^m \cap M^i = \emptyset$ implies that $\dim([R^m, M^{i-1}]) = n - i$ and thus, $k \leqslant i$.

COROLLARY 1. *Let $M^{i-1} = M_\sigma^{i-1}$ $[\overline{M}^{i-1} = M_\sigma^{i-1}]$ have k $[h]$ solutions in $\mathcal{M}$; $M^{i-1} \neq \overline{M}^{i-1}$. Then $k + h \leqslant i + 1$.*

PROOF. Let $\{u_{\sigma_1}, \ldots, u_{\sigma_k}\}$ $[\{u_{\sigma_{k+1}}, \ldots, u_{\sigma_{k+h}}\}]$ be the two sets of solutions in $\mathcal{M}$. Then

$$M^{i-1} = \left(\bigcap_{j=1}^{k} \pi(u_{\sigma_j})\right) \cap M^i \quad \text{and} \quad \overline{M}^{i-1} = \left(\bigcap_{j=k+1}^{k+h} \pi(u_{\sigma_j})\right) \cap M^i.$$

Since $M^i = [M^{i-1}, \overline{M}^{i-1}]$, $M^{i-2} = M^{i-1} \cap \overline{M}^{i-1}$ is an $(i-2)$-flat, and $M^{i-2} = (\bigcap_{j=1}^{k+h} \pi(u_{\sigma_j})) \cap M^i = P^{n-(k+h)} \cap M^i$. As in A.5.9, $[R^m, M^{i-2}] \subseteq P^{n-k-h}$ and $k + h \leqslant i + 1$.

COROLLARY 2. $|U| \geqslant 3$.

PROOF. As $|M| = 2i$, A.5.9 implies that $|U| \geqslant 2$. Since $i \geqslant 2$, we have $|M| = 2i > i + 1$ and by Corollary 1, $|U| \geqslant 3$.

A.5.10. THEOREM. *Under the hypotheses of* A.5.1 *and* A.5.3, $S^{n-1} = Q^{n-1}$.

PROOF. We first prove $Q^{n-1} \subseteq S^{n-1}$. Let the i-flat $M^i \subset Q^{n-1}$. By A.5.6 and A.5.7, we may assume that $M^i \cap R^m = M^i \cap P^{2i} = \emptyset$. Since $M \subset P^{2i}$; $u_\sigma \notin M^i$ and thus, $M_\sigma^{i-1} = \pi(u_\sigma) \cap M^i \subset S^{n-1}$ is an $(i-1)$-flat for each $u_\sigma \in M$. By A.5.9 Corollary 2, $|U| \geqslant 3$ and thus, $M^i \subset S^{n-1}$ by 2.1.5.

The preceding argument is symmetric in S^{n-1} and Q^{n-1}.

We collect our results.

A.5.11. THEOREM. *A nondegenerate* S^{n-1} *with* ind $S^{n-1} > 0$ *is a quadric*; $n \geqslant 3$.

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