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# $SK_1$ OF $n$ LINES IN THE PLANE

BY

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ABSTRACT. We calculate $SK_1(A)$ where $A$ is the coordinate ring of the reduced affine variety consisting of $n$ straight lines in the plane.

1. **Introduction.** In [9] I calculated $SK_1$ for curves whose irreducible components were isomorphic to the affine line and such that at each intersection point the components meet transversally. Here I would like to consider plane curves whose irreducible components are all isomorphic to the affine line and such that at each intersection point the components have distinct tangent directions.

First consider three straight lines through the origin. Let $A$ be a commutative regular ring. Then Dennis and Krusemeyer [2] have calculated $K_2A[X, Y]/(XY)$. Using their result and the calculation sketched in §4 of [9] it can be shown that $K_1A[X, Y]/XY(X - Y) = K_1(A) \oplus \Omega_A$. Here and throughout the paper $\Omega_A$ denotes the absolute differentials $\Omega^1_{A/Z}$ ($Z$ = integers). On the other hand it is proved in [1] that if $k$ is a field (char $k = 0$ or char $k \geqslant n - 1$) and Spec $B$ is the reduced affine variety over $k$ consisting of $n$ straight lines through one point, then Pic $B = {}_{n-1}C_2k^+$ where $k^+$ is the additive group of the field. These results suggest that we ought to have $SK_1(B) = {}_{n-1}C_2\Omega_k$. (Here as well as later if $M$ is an abelian group then $nM$ means the direct sum of $n$ copies of $M$.) For "large" characteristic I prove that this is "almost" true. Unlike the case of Pic, the exact result depends on which lines one has.

In [9] I tried to work with as general a ground ring as possible. Here (except for the above remarks concerning three lines through one point) I will work over a field $k$ of characteristic zero or characteristic $> n - 1$ where $n$ is the largest number of components passing through one point. The reason for this restriction is that I use [4] which requires such a field. Using [5] or [10] some results could perhaps be obtained for more general groundfields (or rings) but would probably be more complicated.

In §2 I make some general remarks about $SK_1$ and the conductor. In §3 I give generators and relations for $SK_1$ of $n$ lines in the plane through the origin.

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The next section uses these generators and relations to obtain more detailed results about $SK_1$ of $n$ lines in the plane through the origin. The final section considers plane curves such that ($\alpha$) all irreducible components are isomorphic to the affine line, ($\beta$) each intersection point is $k$-rational, ($\gamma$) the tangent directions of all components passing through a given intersection point are distinct. This section is motivated by calculations of Pic in [7]. There it is shown that for such curves Pic depends only on the number of intersection points and the number of components passing through each. My results on $SK_1$, although not complete, are sufficient to show that the situation is more complicated with $SK_1$ than with Pic.

2. **General remarks.** Let $X = \operatorname{Spec} A$ be a reduced, connected (not necessarily plane) affine curve over a field $k$, with each irreducible component isomorphic to the affine line. Suppose there are $n$ components and $m$ crossing points (all of which are rational over $k$). Let $Y = \operatorname{Spec} B$ be the normalization of $A$. Then $B = \Pi_{i=1}^{n} k[t_i]$. Let $I$ be the conductor of $A$ in $B$. We have a cartesian square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & B/I \end{array}$$

Since $B$ is a product of polynomial rings we have $K_1(B) = \Pi_{i=1}^{n} k^*$ (where $k^*$ denotes the units of $k$). The rings $A/I$ and $B/I$ are each the product of local Artinian $k$-algebras (with residue class field $k$) supported at the crossing points. For a local ring $SK_1 = 0$. Thus the Mayer-Vietoris sequence [3, p. 246] yields an exact sequence $K_2(A/I) \oplus K_2(B) \xrightarrow{f} K_2(B/I) \longrightarrow SK_1(A) \longrightarrow 0$. For each local Artinian $k$-algebra $R$ (with residue field $k$) take the direct sum decomposition $K_2(R) = K_2(k) \oplus SK_2(R)$ induced by the split surjection $R \longrightarrow k$. We have $K_2(B) = \Pi_{i=1}^{n} K_2(k)$, one copy for each component. The homomorphism $f$ respects these direct sum decompositions, so $SK_1(A)$ is the direct sum of two factors, one a "global" part involving the $K_2(k)$'s, and the other a "local" part involving the $SK_2$'s. There are $m$ copies of $K_2(k)$ from $K_2(A/I)$ one for each intersection point. Suppose an intersection point $P \in X$ has $p$ points in $Y$ lying over it. Then from $K_2(B/I)$ we get $(\Sigma p)$ copies of $K_2(k)$, one for each point of $Y$ that lies over an intersection point of $X$. The mapping $(m + n)K_2(k) \longrightarrow (\Sigma p)K_2(k)$ is as follows: if $Q \in \operatorname{Spec} B$ lies over an intersection point $P \in \operatorname{Spec} A$ the mapping between the corresponding copies of $K_2(k)$ is the identity; if line $l_i$ contains $Q \in \operatorname{Spec} B$ the mapping between the corresponding copies of $K_2(k)$ is the identity. Otherwise the mappings are zero. This is the same mapping as was studied in [9] where we assumed that the $p$ curves through each $P$ crossed transversally. Let $(A/I)_P$ denote the component of $A/I$ concentrated on $P \in \operatorname{Spec} A$, and similarly for $(B/I)_Q$, $Q \in \operatorname{Spec} B$. Then the $SK_2$ part of $f$ splits up as the direct sum of maps $f_P : SK_2(A/I)_P \longrightarrow \bigoplus_Q SK_2(B/I)_Q$ ($Q$ ranging over all points in $Y$ lying over the intersection point $P \in X$). Let $K_2(P) = \operatorname{coker} f_P$. Then we have proved (using formula (1) of [9])

THEOREM 1. *Let $A$ be the coordinate ring of a reduced connected curve over a field $k$. Suppose there are $n$ irreducible components each isomorphic to the affine line and $m$ intersection points all of which are $k$-rational. Let $K_2(P)$ be as defined above. Then*

$$SK_1(A) = \bigoplus_P K_2(P) \oplus (M - m - n + 1)K_2(k)$$

*where $M = \Sigma\, p$.*

The same methods yield similar results for $\operatorname{Pic} A$. For a local Artinian $k$-algebra $R$ with residue field $k$, write $K_1(R) = k^* \oplus \overline{SK}_1(R)$ for the direct sum decomposition induced by the split surjection $R \longrightarrow k$. (Since $R$ is local, $K_1(R) = R^*$, the group of units of $R$. Thus $SK_1(R) = 0$, and $\overline{SK}_1(R)$ represents the extra units in $R$, over those contained in $k$.) Let $A$ be as in Theorem 1, and

$$K_1(P) = \operatorname{coker}\left[\overline{SK}_1(A/I)_P \longrightarrow \bigoplus_Q \overline{SK}_1(B/I)_Q\right]$$

($Q$ ranging over all points in $Y$ lying over the intersection point $P \in X$). Then we have

THEOREM 2. *For $A$ as in Theorem* 1,

$$\operatorname{Pic} A = \bigoplus_P K_1(P) \oplus (M - m - n + 1)k^*.$$

Thus our task is to find the conductor $I$, and then to calculate the groups $K_2(P)$. Now suppose that $\operatorname{Spec} A \subset \operatorname{Spec} k[X, Y]$ is a reduced connected plane curve over a field $k$ satisfying conditions ($\alpha$), ($\beta$), ($\gamma$) given in the introduction. Then it is proved in [7] that the conductor $I$ of $A$ in its normalization $\Pi_{i=1}^n k[t_i]$ is $I = \Pi_P m_P^{p-1}$ where $m_P = (X - a, Y - b)A$ is the maximal ideal of the point $P$ and $p$ is the number of irreducible components through $P$. The calculation of $K_2(P)$ in various cases will be done in the rest of the paper.

3. **$SK_1$ of $n$ lines through one point.** Let $\operatorname{Spec} A$ consist of $n$ distinct lines through the origin. Then by a suitable choice of coordinates we have

$$A = k[X, Y]/XY(Y - X)(Y - \alpha_4 X) \cdots (Y - \alpha_n X)$$

where $0, 1, \alpha_4, \ldots, \alpha_n$ are distinct elements of $k$. We write the lines in this form (instead of, for example, $Y = \alpha_i X$, $1 \leqslant i \leqslant n$) so we can see more clearly what is happening in the first new case, $n = 4$.

The homomorphism $A \longrightarrow B = \Pi_{i=1}^n k[t_i]$ can be defined by $\pi_1(Y) = t_1$, $\pi_1(X) = 0$, and $\pi_i(X) = t_i$, $\pi_i(Y) = \alpha_i t_i$, $i \geqslant 2$ ($\alpha_2 = 0$, $\alpha_3 = 1$ and $\pi_i$ denotes projection onto the $i$th factor). We have seen in §2 that the conductor $I$ of $A$

in $B$ is $I = (X^{n-1}, X^{n-2}Y, \ldots, Y^{n-1})A$. Thus

$$A/I = k[X, Y]/(X^{n-1}, X^{n-2}Y, \ldots, XY^{n-2}, Y^{n-1})$$

and

$$B/I = \prod_{i=1}^{n} k[t_i]/t_i^{n-1}.$$

By Theorem 1 $SK_1(A) = \operatorname{coker} f: SK_2(A/I) \longrightarrow \oplus_{i=1}^{n} SK_2 k[t_i]/t_i^{n-1}$. For any reduced variety consisting of straight lines in the plane, the $K_2(P)$'s will all be of this type. Thus the calculations of this and the next section, together with Theorem 1 yield a description of $SK_1$ for any reduced affine variety consisting of straight lines in the plane.

The group $SK_2(k[t]/t^{n-1})$ has been calculated by Graham in [4, p. 485] if $\operatorname{char} k = 0$ or if $\operatorname{char} k > n-1$. He shows that $SK_2(k[t]/t^{n-1}) = \Omega_k[t]/t^{n-2}\Omega_k[t]$. The ring $k[t]/t^{n-1}$ is local so $K_2(k[t]/t^{n-1})$ is generated by Steinberg symbols $\{xu, yv\}$ [3, p. 252] ($x, y \in k^*$; $u, v \in k[t]/t^{n-1}$ with constant term one). The projection $\pi: K_2(k[t]/t^{n-1}) \longrightarrow SK_2(k[t]/t^{n-1}) = \Omega_k[t]/t^{n-2}\Omega_k[t]$ is given explicitly by

$$\pi\{xu, yv\} = -\frac{v'}{v}\frac{dx}{x} + \frac{u'}{u}\frac{dy}{y} + \frac{u'}{u}\frac{Dv}{v} - \frac{v'}{v}\frac{Du}{u} \tag{1}$$

[4, pp. 486 and 481]. Here $'$ denotes differentiation with respect to $t$, and $D$ means that we apply $d: k \longrightarrow \Omega_k$ to each coefficient of a power series in $k[[t]]$. I will equate a symbol in $SK_2(k[t]/t^{n-1})$ with its image under $\pi$.

I do not know what the group $K_2(A/I)$ is. However $A/I$ is local so $K_2(A/I)$ is generated by symbols. Every element in $A/I$ can be written (uniquely) in the form $\alpha \exp(g_1)\exp(g_2) \cdots \exp(g_{n-2})$ where exp denotes the exponential function and $g_i$ is homogeneous of degree $i$. Thus $SK_2(A/I)$ is generated by the Steinberg symbols $\{\alpha, \exp aX^pY^q\}$ and $\{\exp aX^pY^q, \exp bX^rY^s\}$ where $\alpha, a, b \in k^*$. It is easily checked that $(\exp at^i)' = ait^{i-1}\exp at^i$ and $D \exp at^i = t^i(da)\exp at^i$. Therefore from (1) we get

$$\begin{aligned} \{\alpha, \exp at^i\} &= -iat^{i-1}\, d\alpha/\alpha, \\ \{\exp at^i, \exp bt^j\} &= t^{i+j-1}\,(iadb - jbda). \end{aligned} \tag{2}$$

From this and the definition of the projections $\pi_i$ (using the convention $0^0 = 1$) we have

$$f\{\alpha, \exp aX^pY^q\} = -(p+q)a(0^p, 0^q, \alpha_3^q, \ldots, \alpha_n^q)t^{p+q-1}\, d\alpha/\alpha,$$

$0 < p + q < n - 1$. Here as well as later we write $t = (t_1, t_2, \ldots, t_n)$. Taking sums of such expressions we see that the contribution to Image $f$ from the symbols $\{\alpha, \exp aX^pY^q\}$ is generated (as an abelian group) by elements of the form

$$c(0^{j+1-m}, 0^m, \alpha_3^m, \alpha_4^m, \ldots, \alpha_n^m)t^j \tag{3}$$

for $0 \leqslant j \leqslant n-3$, $0 \leqslant m \leqslant j+1$, $c \in \Omega_k$ arbitrary.

Similarly

$$
\begin{aligned}
&f\{\exp aX^pY^q, \exp bX^rY^s\} \\
&\quad = (0^{p+r}, 0^{q+s}, 1, \alpha_4^{q+s}, \ldots, \alpha_n^{q+s})((p+q)adb - (r+s)bda)t^{p+q+r+s-1} \\
&\qquad + (0, 0, 0, \alpha_4^{q+s-1}\, d\alpha_4, \ldots, \alpha_n^{q+s-1}\, d\alpha_n)(sp - qr)abt^{p+q+r+s-1}.
\end{aligned}
$$

The first vector is of type (3) so we need only consider the second type. If we set $q = r = 0$, $ps > 0$, we see that Image $f$ contains all elements of the form

$$\alpha(0, 0, 0, \alpha_4^m\, d\alpha_4, \ldots, \alpha_n^m\, d\alpha_n)t^j \tag{4}$$

for $1 \leqslant j \leqslant n-3$, $0 \leqslant m \leqslant j-1$, $\alpha \in k$ arbitrary. We cannot increase the range of $j$ and $m$ for arbitrary $p, q, r, s$. For we always have $0 < p+q$, $0 < r+s$ so $j = 0$ is impossible. It is obvious that $m \leqslant j$. If $m = j$ then $p = r = 0$, and the term $sp - qr$ vanishes. For fixed $j$ the $j+2$ rows $(0^{j+1-m}, 0^m, 1, \alpha_4^m, \ldots, \alpha_n^m)$, $0 \leqslant m \leqslant j+1$, occuring in (3) are linearly independent over $k$. Therefore from $t^j$ we are left with $(n-j-2)\Omega_k$, and $\Sigma_{j=0}^{n-3}(n-j-2) = \Sigma_{j=1}^{n-2} j = {}_{n-1}C_2$. Therefore we have proved

THEOREM 3. *Let $k$ be a field with* char $k = 0$ *or* char $k > n-1$. *Let*

$$A = k[X, Y]/XY(Y-X)(Y-\alpha_4X)\cdots(Y-\alpha_nX).$$

*Then $SK_1(A) = ({}_{n-1}C_2\Omega_k)/V$ where $V$ is a finite-dimensional vector space over $k$.*

If $n = 3$ we saw in the introduction that $V = 0$, and if all the $d\alpha_i$ are zero $V = 0$ also. The next section will investigate dim $V$ more carefully.

4. **The dimension of $V$.** Let Spec $A$ consist of $n$ distinct lines through the origin, with notation as in §3. Both the generators and the relations for $SK_1(A)$ are homogeneous in $t$. Thus $SK_1(A)$ is graded by the power $j$ of $t$, $0 \leqslant j \leqslant n-3$. The degree $j$ part of $SK_1(A)$ is then $n\Omega_k$ factored out by the subgroup generated by elements of types (3) and (4) (for $j$ fixed and $m$ as indicated above). The cases $m = 0$ and $m = j+1$ of (3) merely eliminate the first two copies of $\Omega_k$. Thus we are left with $(n-2)\Omega_k$ factored out by the subgroup consisting of linear combinations of the rows of the matrix $M$,

$$
M = \begin{vmatrix}
1 & \alpha_4 & \alpha_5 & \cdots & \alpha_n \\
 & \vdots & & & \\
1 & \alpha_4^j & \alpha_5^j & \cdots & \alpha_n^j \\
0 & d\alpha_4 & d\alpha_5 & \cdots & d\alpha_n \\
0 & \alpha_4 d\alpha_4 & \alpha_5 d\alpha_5 & \cdots & \alpha_n d\alpha_n \\
 & & \vdots & & \\
0 & \alpha_4^{j-1}\, d\alpha_4 & \alpha_5^{j-1}\, d\alpha_5 & \cdots & \alpha_n^{j-1}\, d\alpha_n
\end{vmatrix}
$$

where as usual the first $j$ rows can be multiplied by arbitrary $c \in \Omega_k$ and the last $j$ rows can be multiplied by arbitrary $\alpha \in k$. Let $W_j$ be the subgroup spanned by the first $j$ rows. These rows are linearly independent over $k$, so $(n-2)\Omega_k/W_j = (n-j-2)\Omega_k$. Let $V_j$ be the image in $(n-2)\Omega_k/W_j$ of the $k$-subspace of $(n-2)\Omega_k$ spanned by the last $j$ rows of $M$. Then $\bigoplus_{j=0}^{n-3} V_j$ was denoted $V$ in §3, and $SK_1(A) = ({}_{n-1}C_2\Omega_k)/V$.

Clearly $\dim_k V_j \leqslant j$. By row reducing the top and bottom half of $M$ separately and using the first row to eliminate another copy of $\Omega_k$ we end up with $(n-3)\Omega_k$ factored out by the subgroup consisting of linear combinations of the rows of $M_1$, where the first $j-1$ rows can be multiplied by $c \in \Omega_k$ and the last $j$ rows can be multiplied by $\lambda \in k$.

$$M_1 = \left|\begin{array}{cccc|c|c} \multicolumn{4}{c|}{I_{j-1}} & D_1 & D_2 \\ \hline d\alpha_4 & & & 0 & & \\ & d\alpha_5 & \cdots & & 0 & B \\ & & \cdots & & & \\ 0 & & & d\alpha_{j+2} & & \\ & & & & d\alpha_{j+3} & \end{array}\right|.$$

Here $I_{j-1}$ is the $(j-1) \times (j-1)$ identity matrix, $D_1$ is a column vector over $k$ of length $j-1$, and $D_2$ is a $(j-1) \times (n-j-3)$ matrix over $k$. $B$ is a $j \times (n-j-3)$ matrix over $\Omega_k$ of the form

$$B = B_1 \left|\begin{array}{cccc} d\alpha_{j+4} & & 0 & \\ & d\alpha_{j+5} & & \\ 0 & & \cdots & \\ & & & d\alpha_n \end{array}\right|.$$

where $B_1$ is a $j \times (n-j-3)$ matrix over $k$. (If $j = n-3$ then $D_2$ and $B$ are absent.)

Every $j \times j$ minor of the first $j$ rows of $M$ is nonzero, hence so is every $(j-1) \times (j-1)$ minor of the first $j-1$ rows of $M_1$. Thus all the entries of $D_1$ and $D_2$ are nonzero. Similarly all the entries of $B_1$ are nonzero.

Let $d = \dim_k(d\alpha_4, \ldots, d\alpha_n)$, and suppose the $\alpha_i$ are ordered so that the first $d$ of the $d\alpha_i$ are linearly independent over $k$. Let the last $j$ rows of $M_1$ be $e_1, \ldots, e_j$. We wish to know for which $\lambda_i \in k$ $(1 \leqslant i \leqslant j)$ $\lambda_1 e_1 + \cdots + \lambda_j e_j$ is a linear combination of the first $j-1$ rows of $M_1$. If it is such a linear combination the first row of $M_1$ must be multiplied by $\lambda_1 d\alpha_4$, the second by

$\lambda_2 d\alpha_5, \ldots$ the $(j-1)$st by $\lambda_{j-1} d\alpha_{j+2}$. Let $D_1^t = (d_1, d_2, \ldots, d_{j-1})$. Then $\lambda_1 d_1 d\alpha_4 + \lambda_2 d_2 d\alpha_5 + \cdots + \lambda_{j-1} d_{j-1} d\alpha_{j+2} = \lambda_j d\alpha_{j+3}$. The $d_i$'s are all nonzero, so if $d \geqslant j$ there are no nontrivial solutions, i.e. $\dim V_j = j$. If $d < j$ the vector space of solutions $(\lambda_1, \ldots, \lambda_j)$ of the above equation will have dimension $j - d$. Unless $j = n - 3$ there are other conditions on the $\lambda$'s coming from $D_2$ and $B$ that must be satisfied simultaneously, so the vector space of solutions $(\lambda_1, \ldots, \lambda_j)$ has dimension $\leqslant j - d$. Therefore if $d < j$, $\dim_k V_j \geqslant d$. Altogether we have $\dim_k V_j \geqslant \inf(j, d)$. We have already observed that $\dim_k V_j \leqslant j$. If $d' =$ number of nonzero $d\alpha_i$ we have furthermore $\dim_k V_j \leqslant \inf(j, d')$. I will try to give a sharper upper bound on $\dim_k V_j$. If $d \geqslant j$ we have already seen that $\dim V_j = j$. Therefore assume that $d < j$. Row reduce $M_1$ to clear out the lower left-hand block. This leaves us with

$$M_2 = \begin{vmatrix} I_{j-1} & D_1 & D_2 \\ & & \\ 0 & & D \end{vmatrix}$$

where $D$ is a $j \times (n-j-2)$ matrix. Clearly no linear combination of the first $j - 1$ rows of $M_1$ can lie in the $k$-vector space spanned by the rows of $D$. Therefore $\dim V_j =$ dimension of the $k$-vector space spanned by the rows of $D$. We have assumed that $\dim_k(d\alpha_4, \ldots, d\alpha_n) = d$ and that the first $d$ of the $d\alpha_i$ are linearly independent. Express the rest of the $d\alpha_i$ as $k$-linear combinations of $d\alpha_4, \ldots, d\alpha_{d+3}$ (recall that $d < j$). Then $D = A_1 d\alpha_4 + A_2 d\alpha_5 + \cdots + A_d d\alpha_{d+3}$ where each $A_i$ is a $j \times (n-j-2)$ matrix with coefficients in $k$. There are surjections row space of $D \longrightarrow$ row space of $A_i$, and the intersection of the kernels of these surjections is zero. Thus $\dim V_j \leqslant \Sigma_{i=1}^d \operatorname{rank} A_i \leqslant d \inf(j, n-j-2)$. Of course $\dim V_j \leqslant j$ so altogether we have

$$\dim V_j \leqslant \inf(j, d \inf(j, n-j-2)) = \inf(j, d(n-j-2)),$$

so

$$\inf(j, d) \leqslant \dim_k V_j \leqslant \inf(j, d(n-j-2)). \tag{5}$$

I expect that "in general" we will have $\dim_k V_j = \inf(j, d(n-j-2))$. However if $d' < n-3$ we can try to improve the upper estimate on $\dim_k V_j$. If $d' < j$ then, in $M_1$, $d\alpha_{j+3}$ and $B$ are both zero. Thus the only nonzero entries in $D$ come from the final row reduction. $A_i$ can have nonzero entries only in row $i$, and in rows $d+1$ through $d'$. Therefore $\operatorname{rank} A_i \leqslant \inf(d'-d+1, n-j-2)$, so

$$\begin{aligned} d = \inf(j, d) &\leqslant \dim_k V_j \\ &\leqslant \inf(d', d \inf(d'-d+1, n-j-2)) \qquad (d \leqslant d' < j). \end{aligned} \tag{6}$$

In particular if $d = d' < j$ then $\dim V_j = d$. If $j \leqslant d' < n-3$ then $A_i$ can have nonzero entries in the first $d' - j + 1$ columns or in rows $i$ and $d + 1$ through $j$. That is, in $d' - j + 1$ columns or $j - d + 1$ rows. Thus

$$\operatorname{rank} A_i \leqslant \inf(j, n - j - 2, d' - d + 2),$$

and

$$\begin{aligned} &\inf(j, d) \leqslant \dim_k V_j \\ (7) \qquad &\qquad \leqslant \inf(j, d \inf(n - j - 2, d' - d + 2)) \qquad (j \leqslant d' < n - 3) \end{aligned}$$

If $d' = n - 4$ this is the same as (5), but if $d' < n - 4$ (7) is an improvement over (5).

Note that if $j = n - 3$ then (5) says that dim $V_j = d$, as is easy to see directly. Perhaps my upper bounds are not the best possible, but they are sufficient to indicate the flavor of what is going on. Any particular case is a problem of linear algebra, assuming that one can work effectively with the $d\alpha_i$. I will conclude this section by giving some explicit examples.

If char $k = 0$, then $\Omega_k$ is a vector space over $k$ of dimension trd$(k/Q)$. If the latter is infinite, then $\dim_k V_j$ is of little interest if one's goal is the computation of $SK_1(A)$. To get interesting examples let $k = Q(X_1, \ldots, X_N)$ where the $X_i$ are independent indeterminants. Let $n \leqslant N + 3$, and $\alpha_4 = X_1, \alpha_5 = X_2, \ldots, \alpha_n = X_{n-3}$. Then $\dim_k V_j = j$ for all $j$, and

$$\dim_k V = \sum_{j=1}^{n-3} j = (n - 2)(n - 3)/2.$$

But $\Omega_k$ is a vector space over $k$ of dimension $N$. Hence $SK_1(A)$ is a $k$-vector space of dimension $N(n - 1)(n - 2)/2 - (n - 2)(n - 3)/2$.

Now suppose $N = 1$, $X = X_1$, and let $\alpha_4 = X$, $\alpha_5 = X^2, \ldots, \alpha_n = X^{n-3}$. Here $d = 1$, and the matrix $M$ is as indicated.

$$M = \begin{vmatrix} 1 & X & X^2 & \cdots & X^{n-3} \\ 1 & X^2 & X^4 & \cdots & X^{(n-3)2} \\ & & \vdots & & \\ 1 & X^j & X^{2j} & \cdots & X^{(n-3)j} \\ 0 & dX & 2XdX & \cdots & (n-3)X^{n-4}dX \\ 0 & XdX & 2X^3dX & & (n-3)X^{(n-3)2-1}dX \\ & & \vdots & & \\ 0 & X^{j-1}dX & 2X^{2j-1}dX & \cdots & (n-3)X^{(n-3)j-1}dX \end{vmatrix}.$$

However $\Omega_k = k$ and $dX$ is a basis of $\Omega_k/k$ so the $j$th component of $SK_1(A)$ is $(n - 2)k$ factored out by the subspace generated by the rows of $M'$. To obtain $M'$ the last $j - 1$ rows have been divided by a suitable power of $X$.

$$M' = \begin{vmatrix} 1 & X & X^2 & \cdots & X^{n-3} \\ 1 & X^2 & X^4 & \cdots & X^{(n-3)2} \\ & & \vdots & & \\ 1 & X^j & X^{2j} & \cdots & X^{(n-3)j} \\ 0 & 1 & 2X & \cdots & (n-3)X^{n-4} \\ 0 & 1 & 2X^2 & \cdots & (n-3)(X^2)^{n-4} \\ & & \vdots & & \\ 0 & 1 & 2X^j & \cdots & (n-3)(X^j)^{n-4} \end{vmatrix}.$$

$M'$ is of rank its smallest dimension, i.e. $\inf(2j, n-2)$. To show this it is sufficient to prove that a square matrix $(2j \times 2j)$ of this type is nonsingular. One way to do this is to consider the functions $\exp(a_1 Y), \exp(a_2 Y), \ldots, \exp(a_j Y)$, $Y\exp(a_1 Y), \ldots, Y\exp(a_j Y)$, (exp denoting the exponential function). For distinct real $a_i$ these functions are a basis for the solutions of a differential equation with constant coefficients of order $2j$. Let $r_1 \exp(a_1 Y) + \cdots + r_{2j} Y\exp(a_j Y)$ be a linear combination of the functions. If we successively differentiate and evaluate at zero we get (with $a_i = X^i$) (the transpose of) the above as coefficient matrix. However no solution of a constant coefficient differential equation of order $2j$ can vanish together with its first $2j-1$ derivatives, at zero. Therefore there can be no nontrivial solution for the $r_i$. Hence the coefficient matrix must be nonsingular, as required. The idea of a purely algebraic proof can be found in [8]. Therefore $\dim V_j = \inf(2j, n-2) - j = \inf(j, n-j-2)$ which is the upper limit allowed by (5) in the case $d = 1$.

On the other hand let $\alpha_4 = X, \alpha_5 = a_5 X, \ldots, \alpha_n = a_n X$ with the $a_i$ distinct rational numbers. In this case similar reduction of the matrix $M$ shows that $\dim V_j = 1$, which is the lower limit allowed by (5).

5. **Further calculations of $K_2(P)$.** Here we consider reduced connected plane curves over a field $k$ such that ($\alpha$) all irreducible components are isomorphic to the affine line, ($\beta$) each intersection point is $k$-rational, ($\gamma$) the tangent directions of all the components through a given intersection point are distinct. According to Theorem 1 it suffices to calculate $K_2(P)$ for each intersection point $P$. If we translate $P$ to the origin and use the description of the conductor given in §2 we see that

$$K_2(P) = \operatorname{coker}\left(SK_2k[X, Y]/(X, Y)^{p-1} \xrightarrow{f} \bigoplus_{i=1}^{p} SK_2k[t_i]/t_i^{p-1}\right).$$

Assume that the slopes of the $p$ components through $P$ are $\infty, 0, 1, \alpha_4, \ldots, \alpha_p$

so that the components through $P$ can be written in parametric form $X = f_1(t)$, $Y = t_1 + g_1(t)$ for the first component, and $X = t_i + f_i(t)$, $Y = \alpha_i t_i + g_i(t)$, $2 \leqslant i \leqslant p$ $(\alpha_2 = 0, \alpha_3 = 1)$. The $f_i, g_i$ contain only terms of degree $\geqslant 2$. We know that $SK_2k[X, Y]/(X, Y)^{p-1}$ is generated by symbols. The map $f$ is induced by the above parametric equations. Thus using Graham's calculation of $SK_2k[t]/t^{p-1}$ and formulas (2) we can obtain generators and relations for $K_2(P)$, as was done in §§3, 4 for straight lines.

The case $p = 3$ is the simplest. Because of the truncation $t_i^2 = 0$ the $f_i, g_i$ are all zero, and $K_2(P)$ is the same as for three straight lines through the origin. Thus we have

THEOREM 4. *Let $X = \operatorname{Spec} A$ be a plane curve satisfying conditions $\alpha$, $\beta$, $\gamma$ at the beginning of* §5, *with at most three irreducible components passing through any point and* char $k > 2$. *Then*

$$SK_1(A) = N\Omega_k \oplus (M - m - n + 1)K_2(k)$$

*where $M$, $m$, $n$ are as defined in Theorem* 1, *and $N$ is the number of points that have three components passing through them.*

I will not write out the case $p = 4$, as the method is illustrated adequately by my example for $p = 5$. This case differs from that of four straight lines through a point in that the generators of image $f$ need not be homogeneous. However the conclusion is the same as in the straight line case, i.e. that $K_2(P) = 3\Omega_k/V$ where $V$ is a one-dimensional $k$-subspace if $d\alpha_4 \neq 0$ and $V = 0$ if $d\alpha_4 = 0$.

Now let $p = 5$. Rather than trying to do the general case I will consider a simplified example to show that phenomena of a different nature indeed can occur. Consider the homomorphism $k[X, Y]/(X, Y)^4 \longrightarrow \Pi_{i=1}^5 k[t_i]/(t_i^4)$ defined by $X \longrightarrow 0$, $Y \longrightarrow t_1$ in the first coordinate, $X \longrightarrow t_2$, $Y \longrightarrow 0$ in the second, and $X \longrightarrow t_i + \lambda_i t_i^2$, $Y \longrightarrow \alpha_i t_i$ for $3 \leqslant i \leqslant 5$, $\alpha_3 = 1$.

As in §3, $SK_2k[X, Y]/(X, Y)^4$ is generated by elements of the form $\{\alpha, \exp aX^pY^q\}$ and $\{\exp aX^pY^q, \exp bX^rY^s\}$ with $\alpha, a, b \in k^*$. The image of $\{\exp aX^pY^q, \exp bX^rY^s\}$ in $K_2k[t_i]/(t_i^4)$ is of the form $\{\exp \alpha t_i^{p+q}, \exp \beta t_i^{r+s}\}$ + symbols involving higher powers of $t_i$. By (2) under the isomorphism $SK_2k[t_i]/(t_i^4) \simeq \Omega_k[t_i]/t_i^3\Omega_k[t_i]$ the element $\{\exp \alpha t_i^{p+q}, \exp \beta t_i^{r+s}\}$ is identified with $t_i^{p+q+r+s-1}((p + q)\alpha\, d\beta - (r + s)\beta\, d\alpha)$. Thus we need only consider $p + q + r + s \leqslant 3$.

If only $Y$ occurs or if $p + q = 3$ in the first case or $p + q + r + s = 3$ in the second then the image is the same as in the straight line case because the $\lambda_i$ disappear due to the truncation. Thus the only new cases where we have to calculate the image are $\{\alpha, \exp aX\}$, $\{\alpha, \exp aX^2\}$, $\{\alpha, \exp aXY\}$, $\{\exp aX, \exp bX\}$ and $\{\exp aX, \exp bY\}$. In each case the calculation is straightforward using (2), so I will give only the result. (As before $t = (t_1, \ldots, t_5)$.)

$$\text{(a)}\quad f\{\alpha, \exp aX\} = [(0, -1, -1, -1, -1) + (0, 0, -2\lambda_3, -2\lambda_4, -2\lambda_5)t]\, a\, d\alpha/\alpha,$$

$$\text{(b)}\quad \begin{aligned} f\{\alpha, \exp aX^2\} = [&(0, -2, -2, -2, -2)t \\ &+ (0, 0, -6\lambda_3, -6\lambda_4, -6\lambda_5)t^2]\, a\, d\alpha/\alpha, \end{aligned}$$

$$\text{(c)}\quad \begin{aligned} f\{\alpha, \exp aXY\} = [&(0, 0, -2, -2\alpha_4, -2\alpha_5)t \\ &+ (0, 0, -3\lambda_3, -3\lambda_4\alpha_4, -3\lambda_5\alpha_5)t^2]\, a\, d\alpha/\alpha, \end{aligned}$$

$$\text{(d)}\quad \begin{aligned} f\{\exp aX, \exp bX\} = [&(0, 1, 1, 1, 1)t \\ &+ (0, 0, 3\lambda_3, 3\lambda_4, 3\lambda_5)t^2]\,(adb - bda), \end{aligned}$$

$$\text{(e)}\quad \begin{aligned} f\{\exp aX, \exp bY\} = {}&(0, 0, 1, \alpha_4, \alpha_5)(adb - bda)t + (0, 0, 0, d\alpha_4, d\alpha_5)abt \\ &+ (0, 0, \lambda_3, \alpha_4\lambda_4, \alpha_5\lambda_5)(2adb - bda)t^2 \\ &+ (0, 0, -d\lambda_3, 2\lambda_4 d\alpha_4 - \alpha_4 d\lambda_4, 2\lambda_5 d\alpha_5 - \alpha_5 d\lambda_5)abt^2. \end{aligned}$$

Note that (d) is a sum of expressions of type (b) so we need not consider (d) any further. However

$$\begin{aligned} f\{\exp aX, \exp bY\} = {}&(0, 0, 1, \alpha_4, \alpha_5)(adb - bda)t \\ &+ 1.5(0, 0, \lambda_3, \alpha_4\lambda_4, \alpha_5\lambda_5)(adb - bda)t^2 \\ &+ 0.5(0, 0, \lambda_3, \alpha_4\lambda_4, \alpha_5\lambda_5)d(ab)t^2 \\ &+ (0, 0, 0, d\alpha_4, d\alpha_5)abt \\ &+ (0, 0, -d\lambda_3, 2\lambda_4 d\alpha_4 - \alpha_4 d\lambda_4, 2\lambda_5 d\alpha_5 - \alpha_5 d\lambda_5)abt^2. \end{aligned}$$

The first two terms are the sum of expressions of type (c), so we are left with elements of the form:

$$\text{(f)}\quad \begin{aligned} &(0, 0, 0, d\alpha_4, d\alpha_5)abt \\ &\quad + (0, 0, -d\lambda_3, 2\lambda_4 d\alpha_4 - \alpha_4 d\lambda_4, 2\lambda_5 d\alpha_5 - \alpha_5 d\lambda_5)abt^2 \\ &\quad + 0.5(0, 0, \lambda_3, \alpha_4\lambda_4, \alpha_5\lambda_5)t^2\, d(ab). \end{aligned}$$

In addition to the generators (f) we have obtained as generators linear combinations of the rows of the matrix $M_2$, where all rows except the last two can be multiplied by $c \in \Omega_k$, $c$ arbitrary, and the last two rows can be multiplied by $\lambda \in k$, $\lambda$ arbitrary.

$$M_2 = \begin{vmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 2\lambda_3 & 2\lambda_4 & 2\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & \alpha_4 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 3\lambda_3 & 3\lambda_4 & 3\lambda_5 \\
 & & & & & 0 & 0 & 1 & \alpha_4 & \alpha_5 & 0 & 0 & 1.5\lambda_3 & 1.5\alpha_4\lambda_4 & 1.5\alpha_5\lambda_5 \\
 & & & & & 1 & 0 & 1 & \alpha_4^2 & \alpha_5^2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & 0 & 0 & 1 & \alpha_4 & \alpha_5 \\
 & & & & & & & & & & 0 & 0 & 1 & \alpha_4^2 & \alpha_5^2 \\
 & & & & & & & & & & 1 & 0 & 1 & \alpha_4^3 & \alpha_5^3 \\
 & & & & & & & & & & 0 & 0 & 0 & d\alpha_4 & d\alpha_5 \\
 & & & & & & & & & & 0 & 0 & 0 & \alpha_4 d\alpha_4 & \alpha_5 d\alpha_5
\end{vmatrix}.$$

(The first five columns are of degree 0 in $t$, the second five are of degree 1, and the last five are of degree 2. Within each block of five, the components are in order.) Omitted entries are zero. The first row is (a), the second comes from $\{\alpha, \exp aY\}$. The next two are (b) and (c), and the fifth is $f\{\alpha, \exp aY^2\}$. The last six come from $\{\alpha, \exp X^p Y^q\}$ $(p + q = 3)$ and $\{\exp X^p Y^q, \exp X^r Y^s\}$ (as in §4 for five straight lines through one point).

Now consider a particular case. Suppose that $\lambda_3, \lambda_4, \lambda_5, \alpha_4, \alpha_5$ are all rational (i.e. $d\lambda_i = 0$, $d\alpha_i = 0$). Then (f) consists of all elements of the form $(0, 0, 0.5\lambda_3, 0.5\alpha_4\lambda_4, 0.5\alpha_5\lambda_5)t^2 d(ab)$. If the matrix

$$\begin{vmatrix}
1 & \alpha_4 & \alpha_5 \\
1 & \alpha_4^2 & \alpha_5^2 \\
\lambda_3 & \alpha_4\lambda_4 & \alpha_5\lambda_5
\end{vmatrix}$$

is nonsingular, then $K_2(P) \simeq 3\Omega_k \oplus 2\Omega_k \oplus \Omega_k/dk$. Here $K_2(P)$ is not even a $k$-vector space, as it was in all previously considered examples. If the matrix is singular we get $6\Omega_k$.

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