TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 222, 1976

ON KOLMOGOROV'S INEQUALITIES $\|\tilde{f}\|_p \leq C_p \|f\|_1,\ 0 < p < 1$

BY

BURGESS DAVIS[1]

ABSTRACT. Let μ be a signed measure on the unit circle A of the complex plane satisfying $|\mu|(A) < \infty$, where $|\mu|(A)$ is the total variation of μ, and let $\tilde{\mu}$ be the conjugate function of μ. A theorem of Kolmogorov states that for each real number p between 0 and 1 there is an absolute constant C_p such that $((2\pi)^{-1}\int_0^{2\pi}|\tilde{\mu}(e^{i\theta})|^p\,d\theta)^{1/p} \leq C_p|\mu|(A)$. Here it is shown that measures putting equal and opposite mass at points directly opposite from each other on the unit circle, and no mass any place else, are extremal for all of these inequalities, that is, if ν is one of these measures the number $((2\pi)^{-1}\int_0^{2\pi}|\tilde{\nu}(e^{i\theta})|^p\,d\theta)^{1/p}/|\nu|(A)$ is the smallest possible value for C_p. These constants are also the best possible in the analogous Hilbert transform inequalities. The proof is based on probability theory.

1. **Introduction.** Let μ be a signed measure on the unit circle $A = \{e^{i\theta}: 0 \leq \theta < 2\pi\}$ of the complex plane satisfying $|\mu|(A) < \infty$, where $|\mu|(A)$ is the total variation of μ. Let $\tilde{\mu}$ be the conjugate function of μ, that is

$$\tilde{\mu}(e^{i\theta}) = \lim_{r\to 1} \int_0^{2\pi} \operatorname{Im}\left(\frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} - re^{i\theta}}\right) d\mu(e^{i\varphi}), \quad 0 \leq \theta < 2\pi,$$

and define

$$\|\tilde{\mu}\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |\tilde{\mu}(e^{i\theta})|^p\,d\theta\right)^{1/p}.$$

A theorem of Kolmogorov [7, Vol. 1, p. 260] states that for each real number p between 0 and 1 there is an absolute constant C_p such that

$$\|\tilde{\mu}\|_p \leq C_p|\mu|(A). \tag{1.1}$$

Here it is shown that the smallest possible value for C_p is $\|\tilde{\nu}\|_p$, where ν is the measure given by $\nu(1) = ½$, $\nu(-1) = -½$, and $|\nu|\{z \in A: z \neq \pm 1\} = 0$. That

Received by the editors May 2, 1975.

AMS (MOS) subject classifications (1970). Primary 42A40; Secondary 43A46, 60J65.

Key words and phrases. Conjugate function, Brownian motion, Kolmogorov's inequalities.

(1) Supported by a NSF Grant.

is, the measure ν is extremal for all the inequalities (1.1). The constant $\|\nu\|_p$ is also smallest possible in the analogous Hilbert transform inequalities. That it is an upper bound can be shown by an easy modification of the proof in Zygmund [7, Vol. 2, p. 256] of a similar result involving the M. Riesz inequalities, and that it is a lower bound can be shown by examples similar to those of [2, §4].

Pichorides has shown [6] that the best possible value for C_p in order that (1.1) hold for all nonnegative measures is $\|\tilde{\lambda}\|_p$, where λ is a nonnegative measure putting mass one on a single point and no mass anywhere else.

The approach used here is to solve an optimal stopping problem for two dimensional Brownian motion and then apply this solution to the conjugate function inequalities, the same method used in [2], where the weak type inequality for conjugate functions is considered. This is discussed more thoroughly in the next section.

If an equation or inequality holds with probability one but not at all points of the underlying probability space, which usually happens in the following because it involves a conditional expectation or distribution which is only defined up to sets of measure 0, this will not always be explicitly mentioned.

2. **Preliminaries.** Denote by m the measure $d\theta/2\pi$ on A, and let μ be a totally finite signed measure on A. Let $D = \{z: |z| < 1\}$. Define $f_\mu = f$ on $D \cup A$ by

$$f_\mu(z) = \int_A \left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) d\mu(e^{i\theta}), \qquad z \in D,$$

and

$$f_\mu(e^{i\theta}) = \frac{d\mu}{dm}(e^{i\theta}) + i\tilde{\mu}(e^{i\theta}) \quad \text{a.e. } (m),$$

where by $d\mu/dm$ is meant the Radon-Nikodým derivative of that part of μ which is absolutely continuous with respect to m. The following formulas hold (see [7]).

$$\int_0^{2\pi} |\mathrm{Re}\, f(re^{i\theta})|\, dm(e^{i\theta}) \nearrow |\mu|(A) \quad \text{as } r \nearrow 1. \tag{2.1}$$

$$f(0) = \mu(A). \tag{2.2}$$

Now let Z_t, $0 \leqslant t < \infty$, be standard two dimensional complex Brownian motion started at 0, and let $\tau_D = \inf\{t > 0: |Z_t| = 1\}$. It is a result of P. Lévy (see [5, p. 109]) that $f(Z_t)$, $0 \leqslant t \leqslant \tau_D$, is a two dimensional Brownian motion with a time change. More precisely, if $\gamma(s)$ is defined by

$$\gamma(s) = \int_0^s |f'(Z_t)|^2\, dt,$$

then γ is almost surely strictly increasing on $[0, \tau_D]$ and if $B_t^\mu = B_t$ is defined by

$$B_{\gamma(t)} = f(Z_t),$$

then B_s, $0 \leqslant s \leqslant e = \gamma(\tau_D)$, is a standard complex Brownian motion started at $f(0) = \mu(A)$ up to the time $e = e(\mu)$. For convenience define $B_{s+e} - B_e = Z_{s+\tau_D} - Z_{\tau_D}$, $s > e$, so that now B_t, $0 \leqslant t < \infty$, is standard complex Brownian motion. If f is a univalent map of D to some region R of the complex plane then e is the first time B_t leaves R and is thus a stopping time. In general, it is not clear that e is a stopping time for B_s, but in view of the fact that it is the stopping time τ_D for the preimage process Z_t, in manipulations involving the Strong Markov Property e can be treated as if it is a stopping time. For the rest of this section times for which the Strong Markov Property holds will be called quasi-stopping times.

If Γ is any subarc of A, $P(Z_{\tau_D} \in \Gamma) = m(\Gamma)$, so that

$$\|\tilde{\mu}\|_p^p = E|\operatorname{Im} f(Z_{\tau_D})|^p = E|\operatorname{Im} B_e|^p. \tag{2.3}$$

Furthermore,

$$|\mu|(A) = \lim_{t\to\infty} E|\operatorname{Re} B_{\min(e,t)}|. \tag{2.4}$$

For let $\tau_{rD} = \inf\{t > 0\colon |Z_t| = r\}$, and let $e_r = \gamma(\tau_{rD})$. Then

$$E|\operatorname{Re} B_{e_r}| = \int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|\, dm(e^{i\theta}).$$

Since f is bounded in $\{z\colon |z| \leqslant r\}$, $0 < r < 1$, $\operatorname{Re} B_{\min(e_r,t)}$, $0 \leqslant t < \infty$, is a bounded martingale so that

$$E|\operatorname{Re} B_{\min(e_r,t)}| \nearrow E|\operatorname{Re} B_{e_r}| \quad \text{as } t \nearrow \infty. \tag{2.5}$$

Also, for fixed t, $\operatorname{Re} B_{\min(e,s)}$, $0 \leqslant s \leqslant t$, is an L^2 bounded martingale so that $e_r \nearrow e$ a.s. as $r \nearrow 1$ implies

$$E|\operatorname{Re} B_{\min(e_r,t)}| \nearrow E|\operatorname{Re} B_{\min(e,t)}| \quad \text{as } r \nearrow 1. \tag{2.6}$$

Since $E|\operatorname{Re} B_{\min(e,t)}|$ is nondecreasing as $t \longrightarrow \infty$, (2.5), (2.6), and (2.1) imply (2.4).

Let ν be the measure described in the second paragraph of the introduction. Then

$$f_\nu(z) = \frac{1}{2}\left[\frac{1+z}{1-z} + \frac{z-1}{z+1}\right] = \frac{2z}{1-z^2}$$

is a one-to-one map of D onto the doubly slit plane $\{x + iy: x \neq 0$ or $x = 0, -1 < y < 1\} = SP$. Thus e_ν is the first time that B_s^ν leaves SP. In general, let τ_{SP} be the first time a Brownian motion leaves SP. Then, by (2.4),

$$\lim_{t\to\infty} E|Z_{\min(t,\tau_{SP})}| = \lim_{s\to\infty} E|B_{\min(s,e_\nu)}| = |\nu|(A) = 1. \tag{2.7}$$

The central result of this paper can be stated as follows.

THEOREM 2.1. *If μ is a signed measure on A satisfying $|\mu|(A) = 1$ then $\|\tilde{\nu}\|_p^p \geq \|\tilde{\mu}\|_p^p$, $0 < p < 1$.*

In view of the preceding discussion the following theorem implies Theorem 2.1. It is noted that $|\mu|(A) = 1$ implies $-1 \leq \mu(A) \leq 1$ and that $f_\mu(0) = \mu(A)$ is the starting point of B_t^μ.

THEOREM 2.2. *Let $Z_t = X_t + iY_t$ be standard two dimensional Brownian motion and let P_a and E_a denote probability and expectation associated with Z_t given $P(Z_0 = a) = 1$. Let r be a real number, $-1 \leq r \leq 1$. If e is any quasi-stopping time such that $\lim_{t\to\infty} E_r|X_{\min(t,e)}| = 1$ then*

$$E_r|Y_e|^p \leq E_0|Y_{\tau_{SP}}|^p, \qquad 0 < p < 1. \tag{2.8}$$

Since it is easy to show that Theorem 2.2 need only be proved for times e satisfying $X_e \equiv 0$, this will be done now.

PROPOSITION. *Let e be a quasi-stopping time. Let $e' = \inf\{t \geq e: X_t = 0\}$. Then*

$$\lim_{t\to\infty} E_r|X_{\min(t,e)}| = \lim_{t\to\infty} E_r|X_{\min(t,e')}|,$$

and $E_r|Y_{e'}|^p \geq E_r|Y_e|^p$, $0 < p < 1$.

PROOF. Let γ be the first time Brownian motion hits the imaginary axis I. By a result of Kakutani (see [3]), if $h(z)$ is defined by $h(z) = E_z|Y_\gamma|^p$, $\operatorname{Re} z \geq 0$, then $h(z)$ is the minimal positive harmonic function defined on $\operatorname{Re} z > 0$ with boundary values $|y|^p$ at iy, that is, the Poisson integral of these boundary values. This function is easily calculated, since it is $(1 + \cos p\pi)^{-1}$ times the sum of the real part of the principal branch of $\sqrt[p]{iz}$ and the real part of the branch of $\sqrt[p]{-iz}$ for which $-2\pi < \arg z \leq 0$.

We have

$$h(z) = |z|^p\ [\cos p\phi + \cos p(\pi - \phi)]/(1 + \cos p\pi),$$

where ϕ is the angle between z and the negative imaginary axis, $0 \leq \phi \leq \pi$. Extend h to the entire plane by making it symmetric around I. It is not difficult to show that $h(z) \geqslant |z|^p$. It suffices to prove this only on $\{|z| = 1, \operatorname{Re} z \geqslant 0\}$, where

$$\partial h(\phi)/\partial\phi = p(\sin p(\pi - \phi) - \sin p\phi)/(1 + \cos p\pi),$$

which is positive if $0 < \phi < \pi/2$ and negative if $\pi/2 < \phi < \pi$. Together with the fact that $h(1) = h(-1) = 1$, this shows $h(z) \geqslant 1$ if $|z| = 1$. Thus, if e is as in the statement of Theorem 2.2, and $e' = \inf\{t \geqslant e\colon Z_t \in \Gamma\}$,

$$\begin{aligned} E_r|Y_{e'}|^p &= E_rE(|Y_{e'}|^p|Z_t, 0 \leqslant t \leqslant e) = E_rE_{Z_e}|Y_\gamma|^p = E_rh(Z_e) \\ &\geqslant E_r|Z_e|^p \geqslant E_r|Y_e|^p. \end{aligned}$$

Also, $\lim_{t\to\infty}E_r|X_{\min(t,e)}| = \lim_{t\to\infty}E_r|X_{\min(t,e')}|$ since a martingale increases in L^1 norm only when it crosses 0, and X_s and X_e do not have different signs if $e \leqslant s \leqslant e'$. (More formally, $E(X_{\min(t,e')}|X_{\min(t,e)}) = X_{\min(t,e)}$ a.s. since $X_{\min(s,e)}$, $0 \leqslant s \leqslant t$, is an L^2 bounded martingale. Since $X_{\min(t,e')}$ and $X_{\min(t,e)}$ are of the same sign, $E(|X_{\min(t,e')}|\,|X_{\min(t,e)}) = |X_{\min(t,e)}|$ a.s.) Thus, Proposition 2.1 holds.

3. **A discrete time stopping problem.** The functions p and F, the random variables X_i, $i \geqslant 1$, and the σ-fields $\mathcal{F}_i$, $i \geqslant 1$, will be defined in §4, but for now some of their properties will be given which will enable them to be dealt with. The function $p(s)$, $-\infty < s < \infty$, is an even and nonvanishing probability density function satisfying $p(s) < p(t)$ if $|s| > |t|$, and $F(s)$ is a continuous positive even function on the reals satisfying $\lim_{t\to\infty}F(t) = 0$ and $F(s) < F(t)$ if $|s| > |t|$. The random variables X_i, $i \geqslant 1$, are independent and identically distributed, each with density $p(s)$, and we put $S_0 = 0$, $S_n = \Sigma_{i=1}^n X_i$. The sequence of σ-fields $\mathcal{F}_i$, $i \geqslant 1$, satisfies $\mathcal{F}_i \subset \mathcal{F}_{i+1}$, and $(X_1, \ldots, X_n)$ is $\mathcal{F}_n$ measurable. Furthermore $(X_{n+k}, k \geqslant 1)$ is independent of $\mathcal{F}_n$. By stopping time in this section will be meant any positive integer valued random variable N such that $\{N = k\} \in \mathcal{F}_k$ for each k. If N is a stopping time define

$$AN = \sum_{k=0}^{\infty} E[F(S_k)I(N > k)],$$

where I stands for the indicator function of the set displayed.

If r is a nonnegative real number define the stopping time N_r by

$$N_r = \inf\{k\colon |S_k| \geqslant r\}.$$

Since $P(N_r = k + 1|N_r > k) \geqslant P(|X_1| > 2r) > 0$, we have

$$P(N_r > k) \leqslant [1 - P(|X_1| > 2r)]^k$$

which gives that $EN_r < \infty$ for each r and also that $EN_r \longrightarrow 0$ as $r \longrightarrow 0$ since $P(|X_1| > 2r) \longrightarrow 0$ as $r \longrightarrow 0$. Furthermore, since each S_n has a continuous distribution, $\lim_{t\to s}N_t = N_s$ a.e., and since $N_a \leqslant N_b$ if $a \leqslant b$, the dominated convergence theorem gives $\lim_{t\to s} EN_t = EN_s$. (The dominating variable can be taken to be N_{2s}.) Clearly, $\lim_{t\to\infty} EN_t = \infty$. Finally note that $P(N_a < N_b) > 0$ if $a < b$ since X_1 has a nonvanishing density. Thus EN_t increases continuously from 0 to ∞ as t increases from 0 to ∞.

If X is any continuous random variable and B is a measurable set we define the density of $XI(B)$ to be a function f on the reals satisfying $P(\{a < X < b\} \cap B) = \int_a^b f(t)\,dt$. Thus, if T is a stopping time and if h_i is the density of $S_i I(T > i)$, an alternate formula for AT is

$$AT = F(0) + \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} F(t) h_i(t)\,dt. \tag{3.1}$$

The following lemma is true under only the conditions on p and F given in the first paragraph of this section but is much more easily proved for the application we have in mind. Its proof will be given in §4.

LEMMA 3.0. *Let α be a positive real number. There exists a real valued function $\varphi_\alpha(n)$, $n \geq 1$, which decreases to 0 as $n \to \infty$ such that if N is any stopping time satisfying $EN = \alpha$ and $h_i(t)$ is the density of $S_i I(N > i)$ then*

$$\sum_{i=n}^{\infty} \int_{-\infty}^{\infty} F(t) h_i(t)\,dt < \varphi_a(n).$$

The main result of this section is

THEOREM 3.1. *Let M be a stopping time such that $EM < \infty$. Let r be that number satisfying $EN_r = EM$. Then $AN_r \geq AM$.*

The proof of Theorem 3.1 proceeds via a sequence of lemmas. If f and g are any two positive functions write $f \mathrel{\tilde{\leqslant}} g$ if $\int_{-\delta}^{\delta} f(t)\,dt \leq \int_{-\delta}^{\delta} g(t)\,dt$ for each $\delta > 0$ and $f \mathrel{\tilde{<}} g$ if $f \mathrel{\tilde{\leqslant}} g$ and there is strict inequality between the integrals for some $\delta > 0$. The following two lemmas follow from a standard integration by parts argument, using the fact that $p(s)$ and $F(s)$ are even functions which decrease on $(0, \infty)$. The proofs are not given. The symbol $*$ stands for convolution.

LEMMA 3.1. *If f and g are nonnegative functions and $f \mathrel{\tilde{\leqslant}} g$ and $\int_{-\infty}^{\infty} g(t)\,dt < \infty$ then*

(i) $\int_{-\infty}^{\infty} f(t)F(t)\,dt \leq \int_{-\infty}^{\infty} g(t)F(t)\,dt$.

(ii) $f * p \mathrel{\tilde{\leqslant}} g * p$.

If in addition $f \mathrel{\tilde{<}} g$ then

(iii) $\int_{-\infty}^{\infty} f(t)F(t)\,dt < \int_{-\infty}^{\infty} g(t)F(t)\,dt$.

The proof of the next lemma is also omitted.

LEMMA 3.2. *If f and g are nonnegative functions and $f \mathrel{\tilde{\leqslant}} g$ and if a satisfies*

$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-a}^{a} g(t)\,dt$ then

$$f \stackrel{\sim}{\leqslant} gI[-a, a].$$

Now regard the time M of Theorem 2.1 as fixed. There is a sequence a_1, $a_2, \ldots$ of real numbers in $[0, \infty]$ such that if $K = \inf\{k\colon |S_k| \geqslant a_k\}$ then $P(K > n) = P(M > n)$. The a_i are chosen recursively as follows. Let a_1 satisfy

$$\int_{-a_1}^{a_1} p(s)\,ds = P(M > 1). \tag{3.2}$$

Now if p_2 is the density of $S_2 I(S_1 < a_1)$, define a_2 by $\int_{-a_2}^{a_2} p_2(s)\,ds = P(M > 2)$, and so on. Since $\int_{-\infty}^{\infty} p_2(s)\,ds = \int_{-a_1}^{a_1} p(s)\,ds = P(M > 1) \geqslant P(M > 2)$, such a choice is possible, and this is the unique a_2 that gives $P(M > 2) = P(K > 2)$ given that a_1 is chosen by (3.2), unless a_1 was 0, since p_2 is nonvanishing. The succeeding a_i are chosen in the same manner, and are unique up to the first a_i which is 0, an event which happens only if $P(M > n) = 0$ for some integer n.

LEMMA 3.3. *If K is defined as above then $EK = EM$ and $AK \geqslant AM$.*

PROOF. Let γ_i be the density of $S_i I(M > i)$ and λ_i be the density of $S_i I(K > i)$. Then $\gamma_1 \stackrel{\sim}{\leqslant} \lambda_1$ by Lemma 3.2, where here p plays the role of g in that lemma and γ_1 the role of f. Thus, by Lemma 3.1(ii), $\gamma_1 * p \stackrel{\sim}{\leqslant} \lambda_1 * p$, and again by Lemma 3.2, $\gamma_2 \stackrel{\sim}{\leqslant} \lambda_2$, here with $\lambda_1 * p$ playing the role of g in that lemma and γ_2 the role of f. (Note $\gamma_2 \stackrel{\sim}{\leqslant} \gamma_1 * p \stackrel{\sim}{\leqslant} \lambda_1 * p$, the first $\stackrel{\sim}{\leqslant}$ since $\{K > 2\} \subset \{K > 1\}$.) Continuing in this manner we get $\gamma_i \stackrel{\sim}{\leqslant} \lambda_i$ for each i, and Lemma 3.3 follows from formula (3.1) and Lemma 3.1(i).

LEMMA 3.4. *Let $\theta = \sup AT$, where the supremum is taken over all stopping times T satisfying $ET = EM$. Then there is a sequence of real numbers a_1, $a_2, \ldots$ such that if $Q = \inf\{n\colon |S_n| \geqslant a_n\}$ then $EQ = EM$ and $AQ = \theta$.*

PROOF. Let $T_i = \inf\{k\colon |S_k| \geqslant a_{ik}\}$, $i = 1, 2, \ldots$, satisfy $ET_i = EM$ and $\lim_{i\to\infty} AT_i = \theta$. That such T_i exist is a consequence of Lemma 3.3. We can and do assume $\lim_{i\to\infty} a_{ik} = a_k$ exists (it may be $+\infty$) since if this is not the case diagonalization can be used. Define Q as in the statement of the lemma. Then clearly $\lim_{i\to\infty} P(T_i > k) = P(Q > k)$, $k = 1, 2, \ldots$, since the distributions of S_n are continuous, so that $EQ \leqslant \lim_{i\to\infty} ET_i = \lim EM = EM$. Also, if $h_{i,k}$ is the density of $S_k I(T_i > k)$, and h_k is the distribution of $S_k I(Q > k)$, the boundedness of F and the continuity of the distributions of S_k give

$$\lim_{i\to\infty} \int_{-\infty}^{\infty} h_{i,k}(t) F(t)\,dt = \int_{-\infty}^{\infty} h_k(t) F(t)\,dt.$$

Then, using Lemma 3.0, for fixed n,

$$AQ \geqslant F(0) + \sum_{k=1}^{n} \int_{-\infty}^{\infty} h_k(t)F(t)\,dt$$

$$= F(0) + \lim_{i\to\infty} \sum_{k=1}^{n} \int_{-\infty}^{\infty} h_{i,k}(t)F(t)\,dt$$

$$= \lim_{i\to\infty} \left[AT_i - \sum_{k=n+1}^{\infty} \int_{-\infty}^{\infty} h_{i,k}(t)F(t)\,dt \right]$$

$$\geqslant \theta - \varphi_{EM}(n+1),$$

implying $AQ \geqslant \theta$ so that in fact $AQ = \theta$ and thus $EQ = EM$, for if $EQ < EM$ we could make Q bigger in any manner whatsoever to get a Q' satisfying $EQ' = EM$. But then, since F is nonvanishing and $Q' \geqslant Q$, $AQ' > AQ = \theta$, contradicting the definition of θ.

The proof of Theorem 3.1 will now be completed by showing that $N_r = Q$, where Q is defined in Lemma 3.4. This will be done by showing that if any of the a_i which define Q are not r then there exists a time $\widetilde{Q}$ satisfying $E\widetilde{Q} = EQ$ and $A\widetilde{Q} > AQ = \theta$, a contradiction to the definition of θ. Let γ be the first index such that $a_\gamma \neq r$, and suppose first that $a_\gamma < r$. Let λ be the first index such that $a_\lambda > r$. Such a λ must exist since $EQ = EN_r$. Let

$$p = P(a_\gamma < S_\gamma < r,\ Q > \gamma - 1)$$

and pick $0 < \epsilon_\lambda < a_\lambda$ to satisfy $0 < P(r < S_\lambda < \epsilon_\lambda,\ Q > \lambda) \leqslant p$. If $k \geqslant 0$ let $g_{\lambda+k}$ be the density of $S_{\lambda+k} I(r < S_\lambda < \epsilon_\lambda,\ Q > \lambda + k)$. Let $\hat{Q}$ be Q unless $Q > \lambda$ and $r < S_\lambda < \epsilon_\lambda$, in which case let $\hat{Q} = \lambda$. Then

$$E\hat{Q} = EQ - \sum_{i=\lambda}^{\infty} \int_{-\infty}^{\infty} g_i, \quad \text{and} \quad A\hat{Q} = AQ - \sum_{i=\lambda}^{\infty} \int_{-\infty}^{\infty} g_i(t)F(t)\,dt.$$

Define δ_γ to satisfy $P(D_0) = \int_{-\infty}^{\infty} g_\lambda$, where $D_0 = \{a_\gamma < S_\gamma < \delta_\gamma, Q > \gamma - 1\}$. Note $d_\gamma \leqslant r$. Let $\delta_{\gamma+1}, \delta_{\gamma+2}, \ldots$ satisfy $P(D_k) = \int_{-\infty}^{\infty} g_{\lambda+k}$, where

$$D_k = \{a_\gamma < S_\gamma < \delta_\gamma, \hat{Q} > \gamma - 1, |S_{\gamma+1}| < \delta_{\gamma+1}, \ldots, |S_{\gamma+k}| < \delta_{\gamma+k}\}.$$

Note that if $\hat{Q}$ is replaced by Q in this definition D_k remains the same. Let $f_{\gamma+k}$ be the density of $S_k I(D_k)$, so that $\int_{-\infty}^{\infty} f_{\gamma+k} = \int_{-\infty}^{\infty} g_{\lambda+k} = k \geqslant 0$. Clearly $g_0 \lesssim f_0$, and thus by arguments similar to those used in the proof of Lemma 3.3, $g_{\lambda+k} \lessapprox f_{\gamma+k}$, $k \geqslant 1$. Define $\widetilde{Q} = \hat{Q}$ except on D_0, and on D_0 let

$$\widetilde{Q} = \gamma + k \quad \text{if } |S_{\gamma+1}| < \delta_{\gamma+1}, \ldots, |S_{\gamma+k-1}| < \delta_{\gamma+k-1}, |S_{\gamma+k}| \geqslant \delta_{\gamma+k}.$$

Then

$$E\tilde{Q} = E\hat{Q} + \sum_{i=\gamma}^{\infty} \int_{-\infty}^{\infty} f_i = EQ,$$

and

$$A\tilde{Q} = A\hat{Q} + \sum_{i=\gamma}^{\infty} \int_{-\infty}^{\infty} f_i(t)F(t)\,dt$$

$$> A\hat{Q} + \sum_{i=\lambda}^{\infty} \int_{-\infty}^{\infty} g_i(t)F(t)\,dt = AQ,$$

by Lemma 3.1(i) and (iii).

The other possibility, $a_\gamma > r$, can be handled by a similar construction, which will not be explicitly given. A new stopping time $\hat{Q}$ is constructed which stops with positive probability on $\{Q > \gamma - 1, a_\gamma < |S_\gamma| < r\}$, but is otherwise the same as Q, and then $\tilde{Q}$ is constructed which is the same as $\hat{Q}$ except that it continues with positive probability on $\{\hat{Q} = \lambda, a_\lambda < S_\lambda < r\}$. The details of the construction are very similar to the preceding and as before we get $A\tilde{Q} > AQ$ and $E\tilde{Q} = EQ$.

4. **Application of Theorem 3.1.** In this and the next section the proof of Theorem 2.2 will be completed. To avoid some notation this will be carried out in full only for times e which are stopping times, and not for the slightly more general quasi-stopping times. Since, as already noted, quasi-stopping times can be manipulated as if they were stopping times, this approach does not entail an essential loss of generality. The exponent p will always satisfy $0 < p < 1$. Other notation will be the same as in the statement of Theorem 2.2. E_0 and P_0 will be shortened to E and P. The number δ will satisfy $0 < \delta < 1$ and will be fixed in this section, although later it will be allowed to vary. Class J will be the class of those stopping times T for Z_t which satisfy

(a) $P(T > 0) = 1$.

(b) $P(X_T = 0) = 1$.

(c) There is an $s < T$ such that $P(|X_s| = \delta$ and $|X_t| \neq 0, s \leqslant t < T) = 1$.

Another way to describe class J is the following.

Define $\nu_0 = 0$, and, if $i \geqslant 1$,

$$\mu_i = \inf\{t > \nu_{i-1}: |X_t| = \delta\}, \quad \nu_i = \inf\{t > \mu_i: X_t = 0\}.$$

Then T is in class J if and only if $\Sigma_{i=1}^{\infty} P(T = \nu_i) = 1$. For each real number $s > 0$ let $\beta(s)$ be that stopping time in class J given by $\beta(s) = \inf\{\nu_i: |Y_{\nu_i}| \geqslant s\}$. It will be shown that $\lim_{t\to\infty} E|X_{\min(\beta(s),t)}|$ is a continuous function of s which is strictly increasing from 0 to ∞. This will follow from Lemma 4.1 and the remarks made at the beginning of the last section. Let β stand for that $\beta(s)$ for which the value of this function is 1. The main result of this section is

THEOREM 4.1. *Let $e \in J$ and suppose* $\lim_{t\to\infty} E|X_{\min(e,t)}| = 1$. *Then* $E|Y_e|^p \leqslant E|Y_\beta|^p$.

Before stating the following lemma we remark that inequality (4.1), with a different constant in place of 1.348, can be proved by noting that X_t, $0 \leqslant t \leqslant T$, and Y_t, $0 \leqslant t \leqslant T$, are both martingales which have square function $\sqrt{T}$ under P, and then using the continuous analog of Theorem 6 of [1].

LEMMA 4.1. *Let T be a stopping time for $Z_t = X_t + iY_t$. Let $Y_T^* = \sup_{0\leqslant t\leqslant T} |Y_t|$. There exist positive constants K_p, which do not depend on T, such that* $E(Y_T^*)^p \leqslant K_p \lim_{t\to\infty} E|X_{\min(t,T)}|$.

PROOF. The arguments of [2] imply

$$\lambda P(Y_T^* > \lambda) \leqslant 1.348 \lim_{t\to\infty} E|X_{\min(t,T)}| = w. \tag{4.1}$$

Thus $EY_T^{*p} = \int_0^\infty P(Y_T^{*p} > t)\,dt \leqslant w^p + \int_{w^p}^\infty wt^{-1/p}\,dt = w^p + pw^p/(1-p)$, the desired result.

Now abbreviate μ_1 and ν_1 to μ and ν. Let $p(s)$ be the density under P of Y_ν. Then $p(s) = a * b$, where a is the density of Y_μ and b is the density under P_δ of Y_γ, where γ remains $\inf\{t > 0\colon X_t = 0\}$. Both a and b are the densities of harmonic measures whose exact form is known, and it is easily proved that the p of this section has all the properties claimed for p in §3.

Define $F(t) = E_{it}|Y_\nu|^p - |t|^p$, $-\infty < t < \infty$. This F also has all the properties of the F of §3, which will now be verified. Let $h(z)$ be defined as in §2, and let $\partial h(it)/\partial n$ be the derivative of h in the normal direction to I. This value will be the same as the derivative of h along a circle centered at 0 of radius t and we have

$$\frac{\partial h}{\partial n}(it) = |t|^{p-1}\, p \sin p\pi/(1 + \cos p\pi). \tag{4.2}$$

Since this is positive, h is subharmonic at all nonzero points of I and since $h(0) = 0$, $h(z) > 0$ if $z \neq 0$, h is subharmonic at 0 also.

Let S be the vertical strip $\{x + iy\colon -\delta \leqslant x \leqslant \delta\}$, and let $u(z)$ be the smallest positive function harmonic in the interior of S and continuous on S satisfying $h(z) = u(z)$ on the boundary of S. Then

$$E_{it}|Y_\nu|^p = E_{it}(|Y_\nu|^p|Z_\mu) = E_{it}E_{Z_\mu}|Y_\gamma|^p = E_{it}Eh(Z_\mu) = u(it).$$

Thus, if $g(z) = u(z) - h(z)$, $z \in S$, we have $F(t) = g(it)$. It is easily checked that g is a pure potential since it is superharmonic in S, vanishes at the boundary of

S, and does not grow too fast at infinity. Since g is harmonic inside S except on I, the Riesz mass associated with g is concentrated on I, and it is known that it is proportional to $\partial h/\partial n$ (see [4]). Thus, if $G(z, z_0)$ is the Green function for S, we have that there are positive constants B_p such that

$$g(z) = B_p \int_{-\infty}^{\infty} |t|^{p-1} G(z, it)\, dt. \tag{4.3}$$

Now $G(it, iy)$ is a continuous function of $|t-y|$ which decreases strictly and exponentially to 0 as $|t-y|$ approaches infinity. Thus, considered as a function of y, $G(it, \cdot) \precsim G(is, \cdot)$ if $|t| > |s|$, and thus (4.3) and the analogue of Lemma 3.1(iii) where F is replaced by $|t|^{p-1}$ imply that F is strictly decreasing on $(0, \infty)$. It is easily verified that F has the rest of the properties listed in the first paragraph of §3. Define $S_i = Y_{\nu_i}$, $i \geqslant 1$, $S_0 = 0$, and $\mathsf{F}_i = \sigma(Z_t, t \leqslant \nu_i)$. Then under P the random variables $S_i - S_{i-1}$, $i \geqslant 1$, are independent and identically distributed with density $p(s)$ and furthermore $(S_i - S_{i-1}, i > n)$ is independent of F_n.

There is a one-to-one correspondence in the obvious way between stopping times in class J and stopping times for $(S_i, \mathsf{F}_i, i \geqslant 1)$. If T is in class J, $N(T)$ will stand for the associated stopping time for S_i.

LEMMA 4.2. *If T is in class J then* $\lim_{t\to\infty} E|X_{\min(t,T)}| = \delta EN(T)$.

PROOF. If $n = 0, 1, 2, \ldots$ and $t \geqslant 0$ define

$$Q_n(t) = |X_{\min(t,\nu_{n+1})}| I(\nu_n \leqslant t) I(T > \nu_n).$$

Then $Q_n(t)$ is the absolute value of a martingale. Since $X_{\min(t,\mu_{n+1})}$ and $X_{\min(t,\nu_{n+1})}$ do not have different sign,

$$\begin{aligned} EQ_n(t) &= E|X_{\min(t,\mu_{n+1})}| I(\nu_n \leqslant t) I(T > \nu_n) \nearrow E\delta I(T > \nu_n) \\ &= \delta P(T > \nu_n) \quad \text{as } t \to \infty. \end{aligned}$$

For any $t \geqslant 0$, at most one of $Q_0(t), Q_1(t), \ldots$ is different than 0 since $X_{\nu_k} = 0$ for all k. Also, $|X_{\min(t,T)}| = \Sigma_{n=0}^{\infty} Q_n(t)$, and thus $E|X_{\min(t,T)}|$ increases to $\Sigma_{n=0}^{\infty} \delta P(T > \nu_n) = \delta EN(T)$ as t increases to ∞.

LEMMA 4.3. *If T is in class J and $EN(T) < \infty$ then $AN(T) = E|Y_T|^p$.*

PROOF. We have, for $n \geqslant 0$,

$$E|Y_{\min(T,\nu_{n+1})}|^p - E|Y_{\min(T,\nu_n)}|^p = E(|Y_{\nu_{n+1}}|^p - |Y_{\min(T,\nu_n)}|^p)I(T > \nu_n)$$

$$= EE(|Y_{\nu_{n+1}}|^p - |Y_{\nu_n}|^p|Z_s, s \leqslant \nu_n)I(T > \nu_n)$$

$$= EE_{Z_{\nu_n}}(|Y_\nu|^p - |Y_{\nu_n}|^p)I(T > \nu_n)$$

$$= EF(Y_{\nu_n})I(T > \nu_n) = EF(S_n)I(N(T) > n).$$

Thus $E|Y_{\min(T,\nu_n)}|^p = \Sigma_{k=0}^{n-1} EF(S_n)I(N(T) > k)$. Since $\min(T, \nu_n)$ increases to T as $n \longrightarrow \infty$, the dominated convergence theorem and Lemma 4.1 give $E|Y_T|^p = AN(T)$.

PROOF OF LEMMA 3.0. As in the proof just made it can be shown that, in the notation of Lemma 3.0 with $N = N(T)$,

$$\sum_{i=n}^{\infty} \int_{-\infty}^{\infty} F(t)h_i(t)\,dt = \sum_{i=n}^{\infty} E(|Y_{i+1}|^p - |Y_i|^p)I(T > \nu_i)$$

$$= E(|Y_T|^p - |Y_{\nu_n}|^p)I(T > \nu_n) \leqslant E|Y_T - Y_{\nu_n}|^p I(T > \nu_n).$$

Let $|Y_T - Y_{\nu_n}|I(T > \nu_n) = W$. Note that, on $\{T > \nu_n\}$, $|Y_T - Y_{\nu_n}| \leqslant 2\sup_{0 < t \leqslant T}|Y_t|$ so that (4.1) gives

$$\lambda P(W > \lambda) \leqslant 3 \lim_{t\to\infty} E|X_{\min(t,T)}| = 3EN/\delta = 3\alpha/\delta.$$

Also, since $P(N > k)$ is decreasing in k,

$$P(W > 0) \leqslant P(T > \nu_n) = P(N > n) \leqslant EN/(1 + n) = \alpha/(1 + n).$$

Thus,

$$E|Y_T - Y_{\nu_n}|^p I(T > \nu_n) = EW^p = \int_0^\infty P(W^p > \lambda)\,d\lambda$$

$$\leqslant \int_0^\infty \min\left(\frac{\alpha}{n+1}, \frac{3\alpha\lambda^{-1/p}}{\delta}\right) d\lambda.$$

This last expression may be taken to be $\varphi_\alpha(n)$.

Although δ has been assumed to be fixed in this section to reduce notation, it will soon be necessary to let $\delta \longrightarrow 0$, so the class J of stopping times will be called $J(\delta)$ to signify its dependence on δ, and ν_i will be called $\nu_i(\delta)$. Given $\alpha > \delta$, $\eta(\alpha, \delta)$ will be the stopping time $\inf\{\nu_k(\delta): |Z_{\nu_k(\delta)}| > f(\alpha, \delta)\}$, where $f(\alpha, \delta)$ is that unique nonnegative number satisfying $EN(\eta(\alpha, \delta)) = \alpha/\delta$, which implies $\lim_{t\to\infty} E|X_{\min(t,\eta(\alpha,\delta))}| = \alpha$ by Lemma 4.2. Also, Lemma 4.3 guarantees that, if δ is fixed, $E|Y_{\eta(\alpha,\delta)}|^p$ increases as α increases. Together with Theorem 3.1 and Lemmas 4.2 and 4.3 this gives

THEOREM 4.1. *Let* $0 < \delta < 1$ *and* $\alpha > \delta$ *be real numbers. If* $T \in J(\delta)$ *and* $\lim_{t\to\infty} E|X_{\min(t,T)}| \leqslant \alpha$, *then* $E|Y_T|^p \leqslant E|Y_{\eta(\alpha,\delta)}|^p$.

5. **Proof of Theorem 2.2.** Theorem 2.2 will first be proved for the special case $r = 0$ in (2.8), which amounts to proving Theorem 2.1 for the special case $\mu(A) = 0$. For $0 < \delta < \infty$ define the stopping time θ_δ by $\theta_\delta = \inf\{\nu_i(\delta): |Y_{\nu_i(\delta)}| \geqslant 1 + \delta\}$, and if $a > 0$ define $\tau_{aSP} = \inf\{t \geqslant 0: X_t = 0 \text{ and } |Y_t| > a\}$. Note $\tau_{1SP} = \tau_{SP}$. Then linearity gives $\lim_{t\to\infty} E|X_{\min(t,\tau_{aSP})}| = a$. Since $\theta_\delta \geqslant \tau_{(1+\delta)SP}$,

$$\lim_{t\to\infty} E|X_{\min(\theta_\delta,t)}| \geqslant 1 + \delta. \tag{5.1}$$

Arguments of §3 together with Lemma 4.2 imply $\lim_{t\to\infty} E|X_{\min(\theta_\delta,t)}| < \infty$. Since $P(Z_t \text{ ever hits } \pm i) = 0$, $P(\lim_{\delta\downarrow 0}\theta_\delta = \tau_{SP}) = 1$, and thus using the dominated convergence theorem via Lemma 4.1 we get

$$\lim_{\delta\downarrow 0} E|Y_{\theta_\delta}|^p = E|Y_{\tau_{SP}}|^p. \tag{5.2}$$

Now let e be any stopping time for Z_t satisfying $|Y_e| = 0$ and $\lim_{t\to\infty} E|X_{\min(t,e)}| = 1$. Define $a(\delta) = \inf\{t > e: |X_t| = \delta\}$ and $b(\delta) = \inf\{t > a(\delta): X_t = 0\}$. Then $b(\delta)$ is in class $J(\delta)$ and an argument similar to that used to prove Lemma 4.2 gives

$$\lim_{t\to\infty} E|X_{\min(t,b(\delta))}| = \lim_{t\to\infty} E|X_{\min(t,e)}| + \delta = 1 + \delta. \tag{5.3}$$

Also, since clearly $Y_{b(\delta)} \longrightarrow Y_e$ in probability,

$$\varliminf_{\delta\downarrow 0} E|Y_{b(\delta)}|^p \geqslant E|Y_e|^p. \tag{5.4}$$

Theorem 4.1 with $\alpha = \lim_{t\to\infty} E|X_{\min(t,\theta_\delta)}|$ together with (5.1) and (5.3) imply

$$E|Y_{\theta_\delta}|^p \geqslant E|Y_{b(\delta)}|^p, \qquad 0 < \delta < \infty.$$

This together with (5.2) and (5.4) gives

$$E|Y_{\tau_{SP}}|^p \geqslant E|Y_e|^p,$$

completing the proof of Theorem 2.2 in the case $r = 0$.

The following proposition can be proved by essentially minor modifications in the proof of the above result, including both §§3 and 4, and its proof is not given.

PROPOSITION 5.1. *Let* $-1 \leqslant r \leqslant 1$ *be a real number and let* e *be a stopping time satisfying* $\lim_{t\to\infty} E_r|X_{\min(t,e)}| = 1$. *Then*

$$E_r|Y_e|^p \leqslant E_r|Y_{\tau_{m(r)SP}}|^p,$$

where $m(r)$ *satisfies*

$$\lim_{t\to\infty} E_r|X_{\min(t,\tau_{m(r)SP})}| = 1.$$

The proof of Theorem 2.2 will be completed by showing that

$$E_r|Y_{\tau_{m(r)SP}}|^p \leqslant E|Y_{\tau_{SP}}|^p, \qquad -1 \leqslant r \leqslant 1. \tag{5.5}$$

Note $m(0) = 1$. It is not hard to show that $m(\pm 1) = 0$. Inequality (5.5) will be shown by constructing a stopping time $\psi = \psi_r$ satisfying $\lim_{t\to\infty} E|X_{\min(t,\psi)}| = 1$ and

$$E|Y_\psi|^p = E_r|Y_{\tau_{m(r)SP}}|^p. \tag{5.6}$$

Since Theorem 2.2 is already proved for $r = 0$ this and Proposition 5.1 establish (5.5). Let $\alpha = \inf\{t \geqslant 0: |X_t| = r/2\}$ and let $\psi = \inf\{t > \alpha: X_t = 0 \text{ and } |Y_t| \geqslant m(r)\}$. Note that if Z_t starts at r then ψ and $\tau_{m(r)SP}$ are the same. Also note that the distributions of Z_α under P_r and P are the same, and thus the distributions of Y_ψ under P and under P_r are the same, which gives (5.6). For a fixed $t > 0$, the distributions of $|X_{\min(t,\psi)}I(t \geqslant \alpha)|$ under P and P_r are the same, as are the distributions of $|X_{\min(t,\psi)} - r/2|I(t < \alpha)$. Thus

$$\left|E_r|X_{\min(t,\psi)}| - E|X_{\min(t,\psi)}|\right| \leqslant rP(t < \alpha) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty,$$

implying

$$\lim_{t\to\infty} E|X_{\min(t,\psi)}| = \lim_{t\to\infty} E_r|X_{\min(t,\psi)}|$$
$$= \lim_{t\to\infty} E_r|X_{\min(t,\tau_{m(r)SP})}| = 1.$$

Albert Baernstein II has observed that the proof here almost unchanged gives that $\|\nu\|_p$ is also the best constant in the inequality $\|f + i\tilde{f}\|_p \leqslant K_p\|f\|_1$, $0 < p < 1$.

REFERENCES

1. D. L. Burkholder, *Martingale transforms,* Ann. Math. Statist. 37 (1966), 1494–1504. MR 34 #8456.
2. B. Davis, *On the weak type* (1, 1) *inequality for conjugate functions,* Proc. Amer. Math. Soc. 44 (1974), 307–311. MR 50 #879.
3. J. L. Doob, *Semimartingales and subharmonic functions,* Trans. Amer. Math. Soc. 77 (1954), 86–121. MR 16, 269.
4. O. D. Kellogg, *Foundations of potential theory,* Springer, Berlin, 1929.
5. H. P. McKean, Jr., *Stochastic integrals,* Probability and Math. Statist., no. 5, Academic Press, New York and London, 1969. MR 40 #947.
6. S. K. Pichorides, *On the best values of the constants in the theorems of M. Riesz, Zygmund, and Kolmogorov,* Studia Math. 44 (1972), 165–179. MR 47 #702.
7. A. Zygmund, *Trigonometric series,* Vols. I, II, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR 21 #6498.

DEPARTMENT OF STATISTICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907