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BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH L^2 INITIAL FUNCTIONS

BY

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ABSTRACT. Existence results are given for boundary value problems for vector systems of functional differential equations with L^2 initial functions. The proofs are essentially constructive and lead to computational methods in important cases.

1. **Introduction.** Let $r > 0$ and $N \geqslant 1$ be chosen and define $X = L^2[-r, 0] \times R^N$ where functions in $L^2[-r, 0]$ map $[-r, 0]$ into R^N and if $\{\phi, h\}$ is in X, $\|\{\phi, h\}\|_X^2 \equiv \int_{-r}^0 |\phi|^2 dx + |h|^2$. Let M and N be bounded linear mappings from X into X, let L be an $N \times N$ matrix, and as an example, let F be a Lipschitz continuous mapping from $L^2[-r, 0]$ into R^N. The purpose of this paper is to provide constructive existence proofs for solutions to the boundary value problem

$$\dot{x}(t) = Lx(t) + F(x_t), \qquad 0 \leqslant t \leqslant b, \tag{1.1}$$

with the boundary conditions

$$M\{x_0, x(0)\} + N\{x_b, x(b)\} = \{\psi, k\} \tag{1.2}$$

where for $t \geqslant 0$, $x_t \in L^2[-r, 0]$ is defined by $x_t(\theta) = x(t + \theta)$ for almost all $\theta \in [-r, 0]$, and $x(b) = x_b(0)$.

By constructive is meant here that either a finite difference method will be used to show existence by actually establishing existence and convergence of the difference approximations, or a contraction mapping argument will be used. Thus the proofs provide the basis for actual numerical computation.

The existence results obtained here are partial extensions of results of Waltman and Wong [7], Fennell and Waltman [2], and Grimm and Schmitt [3] where boundary value problems for functional differential equations are studied but with continuous initial functions. The difference method studied here is a variation of Euler's one-step method and can be implemented as a shooting method. Thus our results also partially extend those of deNevers and Schmitt [1].

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We remark that for smooth boundary value problems for functional equations, high order difference type methods can be obtained for a class of second order problems using results of Reddien and Travis [5]. The problems studied here are general vector systems however.

We use here recent results for initial value problems for functional differential equations with L^2 initial functions obtained by Webb [9]. We also will use some results for the numerical solution of such initial value problems obtained recently by Reddien and Webb [6]. However, the references to this last paper are minimal and the results given here are essentially self-contained. §2 contains the existence result using a contraction mapping argument and also definitions and lemmas needed to obtain the existence result by finite differences. This result is contained in §3. An application is given in §4.

2. **Preliminaries.** Two classes of nonlinear terms will be considered: Case I: F is Lipschitz continuous with Lipschitz constant β as a mapping from $L^2[-r, 0]$ into R^N, and Case II: F has the form $F(\phi) = g(\int_{-r}^0 d\eta(\theta)\phi(\theta))$ where $g: R^N \longrightarrow R^N$ is Lipschitz continuous with Lipschitz constant β, the domain of F is the continuous functions defined on $[-r, 0]$, η: $[-r, 0] \longrightarrow \mathrm{Lip}(X, X)$ (the Lipschitz continuous mappings from X into X), η is of bounded variation, $\eta(0) = 0$ and $\lim_{\theta \to -r} \tau(\theta) \neq 0$ where $\tau(\theta) \stackrel{\text{def}}{=} \int_{-r}^\theta |d\eta|$ (the total variation of η between $-r$ and θ). As a mapping from $L^2[-r, 0]$ into R^N, such an F may be only densely defined and discontinuous, but this class of F includes equations of delay type not included in Case 1.

In either case, it has been shown that the initial value problem $\dot{x} = Lx(t) + F(x_t)$, $t \geqslant 0$, and $\{x_0, x(0)\} = \{\phi, h\}$ has an associated solution semigroup $T_1(t)$ (defined in a generalized sense in Case II, see Webb [9]). In Case II, it is convenient to renorm X. Define $\|\{\phi, h\}\|^2_{X_\mu} = \int_{-r}^0 |\phi|^2 d\mu + |h|^2$ where $d\mu(\theta) = \tau(\theta)d\theta$. X_μ will represent the set X with this norm. Note that the two norms are equivalent on X.

We also associate a semigroup with the equation $\dot{x} = Lx_t(0)$, $t \geqslant 0$, and $\{x_0, x(0)\} = \{\phi, h\}$ as follows. The equation is actually an ordinary differential equation with solution $x(t) = e^{tL}h$, which is independent of ϕ. Define $T_2(t)\{\phi, h\} = \{\widetilde{\phi}, e^{tL}h\}$, $t \geqslant 0$, where $\widetilde{\phi}(t + s) = e^{(t+s)L}h$ for $t + s \geqslant 0$ and $\widetilde{\phi}(t + s) = \phi(t + s)$ otherwise, $-r \leqslant s \leqslant 0$. $T_2(t)$ forms a semigroup of bounded linear operators on either X or X_μ.

Using these semigroups, the boundary value problem (1.1)–(1.2) may be written in several ways, first as

$$M\{\phi, h\} + NT_1(b)\{\phi, h\} = \{\psi, k\}, \tag{2.1}$$

where $\{\phi, h\}$ is considered to be in X or X_μ, depending on whether or not the F

generating the semigroup $T_1(t)$ is of Case I or Case II. With the boundary value problem formulated as in equation (2.1), it is possible to give existence results using the contraction mapping theorem. We give two possibilities in the next theorem. The Lipschitz constants defined in the theorem are with respect to X or X_μ depending on whether or not the mapping F generating the semigroup $T_1(t)$ is of Case I or Case II respectively.

THEOREM 2.1. (i) *Let $(M+N)^{-1}$ exist with the domain of $(M+N)^{-1}$ containing the range of $N(T_1(b)-I)$, let $\{\psi, k\}$ be in the domain of $(M+N)^{-1}$ and let $\|(M+N)^{-1}N(T_1(b)-I)\|_{\text{Lip}} < 1$. Then solutions to* (2.1) *exist and are unique.* (ii) *Let $(M+NT_2(b))^{-1}$ exist and have domain containing the range of $N(T_1(b)-T_2(b))$, let $\{\psi, k\}$ be in the domain of $(M+NT_2(b))^{-1}$, and let $\|(M+NT_2(b))^{-1}N(T_1(b)-T_2(b))\|_{\text{Lip}} < 1$. Then solutions to* (2.1) *exist and are unique.*

PROOF. For example, (2.1) may be written as

$$M\{\phi, h\} + NT_2(b)\{\phi, h\} + N(T_1(b)-T_2(b))\{\phi, h\} = \{\psi, k\}$$

and then using (ii) as

$$\begin{aligned}\{\phi, h\} + (M+NT_2(b))^{-1}N(T_1(b)-T_2(b))\{\phi, h\}& \\ = (M+NT_2(b))^{-1}\{\psi, k\}.&\end{aligned} \tag{2.2}$$

Equation (2.2) has the form $x + Sx = y$ with $\|S\|_{\text{Lip}} < 1$ in a Banach space, and so it is solvable uniquely.

REMARK. It follows from results of Webb [9] that in Case I, $\|T_1(b)\|_{\text{Lip}} \leq e^{\omega b}$ where $\omega = \|L\|_N + \beta + ½$ and $\|L\|_N$ represents the Euclidean matrix norm of the $N \times N$ matrix L. In Case II, it follows from [9] that $\|T_1(b)\|_{\text{Lip}} \leq e^{\omega b}$ where $\omega = \tau(0)(1 + (\beta + \|L\|_N)^2)/2$. Thus the hypotheses of Theorem 2.1 are actually on M, N, the length of the interval and the magnitude of the constants β, $\|L\|_N$ and $\tau(0)$. Theorem 2.1 is an extension of Theorem 4.1 of [7]. We note also the following result. See also Remark 1 in [7].

LEMMA 2.2. *$(M+NT_2(b))^{-1}$ exists with domain equal X (X_μ) if and only if the problem*

$$\dot{x} = Lx_t(0), \qquad 0 \leq t \leq b, \tag{2.3}$$

and the boundary conditions

$$M\{x_0, x(0)\} + N\{x_b, x(b)\} = \{\psi, k\} \tag{2.4}$$

are uniquely solvable for each $\{\psi, k\}$ in $X(X_\mu)$.

PROOF. Equation (2.3)–(2.4) is equivalent to

$$M\{\phi, h\} + NT_2(b)\{\phi, h\} = \{\psi, k\},$$

giving the result.

Our next existence result, Theorem 3.2, will relax these conditions on M, N and F. The small Lipschitz constants required by the contraction mapping theorem will be traded for almost uniform boundedness on F and a compactness argument. First, a sequence of lemmas will be given.

Define $t_n = r/n$ and let $\chi_j^n = \chi_{[-t_n j, -t_n(j-1))}$ for $j = 1, 2, \ldots, n$, χ_A denoting the characteristic function of the set A. Let X_n denote the subspace of $X\,(X_\mu)$ defined by

$$X_n = \left\{ \{\phi, h\} : \phi = \sum_{j=1}^{n} v_j \chi_j^n,\ v_j \in R^n,\ j = 1, \ldots, n \right\}.$$

Let P_n be a mapping from $X\,(X_\mu)$ into X_n defined by

$$P_n\{\phi, h\} = \left\{ \sum_{j=1}^{n} v_j \chi_j^n,\ h \right\}$$

where

$$v_j = \frac{n}{r} \int_{-t_n j}^{-t_n(j-1)} \phi(s)\, ds.$$

Define $\pi_1\{\phi, h\} = \phi$. In Case I we assume $\{F_n\}$ is a sequence of Lipschitz continuous mappings from $\pi_1 X_n$ to R^N with Lipschitz constants $\beta_n \longrightarrow \beta$ and with $F_n(\pi_1 P_n\{\phi, h\}) \longrightarrow F(\phi)$ as $n \longrightarrow \infty$ for all $\{\phi, h\} \in X$. For $\{\phi, h\} = \{\Sigma_{j=1}^n v_j \chi_j^n, h\}$, define

$$A_n\{\phi, h\} = \left\{ -\sum_{i=1}^{n} \left(\frac{n}{r}\right)(v_{i-1} - v_i)\chi_i^n,\ -F_n(\phi) - Lh \right\}$$

where $v_0 = h$. We then have the next lemma which is a slight extension of a result in [6]. Define $l = \|L\|_N$ and let $(\cdot\,,\cdot)$ denote the R^N inner product.

LEMMA 2.3. *For each n let $t_n = r/n$ and let $\gamma_n = \beta_n + l + \frac{1}{2}$. Then $I - t_n A_n$ is a Lipschitz continuous mapping from X_n into X_n taken as a subspace of X with Lipschitz constant $|(I - t_n A_n)|_{\mathrm{Lip}} \leqslant 1 + t_n \gamma_n$.*

PROOF. For $\{\phi, h\} = \{\Sigma_{j=1}^n h_j \chi_j^n, h_0\}$ and $\{\psi, k\} = \{\Sigma_{j=1}^n k_j \chi_j^n, k_0\}$, we have

$$\begin{aligned}
&\|(I - t_nA_n)\{\phi, h\} - (I - t_nA_n)\{\psi, k\}\|^2 \\
&\quad = t_n \sum_{i=1}^{n} |h_{i-1} - k_{i-1}|^2 + |h_0 - k_0|^2 \\
&\qquad + 2t_n(h_0 - k_0, F_n(\phi) - F_n(\psi)) + 2t_n(h_0 - k_0, L(h_0 - k_0)) \\
&\qquad + t_n^2|F_n(\phi) - F_n(\psi) + L(h_0 - k_0)|^2 \\
&\quad \leq \|\{\phi, h\} - \{\psi, k\}\|^2 + t_n\beta_n(|h_0 - k_0|^2 + |\phi - \psi|^2) + t_n|h_0 - k_0|^2 \\
&\qquad + 2t_nl|h_0 - k_0|^2 + t_n^2\beta_n^2|\phi - \psi|^2 + t_n^2l^2|h_0 - k_0|^2 \\
&\qquad + t_n^2\beta_nl(|\phi - \psi|^2 + |h_0 - k_0|^2) \\
&\quad \leq (1 + t_n(\beta_n + l + ½))^2\|\{\phi, h\} - \{\psi, k\}\|^2,
\end{aligned}$$

giving the result.

REMARK. Since it follows that $P_n\colon X \longrightarrow X_n$ is a bounded linear projection with $\|P_n\| = 1$, then the estimate

$$|(I - t_nA_n)P_n|_{\text{Lip}} \leq 1 + t_n\gamma_n$$

is valid in X.

We next give a similar lemma for Case II. Recall F has the form $F(\phi) = g(\int_{-r}^{0} d\eta(\theta)\phi(\theta))$. X_n is defined as before but will now be renormed and denoted as $X_{n,\tau}$ when it carries this new norm. With $\tau(\theta) = \int_{-r}^{\theta} |d\eta|$, define

$$\langle\{\phi, h\}, \{\psi, k\}\rangle_{X_{n,\tau}} = t_n \sum_{j=1}^{n} (v_j, w_j)\tau_{j-1}^n + (h, k)$$

where $\tau_j^n = \tau(-t_nj), j = 0, 1, \ldots, n$, and $(\cdot, \cdot)$ denotes the inner product on R^N. P_n is defined as before. The next lemma regarding P_n will be needed and follows directly from definition of the norm on $X_{n,\tau}$.

LEMMA 2.4. *Let $P_n\colon X_\mu \longrightarrow X_{n,\tau}$. Then*

$$\|P_n\|^2 \leq \max\left(1, \tau(0)/\lim_{\theta\to -r} \tau(\theta)\right).$$

We next define how F is to be approximated in this case. Let $F_n\colon \pi_1X_{n,\tau} \longrightarrow R^N$ be given by

$$F_n\left(\sum_{j=1}^{n} v_j\chi_j^n\right) = g_n\left(\sum_{j=1}^{n} (\eta(-(j-1)t_n) - \eta(-jt_n))v_j\right)$$

where $\{g_n\}$ is a sequence of Lipschitz continuous mappings from R^N into R^N with Lipschitz constants β_n satisfying $\beta_n \longrightarrow \beta$, and with $g_n(h) \longrightarrow g(h)$ as $n \longrightarrow \infty$ for all $h \in R^N$. Now define A_n as in Case I. We then have the next lemma which extends a result given in [6].

LEMMA 2.5. *For each n let $t_n = r/n$ and let $\gamma_n = (1 + \beta_n^2 + l_n)(\tau(0) + 1)$. Then the Lipschitz constant of $(I - t_n A_n)$ as a mapping from $X_{n,\tau}$ into $X_{n,\tau}$ satisfies*

$$|(I - t_n A_n)|_{\mathrm{Lip}(X_{n,\tau}, X_{n,\tau})} \leqslant 1 + t_n \gamma_n.$$

PROOF. Let $\{\phi, h\} = \{\Sigma_{i=1}^n h_i \chi_i^n, h_0\}$, $\{\psi, k\} = \{\Sigma_{i=1}^n k_i \chi_i^n, k_0\}$ and $u_i = h_i - k_i$, $i = 0, 1, \ldots, n$. Observe that by the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n (\tau_{i-1}^n - \tau_i^n)|u_i|\right)^2 \leqslant \tau(0) \sum_{i=1}^n |u_i|^2 (\tau_{i-1}^n - \tau_i^n).$$

Combining this inequality with the definition of F_n, it follows that

$$\begin{aligned}
&\|(I - t_n A_n)\{\phi, h\} - (I - t_n A_n)\{\psi, k\}\|^2 \\
&= t_n \sum_{i=1}^n |u_{i-1}|^2 \tau_{i-1}^n + |u_0|^2 + 2t_n(u_0, F_n(\phi) - F_n(\psi) + L(u_0)) \\
&\quad + t_n^2 |F_n(\phi) - F_n(\psi) + Lu_0|^2 \\
&\leqslant t_n \sum_{i=1}^n |u_{i-1}|^2 \tau_{i-1}^n + |u_0|^2 + 2t_n |u_0| \beta_n \sum_{i=1}^n (\tau_{i-1}^n - \tau_i^n)|u_i| \\
&\quad + 2t_n l |u_0|^2 + t_n^2 \beta_n^2 \left(\sum_{i=1}^n (\tau_{i-1}^n - \tau_i^n)|u_i|\right)^2 \\
&\quad + 2t_n^2 l \beta_n \left(\sum_{i=1}^n (\tau_{i-1}^n - \tau_i^n)|u_i|\right) |u_0| + t_n^2 l^2 |u_0|^2 \\
&\leqslant t_n \sum_{i=1}^n |u_{i-1}|^2 \tau_{i-1}^n + |u_0|^2 + t_n(|u_0| \beta_n \tau(0)^{1/2})^2 \\
&\quad + t_n \left((1/\tau(0)^{1/2}) \sum_{i=1}^n (\tau_{i-1}^n - \tau_i^n)|u_i|\right)^2 \\
&\quad + 2t_n l |u_0|^2 + t_n^2 \beta_n^2 \tau(0) \sum_{i=1}^n |u_i|^2 (\tau_{i-1}^n - \tau_i^n) \\
&\quad + t_n^2 \left(l^2 \beta_n^2 |u_0|^2 + \tau(0) \sum_{i=1}^n |u_i|^2 (\tau_{i-1}^n - \tau_i^n)\right) + t_n^2 l^2 |u_0|^2 \\
&\leqslant t_n \sum_{i=1}^n |u_{i-1}|^2 \tau_{i-1}^n + |u_0|^2 + t_n |u_0|^2 \beta_n^2 \tau(0)
\end{aligned}$$

$$+ t_n \sum_{i=1}^{n} (\tau_{i-1}^n - \tau_i^n)|u_i|^2 + 2t_n l |u_0|^2$$

$$+ t_n^2 \beta_n^2 \tau(0) \sum_{i=1}^{n} |u_i|^2 (\tau_{i-1}^n - \tau_i^n) + t_n^2 l^2 \beta_n^2 |u_0|^2$$

$$+ t_n^2 \tau(0) \sum_{i=1}^{n} |u_i|^2 (\tau_{i-1}^n - \tau_i^n) + t_n^2 l^2 |u_0|^2$$

$$\leqslant (1 + t_n(\tau(0) + 2l + \beta_n^2 \tau(0)))|u_0|^2$$

$$+ t_n \sum_{i=1}^{n} |u_i|^2 \tau_{i-1}^n + t_n^2 \beta_n^2 \tau(0) \sum_{i=1}^{n} |u_i|^2 \tau_{i-1}^n + t_n^2 l^2 \beta_n^2 |u_0|^2$$

$$+ t_n^2 \tau(0) \sum_{i=1}^{n} |u_i|^2 \tau_{i-1}^n + t_n^2 l^2 |u_0|^2$$

$$\leqslant (1 + t_n(\tau(0) + 2l + \beta_n^2 \tau(0)) + t_n^2 \beta_n^2 \tau(0) + t_n^2 \tau(0) + t_n^2 l^2 + t_n^2 l^2 \beta_n^2)$$

$$\cdot \|\{\phi, h\} - \{\psi, k\}\|_{X_{n,\tau}}^2$$

$$\leqslant (1 + t_n(1 + \beta_n^2 + l)(\tau(0) + 1))^2 \|\{\phi, h\} - \{\psi, k\}\|_{X_{n,\tau}}^2,$$

which completes the proof.

We will also need later the facts contained in the next three lemmas.

LEMMA 2.6. *Let $K = \{(I - t_n A_n)^{m_n} P_n\{\phi, h\}: m_n = [t/t_n], t_n = r/n, n = 1, 2, \ldots$, and $\|\{\phi, h\}\|_X \leqslant 1\}$ in Case* I *and define K_μ analogously to K but with $\|\{\phi, h\}\|_{X_\mu} \leqslant 1$ in Case* II. (*Note that $\pi_1 K$ or $\pi_1 K_\mu$ is a set of step functions on $[-r, 0]$.*) *If $t \geqslant r$, then $\pi_1 K$* and *$\pi_1 K_\mu$ are uniformly bounded in the supremum norm on $[-r, 0]$ and K is uniformly bounded in X and K_μ is uniformly bounded in X_μ.*

PROOF. Let n be fixed and recall that $P_n\{\phi, h\}\{\Sigma_{j=1}^n v_j \chi_j^n, v_0\}$ where

$$v_j = \frac{n}{r} \int_{-jt_n}^{-(j-1)t_n} \phi(s)\, ds, \quad v_0 = h.$$

Define $\pi_2\{\phi, h\} = h$. For $1 \leqslant j \leqslant m_n - n$, define

$$v_{-j} = h + \frac{r}{n} \sum_{k=1}^{j} (F_n(\pi_1(I - t_n A_n)^{k-1} P_n\{\phi, h\})$$
$$+ L\pi_2(I - t_n A_n)^{k-1} P_n\{\phi, h\}).$$

Then

$$(I - t_nA_n)P_n\{\phi, h\} = \left\{ \sum_{j=1}^{n} v_{j-1}\chi_j^n,\ h + t_nF_n(\pi_1P_n\{\phi, h\}) + t_nL\pi_2P_n\{\phi, h\} \right\}$$

$$= \left\{ \sum_{j=1}^{n} v_{j-1}\chi_j^n,\ v_{-1} \right\},$$

$$(I - t_nA_n)^2P_n\{\phi, h\} = \left\{ \sum_{j=1}^{n} v_{j-2}\chi_j^n,\ v_{-1} + t_nF_n(\pi_1(I - t_nA_n)P_n\{\phi, h\}) + t_nL\pi_2(I - t_nA_n)P_n\{\phi, h\} \right\}$$

$$= \left\{ \sum_{j=1}^{n} v_{j-2}\chi_j^n,\ v_{-2} \right\},$$

$\cdot$
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$$(I - t_nA_n)^{m_n}P_n\{\phi, h\} = \left\{ \sum_{j=1}^{n} v_{j-m_n}\chi_j^n,\ v_{-m_n} \right\}.$$

Therefore, to show the conclusion, it suffices to show the expressions v_{-j}, $j = 0, 1, 2, \ldots, m_n - n$, are bounded independently of n and $\{\phi, h\}$. But for $j = 0, 1, \ldots, m_n$,

$$v_{-j} = \pi_2(I - t_nA_n)^jP_n\{\phi, h\}.$$

Thus in Case I,

$$|v_{-j}| \leq (1 + t_n\gamma_n)^j \|\{\phi, h\}\|_X, \qquad j = 0, 1, \ldots, m_n.$$

In Case II,

$$|v_{-j}| \leq (1 + t_n\gamma_n)^j \max\left(1, \tau(0)/\lim_{\theta \to -r} \tau(\theta)\right)^{1/2} \cdot \|\{\phi, h\}\|_{X_\mu}, \qquad j = 0, 1, \ldots, m_n,$$

from which the result follows.

LEMMA 2.7. *Let K and K_μ be defined as in Lemma 2.6. Then there exists a constant $C > 0$ so that for any $\phi_n = \pi_1(I - t_nA_n)^{m_n}P_n\{\phi, h\}$ in π_1K or π_1K_μ, $m_n = [t/t_n]$, $t_n = r/n$ and $t > r$,*

$$|\phi_n(jt_n) - \phi_n((j-1)t_n)| \leq Ct_n, \qquad j = 2, 3, \ldots, n.$$

PROOF. Using the notation introduced in the proof of Lemma 2.6, it follows in both cases that

$$|v_{j-m_n} - v_{j-m_n+1}| \leq t_n |F_n(\pi_1(I - t_nA_n)^{m_n-j-1}P_n\{\phi, h\})|$$

$$+ t_n l|\pi_2(I - t_nA_n)^{m_n-j-1}P_n\{\phi, h\}|.$$

For Case I, the result follows using this inequality, Lemma 2.6, the uniform Lipschitz continuity of the F_n's and their pointwise convergence to F. Now in Case II, recall

$$|F_n(\phi)| = \left|g_n\left(\sum_{j=1}^{n} (\eta(-(j-1)t_n) - \eta(-jt_n))v_j\right)\right|$$

$$\leq \beta_n \sum_{j=1}^{n} (\tau_{j-1} - \tau_j)|v_j| \leq \beta_n \tau(0) \max_j |v_j|.$$

Thus combining we have

$$|v_{j-m_n} - v_{j-m_n+1}| \leq \beta_n \tau(0) t_n \max_{1 \leq s \leq n} |v_{s-m_n+1}|$$

$$+ t_n l|\pi_2(I - t_nA_n)^{m_n-j-1}P_n\{\phi, h\}|.$$

Now using Lemma 2.6 and Lemma 2.5 with this inequality, the result for Case II follows.

LEMMA 2.8. *Let K and K_μ be defined as in Lemma 2.6. Then for $t > r$, K is pre-compact in X and K_μ is pre-compact in X_μ.*

PROOF. Let $\{\phi_n, h_n\}$ be a sequence in either K or K_μ. Since $\{h_n\}$ forms a bounded sequence in R^N in either case, $\{h_n\}$ has a convergent subsequence so we need only show $\{\phi_n\}$ has a subsequence converging in $L^2[-r, 0]$ in Case I and $(L^2[-r, 0], \mu)$ in Case II to complete the proof. If the sequence $\{\phi_n\}$ has been chosen so that some mesh size r/k repeats infinitely often, then convergence of a subsequence of $\{\phi_n\}$ follows immediately since the problem is finite dimensional. Now assume that $\{f_n\}$ is a sequence of step functions in $L^2[-r, 0]$ or $(L^2[-r, 0], \mu)$ satisfying $|f_n(ir/n) - f_n((i-1)r/n)| \leq cr/n$ for some constant $c > 0$ and independent of n with f_n having jumps only at the points ir/n, $i = 1, 2, \ldots, n/r$. (For simplicity in what follows we assume the sequence of integers $\{n\}$ is such that $n \bmod r = 0$.) In addition, assume the sequence $\{f_n\}$ is pointwise uniformly bounded. Now let $\epsilon > 0$ and choose N_1 so that $cr/N_1 < \epsilon$. Let $\delta = r/N_1$. Let n be $\geq N_1$ and let x, y be in $[-r, 0]$ with $|x - y| \leq \delta$. Then

$$|f_n(x) - f_n(y)| \leq \left(\left[\frac{\delta n}{r}\right] + 1\right)\frac{r}{n} \cdot c \leq 2\frac{r}{N_1}c = 2\epsilon.$$

Let $\{x_i\}$ be a countable dense set in $[-r, 0]$. Since $\{f_n\}$ forms a uniformly bounded sequence, then $\{f_n(x_i)\}_{n \geq 1}$ forms a bounded sequence of real numbers, so by the usual diagonal argument we extract a subsequence which we denote by $\{f_n\}$ that converges on this countable dense set. There exists a natural number K so that the intervals $|x - x_j| \leq \delta$, $j = 1, \ldots, K$, cover $[-r, 0]$. We now choose N_1 larger if necessary so that

$$|f_n(x_j) - f_m(x_j)| \leq \epsilon \qquad \text{for } n, m \geq N_1, j = 1, \ldots, K.$$

Now let $x \in [-r, 0]$ be arbitrary. There exists an index j so that $|x - x_j| \leq \delta$. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x_j) - f_m(x_j)| \\ &\quad + |f_n(x) - f_n(x_j)| + |f_m(x_j) - f_m(x)| \\ &\leq 5\epsilon, \qquad n, m \geq N_1. \end{aligned}$$

Thus for each x, $\{f_n(x)\}$ forms a Cauchy sequence, actually uniform in x. Therefore $\{f_n\}$ converges uniformly to some function f and uniform convergence implies $L^2[-r, 0]$ convergence or $(L^2[-r, 0], \mu)$ convergence. Since by the previous two lemmas the functions $\{\phi_n\}$ have the properties assumed for the functions $\{f_n\}$, the proof is complete.

3. **Existence by finite differences.** We introduce the notation

$$T_1^n(t) = (I - t_n A_n)^{[tn/r]}$$

where A_n is defined as in §2. We also want to obtain a similar notation for the problem $\dot{x} = Lx_t(0)$. Let $\{\phi, h\} = \{\Sigma_{j=1}^n h_j \chi_j^n, h_0\}$ and define

$$A_n^2\{\phi, h\} = \left\{ \sum_{j=1}^n \frac{n}{r}(h_{j-1} - h_j)\chi_j^n, Lh_0 \right\}.$$

Then

$$(I + t_n A_n^2)\{\phi, h\} = \left\{ \sum_{j=1}^n h_{j-1}\chi_j^n, (I + t_n L)h_0 \right\} = \left\{ \sum_{j=1}^n h_{j-1}\chi_j^n, h_{-1} \right\}$$

and

$$(I + t_n A_n^2)^k\{\phi, h\} = \left\{ \sum_{j=1}^n h_{j-k}\chi_j^n, h_{-k} \right\},$$

where $h_{-k} = (I + t_n L)^k h_0$. We now define $T_2^n(t) = (I + t_n A_n^2)^{[tn/r]}$. It follows in an identical manner to the proofs of Lemmas 2.3 and 2.5 that $\|(I + t_n A_n^2)\|_X \leq (1 + t_n(l + ½))$ and $\|(I + t_n A_n^2)\|_{X_\mu} \leq (1 + t_n(1 + l^2)(1 + \tau_0))$.

Using equation (2.1) as a basis, the numerical method will be to choose $\{\phi_n, h_n\}$ in X (X_μ) so that

$$M\{\phi_n, h_n\} + NT_1^n(b)P_n\{\phi_n, h_n\} = P_n\{\psi, k\}. \tag{3.1}$$

To simplify notation we will later write z_n for $\{\phi_n, h_n\}$. Note that equation (3.1) is not fully discretized since M and N have not been approximated and z_n is arbitrary in X (X_μ). An example will be given in §4 to show how in important cases (3.1) is fully discretized.

We will assume (H1) that $(M + NT_2(b))^{-1}$ exists with domain containing the range of $N(T_1(b) - T_2(b))$ and $(M + NT_2(b))^{-1}$ is bounded, and also that $(M + NT_2^n(b)P_n)^{-1}$ exists for all n sufficiently large with domain containing the range of $N(T_1^n(b)P_n - T_2^n(b)P_n)$ and that $\{(M + NT_2^n(b)P_n)^{-1}\}$ is uniformly bounded in norm. We also assume that $\{\psi, h\}$ is in the domain of $(M + NT_2(b))^{-1}$ and that $P_n\{\psi, h\}$ is in the domain of $(M + NT_2^n(b)P_n)^{-1}$. These hypotheses are on the matrix L and the operators M and N and can be verified directly in several important examples, one of which will be given in §4. We further assume (H2) that in Case I, F_n takes bounded sets in $\pi_1 X_n$ with the induced $\pi_1 X$ norm uniformly in n into bounded sets in R^N and that

$$\limsup_{\|\{\phi_n,0\}\|_X \to\infty;\phi_n\in\pi_1X_n} \frac{|F_n(\phi_n)|}{\|\{\phi_n, 0\}\|_X} \leqslant \delta$$

holds uniformly for all n large and any $\delta > 0$. In Case II we assume F_n takes bounded sets in $\pi_1 X_n$ with the induced $\pi_1 X_\mu$ norm uniformly in n into bounded sets in R^N and that

$$\limsup_{\|\{\phi_n,0\}\|_{X_\mu} \to\infty;\phi_n\in\pi_1X_n} \frac{|F_n(\phi_n)|}{\|\{\phi_n, 0\}\|_{X_\mu}} \leqslant \delta$$

holds uniformly for all n large and any $\delta > 0$. Note that this condition would be satisfied in Case I if $\{F_n\}$ were totally uniformly bounded and in Case II with a similar condition on $\{g_n\}$. We note that the induced $\pi_1 X_\mu$ norms on $\pi_1 X_n$ are uniformly equivalent in n to the $\pi_1 X_{n,\tau}$ norms so that the preceding assumption could just as well have been expressed in terms of the $\pi_1 X_{n,\tau}$ norms.

Define $s_n(t, \{\phi, h\}) = F_n(\pi_1 T_1^n(t)P_n\{\phi, h\})$ and define $Z^n(jt_n)$ as a map from X_n into X_n by $Z^n(jt_n)P_n\{\phi, h\} = \{0, s_n(jt_n, \{\phi, h\})\}$.

LEMMA 3.1.

$$\begin{aligned} T_1^n(t)P_n\{\phi, h\} &= \prod_{j=0}^{[nt/r]} (I + t_nA_n^2 + t_nZ^n(jt_n))P_n\{\phi, h\} \\ &= (I + t_nA_n^2)^{[nt/r]}P_n\{\phi, h\} \\ &\quad + t_n \sum_{j=0}^{m-1} (I + t_nA_n^2)^{m-j-1} Z^n(jt_n)P_n\{\phi, h\} \end{aligned}$$

where $m = [tn/r]$.

PROOF. The first equality follows from the definition of $T_1^n(t)P_n$. Let $S = I + t_nA_n^2$. S is a linear mapping in X_n. The second equality is evidently correct for $m = 1$. Let

$$\prod_{j=0}^{m-2} (S + t_nZ^n(jt_n)) = S^{m-1} + t_n \sum_{j=0}^{m-2} S^{m-j-2}Z^n(jt_n).$$

Then

$$(S + t_nZ^n((m-1)t_n)) \prod_{j=0}^{m-2} (S + t_nZ^n(jt_n))$$
$$= S^m + t_n \sum_{j=0}^{m-2} S^{m-j-1}Z^n(jt_n) + t_nZ^n((m-1)t_n),$$

giving the result.

REMARK. This formula can be viewed as a discretization of the variation of parameters formula given in Hale [4].

We now give the main result of this section.

THEOREM 3.2. *Let hypotheses* (H1) *and* (H2) *hold. Then the solution to*

$$(3.1)' \qquad Mz_n + NT_1^n(b)P_nz_n = P_n\widetilde{\psi}, \quad z_n = \{\phi_n, h_n\}, \quad \widetilde{\psi} = \{\psi, k\},$$

exists for all n sufficiently large and has a strong limit point in X (X_μ) which is a solution to (2.1). *Moreover, any sequence of solutions of* (3.1)$'$ *that converges can only converge to a solution of* (2.1).

PROOF. Rewrite equation (3.1)$'$ as

$$(M + NT_2^n(b)P_n)z_n + N(T_1^n(b)P_n - T_2^n(b)P_n)z_n = P_n\widetilde{\psi}$$

and further as

$$(3.2)\ z_n + (M + NT_2^n(b)P_n)^{-1}N(T_1^n(b)P_n - T_2^n(b)P_n)z_n = (M + NT_2^n(b)P_n)^{-1}P_n\widetilde{\psi}.$$

Define $S_n = (M + NT_2^n(b)P_n)^{-1}N(T_1^n(b)P_n - T_2^n(b)P_n)$ and $y_n = (M + NT_2^n(b))^{-1} \cdot P_n\widetilde{\psi}$ for later reference. From Lemma 3.1 we have

$$(3.3) \qquad T_1^n(b)P_nz - T_2^n(b)P_nz = t_n \sum_{j=0}^{m-1} (I + t_nA_n^2)^{m-j-1}Z_n(jt_n)P_nz$$

where $m = [bn/r]$, $z = \{\phi, h\}$. Let $z_j = \{\phi_j, h_j\}$ be any sequence in X (or X_μ) with $\|z_j\|_X \longrightarrow \infty$ (or $\|z_j\|_{X_\mu} \longrightarrow \infty$). Define

$$\beta_j = \sup_{0<k<[bn/r];n=1,2,\dots} \|\pi_1T_1^n(kt_n)P_nz_j\|_{\pi_1X}$$

and analogously define $\beta_{j,\mu}$ with the norm taken in X_μ. Recall that $P_n\{\phi, h\} = \{\Sigma_{j=1}^n v_j\chi_j^n, v_0\}$ where

$$v_0 = h, \quad v_j = \frac{n}{r}\int_{-jt_n}^{-(j-1)t_n} \phi(s)\,ds$$

and

$$(I - t_nA_n)^k P_n\{\phi, h\} = \left\{\sum_{j=1}^n v_{j-k}\chi_j^n, v_{-k}\right\}$$

where the v_i are explicitly given in Lemma 2.6. Thus $\pi_1 T_1^n(kt_n)P_n z_j = \Sigma_{s=1}^n v_{s-k}\chi_s^n$. For $s-k>0, v_{s-k} = v_j, j = s-k$. For $s-k \leq 0$,

$$v_{s-k} = \pi_2(I - t_nA_n)^{k-s}P_n z_j, \qquad \pi_2\{\phi, h\} = h.$$

In both Case I and Case II, we have for $s-k \leq 0$ that $|v_{s-k}| \leq c\|z_j\|$ for some constant c independent of n using the proof of Lemma 2.6. Thus in Case I,

$$\begin{aligned}\|\pi_1 T_1^n(kt_n)P_n z_j\|_{L^2[-r,0]} &\leq \left\|\sum_{\substack{s-k>0\\1\leq s\leq n}} v_{s-k}\chi_s^n\right\| + \left\|\sum_{\substack{s-k\leq 0\\1\leq s\leq n}} v_{s-k}\chi_s^n\right\|\\ &\leq \|\pi_1 z_j\| + \left(\sum_{s-k\leq 0} |v_{s-k}|^2 t_n\right)^{1/2} \leq \|z_j\|_X + \left(c\|z_j\|_X^2 \cdot \sum_{s-k\leq 0} t_n\right)^{1/2}\\ &\leq (1 + (rc)^{1/2})\|z_j\|_X\end{aligned}$$

and in Case II,

$$\begin{aligned}\|\pi_1 T_1^n(kt_n)P_n z_j\|_{(L^2[-r,0],\mu)} &\leq \|z_j\|_{X_\mu} + \left\|\sum_{s-k\leq 0} v_{s-k}\chi_s^n\right\|_{(L^2[-r,0],\mu)}\\ &\leq \|z_j\|_{X_\mu} + \tau(0)^{1/2} r^{1/2} c\|z_j\|_{X_\mu}.\end{aligned}$$

Thus we may take the supremum over n in the definition of β_j and $\beta_{j,\mu}$ and obtain finite numbers. Moreover, the bounds obtained depend on $\|z_j\|$ and not z_j itself.

Now let $\epsilon > 0$. By assumption there exist numbers $R > 0$, $\rho > 0$ and a positive integer N_1 so that in Case I, $\|\pi_1 P_n\{\phi, 0\}\|_{\pi_1 X} < \rho$ implies $|F_n(\pi_1 P_n\{\phi, 0\})| < R$ and $\|\pi_1 P_n\{\phi, 0\}\|_{\pi_1 X} > \rho$ and $n \geq N_1$ implies $|F_n(\pi_1 P_n\{\phi, 0\})| \leq \epsilon\|\pi_1 P_n\{\phi, 0\}\|_{\pi_1 X}$. In Case II identical statements hold with the norm in $\pi_1 X$ replaced by the norm in $\pi_1 X_\mu$. From (3.3) it follows that

$$\|T_1^n(b)P_n z_j - T_2^n(b)P_n z_j\| \leq t_n \sum_{k=0}^{m-1} \|(I + t_nA_n^2)^{m-k-1} Z^n(kt_n)P_n z_j\| \tag{3.4}$$

holds in either the X or X_μ norm. In either norm, we have shown the existence of a constant γ so that $\|I + t_n A_n^2\| \leqslant 1 + t_n\gamma$, i.e.

$$\gamma = \max(l + ½, (1 + l^2)(1 + \tau_0)).$$

Thus using (3.4) we have

$$\|T_1^n(b)P_n z_j - T_2^n(b)P_n z_j\| \leqslant t_n\left(\sum_{k=0}^{m-1} (1 + t_n\gamma)^{m-k-1}\|Z^n(kt_n)P_n z_j\|\right).$$

Since $Z^n(kt_n)P_n z_j = \{0, F_n(\pi_1 T_1^n(kt_n)P_n z_j)\}$, it follows that for all $n \geqslant N_1$, in Case I

(3.5a) $$\|T_1^n(b)P_n z_j - T_2^n(b)P_n z_j\|_X \leqslant be^{\gamma b} \max(R, \epsilon\beta_j)$$

and in Case II

(3.5b) $$\|T_1^n(b)P_n z_j - T_2^n(b)P_n z_j\|_{X_\mu} \leqslant be^{\gamma b} \max(R, \epsilon\beta_{j,\mu}).$$

Now we have shown $\beta_j \leqslant c_1\|z_j\|_X$ for some constant c_1 and $\beta_{j,\mu} \leqslant c_2\|z_j\|_{X_\mu}$ for some constant c_2. Combining this with (3.5) we have for $n \geqslant N_1$, in Case I

(3.6a) $$\|T_1^n(b)P_n z_j - T_2^n(b)P_n z_j\|_X \leqslant be^{\gamma b} \max(R, \epsilon c_1\|z_j\|_X)$$

and in Case II

(3.6b) $$\|T_1^n(b)P_n z_j - T_2^n(b)P_n z_j\|_{X_\mu} \leqslant be^{\gamma b} \max(R, \epsilon c_2\|z_j\|_{X_\mu}).$$

Now returning to (3.2), we have that it may be written as

(3.7) $$z_n + S_n z_n = y_n$$

where $\|y_n\|_X \leqslant c_3$. S_n is a completely continuous operator in X in Case I and in X_μ in Case II for each n, because both $T_1^n(b)\, P_n$ and $T_2^n(b)P_n$ are Lipschitz continuous mappings into a finite dimensional subspace, X_n, of the appropriate space and so have finite dimensional range. Now use inequality (3.6) to deduce that for any $0 < \epsilon < 1$, there exist positive numbers ρ, R, and N_1 such that $\|z\| > \rho$ implies $\|S_n z\| \leqslant \epsilon\|z\|$ and $\|z\| \leqslant \rho$ implies $\|S_n z\| \leqslant R$ and these bounds hold uniformly for $n \geqslant N_1$. It then follows from the fixed point theorem of Granas [7] that (3.7) is solvable for each $n \geqslant N_1$. Moreover, these solutions z_n satisfy either $\|z_n\| \leqslant \rho$ or then from (3.7) we have $\|z_n\| \leqslant c/(1 - \epsilon)$. Thus the solutions $\{z_n\}$ remain uniformly bounded in X or X_μ, depending on the case being considered.

We next show that some subsequence $\{z_{n_j}\}$ of $\{z_n\}$ is convergent to say $z \in X\ (X_\mu)$ and z is a solution to (2.1). Define

$$\psi_n = N(T_1^n(b)P_n z_n - T_2^n(b)P_n z_n).$$

From Lemma 2.8, we have that the ψ_n's lie in a pre-compact set and so have a convergent subsequence (which we call $\{\psi_n\}$) satisfying $\psi_n \longrightarrow \hat{\psi}$ in $X\ (X_\mu)$.

Now since

$$(M + NT_2^n(b)P_n)^{-1} - (M + NT_2(b))^{-1}$$
$$= (M + NT_2^n(b)P_n)^{-1}(NT_2(b) - NT_2^n(b)P_n)(M + NT_2(b))^{-1},$$

we may write that

$$\|(M + NT_2^n(b)P_n)^{-1}\psi_n - (M + NT_2(b))^{-1}\hat{\psi}\|$$

$$\leq \|(M + NT_2^n(b)P_n)^{-1}\| \cdot \|\hat{\psi} - \psi_n\| \tag{3.8}$$

$$+ \|(M + NT_2^n(b)P_n)^{-1}\| \cdot \|N\| \cdot \|(T_2(b) - T_2^n(b)P_n)(M + NT_2(b))^{-1}\hat{\psi}\|.$$

The first term on the right in (3.8) goes to zero since $\psi_n \to \hat{\psi}$ in norm. $T_2^n(b)P_n$ converges pointwise on X (X_μ) to $T_2(b)$ since this convergence is simply the convergence of Euler's method for an ordinary differential equation. This convergence will then be uniform on compact sets and so convergence of $\|(T_2(b) - T_2^n(b)P_n)(M + NT_2(b))^{-1}\hat{\psi}\|$ to zero follows.

Now by a similar analysis, $(M + NT_2^n(b)P_n)^{-1}P_n\hat{\psi}$ converges to $(M + NT_2(b))^{-1}\hat{\psi}$. Then from (3.2), it follows that $\{z_n\}$ itself is convergent, say to z. It thus follows again as above and using the fact that $T_1^n(b)P_n$ converges pointwise to $T_1(b)$ [6] that

$$z + (M + NT_2(b))^{-1}N(T_1(b)z - T_2(b)z) = (M + NT_2(b))^{-1}z,$$

i.e., z solves the boundary value problem (2.1). This completes the proof.

REMARKS. We note again that Theorem 3.2 is an extension of a result [7, Theorem 3.1] obtained by Waltman and Wong in two directions. First, we have considered L^2 initial functions where these results applied only to the continuous case, and second, we have given a constructive proof that leads to a finite difference method. It is possible to use the analysis given here to establish similar numerical results for the $C[-r, 0]$ case boundary value problems studied in [7]. Indeed, convergence can be obtained for a simpler P_n given in Webb [8]. We omit the details and statements of theorems as they are analogous to results given here.

4. An application. Consider the equation

$$\dot{x} = Ax(t) + \int_{-1}^{0} g(x(t + s))\,ds, \qquad 0 \leq t \leq 2, \tag{4.1}$$

where x: $[-1, 2] \to R^2$, $x = (x_1x_2)^T$, A is a two-by-two constant matrix, and g: $L^2[-1, 0] \to L^2[-1, 0]$ is measurable, bounded and Lipschitz continuous, and subject to the boundary conditions

$$
\begin{aligned}
x(t) &= \phi(t) = (\phi_1 \phi_2)^T, \qquad t \in [-1, 0], \text{ a.e.,} \\
&x_1(0) = 0, \quad x_2(2) = 0,
\end{aligned} \tag{4.2}
$$

where $\phi \in L^2[-1, 0]$. The equation $M\{\phi, h\} + NT_2(b)\{\phi, h\} = \{\psi, k\}$ in this example becomes $\dot{x} = Ax$, $0 \leqslant t \leqslant 2$, with

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ h_1 \\ h_2 \end{pmatrix}
+
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ (e^{2A}h)_2 \end{pmatrix}
=
\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ k_1 \\ k_2 \end{pmatrix} \tag{4.3}
$$

where $h = (h_1 h_2)^T$, $k = (k_1 k_2)^T$, $\psi = (\psi_1 \psi_2)^T$, $(e^{2A}h)_2$ denotes the second component of $e^{2A}h$ and the elements indicated by " $\cdot$ " play no explicit role. We assume $e^{2A} = (b_{ij})$ and $b_{22} \neq 0$. Then these equations have the unique solution $\phi_1 = \psi_1$, $\phi_2 = \psi_2$, $h_1 = k_1$ and h_2 solves $(e^{2A}h)_2 = k_2$ uniquely. Thus $(M + NT_2(2))^{-1}$ exists and is everywhere defined and the solution formula shows $(M + NT_2(2))^{-1}$ is bounded in either X or X_μ.

The equation

$$
M\{\phi_n, h_n\} + NT_2^n(2)P_n\{\phi_n, h_n\} = P_n\{\psi, k\} \tag{4.4}
$$

submits to a similar analysis. The numerical method (see §3) may be written as $x_i = x_{i-1} + t_n A x_{i-1}$ where $x_0 = h_n$ and x_i is taken to approximate x at it_n, $t_n = 1/n$. Thus $x_i = (I + t_n A)^i h_n$, and (4.4) becomes $\phi_n = P_n \psi$, $(h_n)_1 = k_1$, and $(h_n)_2$ must be such that

$$
((I + t_n A)^{2n} h)_2 = k_2. \tag{4.5}
$$

Since $(I + A/n)^{2n}$ approaches e^{2A} elementwise as $n \longrightarrow \infty$ and $b_{22} \neq 0$, equation (4.5) uniquely defines h_2 for all n large and so $(M + NT_2^n(2)P_n)^{-1}$ exists and is everywhere defined. The solution formula enables us to deduce that $\{(M + NT_2^n(2)P_n)^{-1}\}$ are uniformly bounded in norm.

We thus may apply Theorem 3.2 with $F_n = \int_{-1}^0 g(\cdot)$ and deduce the existence of solutions to equations (4.1)–(4.2). Since second order problems can be written as first order vector systems in the usual way with $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, the results of this example extend in one direction results of deNevers and Schmitt [1]. Here we have more general, but autonomous, functionals on the right-hand side and allow L^2 initial functions.

Finally, in this example, (4.4) gives an implementable numerical method. Let $t_n = 1/n$ and $z_i = t_n i$ for $i = -n, -n + 1, \ldots, 0, 1, 2, \ldots, 2n$. Equation

(4.4) can then be written as follows where x_i is taken to approximate x at z_i and $F_n \stackrel{\text{def}}{=} F = \int_{-1}^{0} g(\cdot)$:

$$\frac{x_i - x_{i-1}}{t_n} = Ax_i + \int_{-1}^{0} g(P_n x)\,ds, \qquad i = 1, \ldots, 2n, \tag{4.6}$$

where $P_n x$ is the piecewise constant function defined on $[z_{i-n-1}, z_{i-1})$ with value at z_{i-j} of x_{i-j}, $x_s = (P_n\phi)(z_s)$ for $z_s < 0$, the first component of x_0 is zero and the second component of x_{2n} is zero. This leads to $4n$ scalar equations in $4n$ unknowns. One could also implement this as a shooting method by doing a search on $x_2(0)$ and solving (4.6) as an initial value problem to see if the computed $x_2(2) = 0$.

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