

INNER PRODUCT MODULES ARISING FROM COMPACT AUTOMORPHISM GROUPS OF VON NEUMANN ALGEBRAS

BY

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ABSTRACT. Let M be a von Neumann algebra of operators on a separable Hilbert space H , and G a compact, strong-operator continuous group of $*$ -automorphisms of M . The action of G on M gives rise to a faithful, ultra-weakly continuous conditional expectation of M on the subalgebra $N = \{A \in M: g(A) = A \forall g \in G\}$, which in turn makes M into an inner product module over N . The inner product module M may be "completed" to yield a self-dual inner product module \bar{M} over N ; our most general result states that the W^* -algebra $A(\bar{M})$ of bounded N -module maps of \bar{M} into itself is isomorphic to a restriction of the crossed product $M \times G$ of M by G . When G is compact abelian, we give conditions for $A(\bar{M})$ and $M \times G$ to be isomorphic and show, among other things, that if G acts faithfully on M , then $M \times G$ is a factor if and only if N is a factor. As an example, we discuss certain compact abelian automorphism groups of group von Neumann algebras.

1. **Crossed products.** Let G be a locally compact group (with left Haar measure dg) acting as a strong-operator continuous automorphism group on M . That is, there is a homomorphism $g \rightarrow g(\cdot)$ of G into the group of $*$ -automorphisms of M such that for each $A \in M$, the map $g \rightarrow g(A)$ of G into M is continuous with respect to the strong-operator topology on M . We recall the construction of the crossed product of M by G as set forth by M. Takesaki in [8].

Let $L^2(G, H)$ denote the set of functions $\Phi: G \rightarrow H$ satisfying (i) for each $\xi \in H$, the complex-valued function $g \rightarrow (\Phi(g), \xi)$ on G is measurable; and (ii) $\int_G \|\Phi(g)\|^2 dg < \infty$. $L^2(G, H)$ is then a Hilbert space with inner product $(\Phi, \Psi) = \int_G (\Phi(g), \Psi(g)) dg$. For each $A \in M$, define a bounded operator \tilde{A} on $L^2(G, H)$ by $(\tilde{A}\Phi)(g) = g^{-1}(A)(\Phi(g))$; for each $g \in G$, let L_g be the unitary operator on $L^2(G, H)$ defined by $L_g\Phi(h) = \Phi(g^{-1}h)$. We let $M \times G$, the crossed product of M by G , denote the von Neumann algebra of operators on $L^2(G, H)$ generated by the operators \tilde{A} ($A \in M$) and L_g ($g \in G$).

Suppose now that G is compact (with normalized Haar measure dg). For $A \in M$, $\xi \in H$, we define the function $A \odot \xi \in L^2(G, H)$ by $(A \odot \xi)(g) = g^{-1}(A)\xi$ for $g \in G$. Notice that $\tilde{B}(A \odot \xi) = BA \odot \xi$ and $L_g(A \odot \xi) = g(A) \odot \xi$

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$\forall B \in M, g \in G$; hence the closed linear span $M \odot H$ of $\{A \odot \xi: A \in M, \xi \in H\}$ is an invariant subspace of $L^2(G, H)$ for $M \times G$. It is also invariant for the commutant $(M \times G)'$ of $M \times G$. To see this, take $T \in (M \times G)'$. For $\xi \in H$, we have $T(I \odot \xi) = TL_g(I \odot \xi) = L_gT(I \odot \xi) \forall g \in G$, whence it follows that $T(I \odot \xi) = I \odot \eta$ for some $\eta \in H$. For $A \in M$, we then have $T(A \odot \xi) = \widetilde{TA}(I \odot \xi) = \widetilde{AT}(I \odot \xi) = \widetilde{A}(I \odot \eta) = A \odot \eta$. We record this observation below.

PROPOSITION 1.2. *If G is compact, the projection of $L^2(G, H)$ on $M \odot H$ belongs to the center of $M \times G$.*

In general, of course, $M \odot H$ may be a proper subspace of $L^2(G, H)$. For a simple example, let M be the algebra of complex 2×2 matrices acting on 2-dimensional Hilbert space H , G the group of unitary 2×2 matrices with determinant 1, and let G act as inner automorphisms on M in the obvious way (i.e. $V(A) = VAV^*$ for $A \in M, V \in G$). Easy calculations show that for each $\xi \in H$, the map $\Phi \in L^2(G, H)$ defined by $\Phi(V) = V^*\xi$ ($V \in G$) is orthogonal to $M \odot H$.

2. Inner product modules. Let N be a von Neumann algebra of operators on H , and let X be a right N -module with vector space structure compatible with that of N . An N -valued inner product on X is a conjugate-bilinear map $\langle \cdot, \cdot \rangle: X \times X \rightarrow N$ satisfying

- (i) $\langle x, x \rangle \geq 0 \quad \forall x \in X$;
- (ii) $\langle x, x \rangle = 0 \quad \text{only if } x = 0$;
- (iii) $\langle x, y \rangle^* = \langle y, x \rangle \quad \forall x, y \in X$;
- (iv) $\langle x \cdot A, y \rangle = \langle x, y \rangle A \quad \forall x, y \in X, A \in N$

(where $x \cdot A$ denotes the right action of $A \in N$ on $x \in X$). A right N -module equipped with an N -valued inner product will be called an *inner product module* over N . Notice that if X is an inner product module over N , the set $\{\langle x, y \rangle: x, y \in X\}$ spans a two-sided ideal of N ; we let $\overline{\langle X, X \rangle}$ denote the ultraweak closure of this ideal. It is shown in [3] that X is normed by $\|x\|_X = \|\langle x, x \rangle\|^{1/2}$ ($x \in X$). We write X' for the set of N -module maps from X to N which are bounded with respect to $\|\cdot\|_X$. We make X' into a right N -module by defining scalar multiplication and right action of N on X' by $(\lambda\tau)(x) = \overline{\lambda}\tau(x)$ and

$$(\tau \cdot A)(x) = A^*\tau(x) \quad (\lambda \in \mathbb{C}, \tau \in X', x \in X, A \in N).$$

We shall regard X as a submodule of X' by identifying $x \in X$ with the map $\langle \cdot, x \rangle: X \rightarrow N$. We call X *self-dual* if $X = X'$. The module X' may be thought of as the “completion” of X in the sense that the inner product on X can be extended to an N -valued inner product on X' in such a way as to make X' self-

dual, with the extended inner product satisfying $\langle x, \tau \rangle = \tau(x) \forall x \in X, \tau \in X'$ (3.2 of [3]).

Suppose now that Y is a self-dual inner product module over N . Each bounded module map $T: Y \rightarrow Y$ possesses an adjoint $T^*: Y \rightarrow Y$ such that $\langle T^*x, y \rangle = \langle x, Ty \rangle \forall x, y \in Y$.

The $*$ -algebra $A(Y)$ consisting of all such T is a W^* -algebra; a norm-bounded net $\{T_\alpha\}$ in $A(Y)$ converges ultraweakly to $T \in A(Y)$ if and only if $\langle T_\alpha x, y \rangle \rightarrow \langle Tx, y \rangle$ ultraweakly in $N \forall x, y \in Y$ (3.10 of [3]). In [4], M. A. Rieffel constructs a faithful normal $*$ -representation of $A(Y)$ on Hilbert space as follows. Let $Y \otimes H$ be the algebraic tensor product of Y with H . Define $[\cdot, \cdot]: Y \otimes H \times Y \otimes H \rightarrow \mathbb{C}$ on elementary tensors by $[x \otimes \xi, y \otimes \eta] = \langle x, y \rangle \xi, \eta$ ($x, y \in Y, \xi, \eta \in H$). Arguing as in 1.7 of [4] (or using 6.1 of [3] to see that for $x_1, x_2, \dots, x_n \in Y$, the $n \times n$ operator matrix $[x_i, x_j]$ is nonnegative) one shows that $[\cdot, \cdot]$ extends to a well-defined positive semidefinite conjugate-bilinear form on $Y \otimes H$. Let $Z = \{w \in Y \otimes H: [w, w] = 0\}$, so Z is a subspace of $Y \otimes H$ and $K_0 = (Y \otimes H)/Z$ is a pre-Hilbert space with inner product $(w_1 + Z, w_2 + Z) = [w_1, w_2]$.

REMARK 2.1. For $x \in Y, B \in N, \xi \in H$, direct computation shows that $(x \cdot B) \otimes \xi + Z = x \otimes B\xi + Z$.

Let K be the Hilbert space completion of K_0 . For $T \in A(Y)$, define a linear map $\theta_0(T): Y \otimes H \rightarrow Y \otimes H$ by

$$\theta_0(T)(x \otimes \xi) = Tx \otimes \xi \quad (x \in Y, \xi \in H).$$

It is shown in 5.3 of [4] that $\theta_0(T)$ induces a bounded linear map $\theta(T): K \rightarrow K$ satisfying $\theta(T)(x \otimes \xi + Z) = Tx \otimes \xi + Z$. One checks without difficulty that θ is a faithful normal $*$ -representation of $A(Y)$ on K .

For $x, y \in Y$, define $x \otimes y: Y \rightarrow Y$ by $x \otimes y(w) = x \cdot \langle w, y \rangle$ ($w \in Y$). One checks that $x \otimes y \in A(Y)$ with $(x \otimes y)^* = y \otimes x$ and $T(x \otimes y) = Tx \otimes y \forall T \in A(Y)$. It follows that $\{x \otimes y: x, y \in Y\}$ spans a two-sided ideal of $A(Y)$.

PROPOSITION 2.2. If $\overline{\langle Y, Y \rangle} = N$, then $\{x \otimes y: x, y \in Y\}$ spans an ultraweakly dense two-sided ideal of $A(Y)$.

PROOF. If the span of $\{x \otimes y: x, y \in Y\}$ is not ultraweakly dense in $A(Y)$, there is a nonzero projection $P \in A(Y)$ such that $P(x \otimes y) = 0 \forall x, y \in Y$. We then have $\langle Px, x \rangle \langle y', y \rangle = \langle P(x \otimes y)y', x \rangle = 0 \forall x, y, y' \in Y$. Since $\overline{\langle Y, Y \rangle} = N$, this forces $\langle Px, x \rangle = 0 \forall x \in X$ and hence $P = 0$, a contradiction.

Now let X be an inner product module, not necessarily self-dual, over N with $\overline{\langle X, X \rangle} = N$. In the sequel, we will need to know that the set $\{x \otimes y: x, y \in X\}$ spans an ultraweakly dense $*$ -subalgebra of $A(X')$. (Here, as elsewhere, we are regarding X as a submodule of X' , so $x \otimes y(\tau) = x \cdot \langle \tau, y \rangle = x \cdot \tau(y)^*$ for

$\tau \in X'$.) It is clear that the span of this set is a *-subalgebra of $A(X')$; unfortunately, the fact that it is ultraweakly dense seems not to follow directly from 2.2, but rather depends on certain details of the construction of the extended inner product in [3]. We will require the following lemma.

LEMMA 2.3. *For each $\tau \in X'$, there is a net $\{x_\alpha\}$ in X such that $\langle \tau - x_\alpha, \tau - x_\alpha \rangle \rightarrow 0$ ultraweakly in N .*

PROOF. Let $s = \{f_1, f_2, \dots, f_n\}$ be a finite set of normal positive linear functionals on N , and set $f = f_1 + f_2 + \dots + f_n$. If we let $Z_f = \{x \in X: f(\langle x, x \rangle) = 0\}$, one checks that Z_f is a subspace of X and that X/Z_f is a pre-Hilbert space with inner product $(x + Z_f, y + Z_f) = f(\langle x, y \rangle)$. Since

$$\tau(x)^* \tau(x) \leq \|\tau\|^2 \langle x, x \rangle \quad \forall x \in X$$

by 2.8 of [3], the map $x + Z_f \rightarrow f(\tau(x))$ is a well-defined bounded linear functional on X/Z_f , so there is a vector τ_f in the Hilbert space completion of X/Z_f such that

$$(x + Z_f, \tau_f) = f(\tau(x)) \quad \forall x \in X.$$

Let $x_s \in X$ be such that $(x_s + Z_f - \tau_f, x_s + Z_f - \tau_f) \leq 1/n$. We have $f(\langle \tau, \tau \rangle) = (\tau_f, \tau_f)$ by the construction of the extended inner product on X' in 3.2 of [3], so $f(\langle \tau - x_s, \tau - x_s \rangle) \leq 1/n$ and hence $f_i(\langle \tau - x_s, \tau - x_s \rangle) \leq 1/n$ for $1 \leq i \leq n$. It is now clear that the net $\{x_s\}$ (indexed by the inclusion-directed family of finite sets of normal positive linear functionals on N) has the required property.

Let $\theta: A(X') \rightarrow B(K)$ (the algebra of bounded operators on K) be the *-representation of $A(X')$ on the Hilbert space K constructed at the beginning of this section.

LEMMA 2.4. *Let $\tau \in X'$ and let $\{x_\alpha\}$ be as in 2.3. Then for each $\xi \in H$, the net $\{x_\alpha \otimes \xi + Z\}$ converges to $\tau \otimes \xi + Z$ in the norm of K . Moreover, for $\psi, \psi' \in X'$, the net $\{\psi' \otimes \psi(x_\alpha)^* \xi + Z\}$ converges to $\psi' \otimes \langle \psi, \tau \rangle \xi + Z$ weakly in K .*

PROOF. The first statement follows from the identity

$$[(\tau - x_\alpha) \otimes \xi, (\tau - x_\alpha) \otimes \xi] = (\langle \tau - x_\alpha, \tau - x_\alpha \rangle \xi, \xi).$$

If $\psi \in X'$ and f is a normal positive linear functional on N , we have

$$|f(\langle \psi, x_\alpha \rangle) - f(\langle \psi, \tau \rangle)| \leq f(\langle \psi, \psi \rangle)^{1/2} f(\langle \tau - x_\alpha, \tau - x_\alpha \rangle)^{1/2},$$

so $\langle \psi, x_\alpha \rangle = \psi(x_\alpha)^* \rightarrow \langle \psi, \tau \rangle$ ultraweakly in N and hence $\psi(x_\alpha)^* \xi \rightarrow \langle \psi, \tau \rangle \xi$ weakly in H . For $\psi' \in X'$, the map $\eta \rightarrow \psi' \otimes \eta + Z$ is a bounded linear transformation from H into K , so $\psi' \otimes \psi(x_\alpha)^* \xi + Z \rightarrow \psi' \otimes \langle \psi, \tau \rangle \xi + Z$ weakly in K .

PROPOSITION 2.5. *Let X be an inner product module over N such that $\overline{\langle X, X \rangle} = N$. Then $\{x \otimes y: x, y \in X\}$ spans an ultraweakly dense $*$ -subalgebra of $A(X')$.*

PROOF. By 2.2, it will suffice to show that if $W \in B(K)$ commutes with all of the operators $\theta(x \otimes y)$ ($x, y \in X$), then W commutes with all of the operators $\theta(\tau \otimes \psi)$ ($\tau, \psi \in X'$). We first show that W commutes with $\theta(\tau \otimes y)$ $\forall \tau \in X', y \in X$. Let $\{x_\alpha\}$ be a net in X approximating τ as in 2.3. For $\psi_1, \psi_2 \in X', \xi_1, \xi_2 \in H$, we have

$$\begin{aligned} & (W\theta(\tau \otimes y)(\psi_1 \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z) \\ &= (W((\tau \cdot \psi_1(y)^*) \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z) \\ &= (W(\tau \otimes \psi_1(y)^*\xi_1 + Z), \psi_2 \otimes \xi_2 + Z) \quad (\text{by 2.1}) \\ &= \lim_\alpha (W(x_\alpha \otimes \psi_1(y)^*\xi_1 + Z), \psi_2 \otimes \xi_2 + Z) \quad (\text{by 2.4}) \\ &= \lim_\alpha (W\theta(x_\alpha \otimes y)(\psi_1 \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z) \\ &= \lim_\alpha (\theta(x_\alpha \otimes y)W(\psi_1 \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z) \\ &= \lim_\alpha (W(\psi_1 \otimes \xi_1 + Z), \theta(y \otimes x_\alpha)(\psi_2 \otimes \xi_2 + Z)) \\ &= \lim_\alpha (W(\psi_1 \otimes \xi_1 + Z), y \otimes \psi_2(x_\alpha)^*\xi_2 + Z) \quad (\text{by 2.1}) \\ &= (W(\psi_1 \otimes \xi_1 + Z), y \otimes \langle \psi_2, \tau \rangle \xi_2 + Z) \quad (\text{by 2.4}) \\ &= (\theta(\tau \otimes y)W(\psi_1 \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z). \end{aligned}$$

We conclude that $\theta(\tau \otimes y)W = W\theta(\tau \otimes y) \forall \tau \in X', y \in Y$. In exactly the same way, we can now show that W commutes with all of the operators $\theta(\tau \otimes \psi)$ ($\tau, \psi \in X'$), which concludes the proof.

In [5], Rieffel investigated the notion of “Morita equivalence” among W^* -algebras. Roughly speaking, two W^* -algebras are *Morita equivalent* if their categories of nondegenerate normal $*$ -representations on Hilbert space are equivalent via a linear functor which is “normal” in an appropriate sense. One of the main results of [5] states that two W^* -algebras N_1 and N_2 are Morita equivalent if and only if there is a self-dual inner product module Y over N_1 with $\overline{\langle Y, Y \rangle} = N_1$ such that $A(Y)$ is isomorphic to N_2 . We conclude this section with a discussion of the relationship between N and $A(Y)$, where Y is a self-dual inner product module over N , which may be thought of as giving a characterization of Morita equivalence in terms of tensor products with type I factors.

Let J be an index set, and let N_j be a copy of N (regarded as a self-dual inner product module over N) for each $j \in J$. Form $X = \text{UDS}\{N_j: j \in J\}$ as in §3 of [3], so X is a self-dual inner product module over N .

PROPOSITION 2.6. $A(X)$ and $N \otimes B(l^2(J))$ are isomorphic.

PROOF. This follows routinely from 1.22.14 of [6], where for each $j \in J$ we let V_j be the partial isometry in $A(X)$ which takes a J -tuple $\{B_k\}_{k \in J} \in X$ to the J -tuple with B_j in the j_0 -slot and 0's elsewhere ($j_0 \in J$ fixed).

PROPOSITION 2.7. Let Y be a self-dual inner product module over N such that $\langle Y, Y \rangle = N$. There is a Hilbert space H_1 and a projection $E \in N \otimes B(H_1)$ with central cover $I \otimes I_{H_1}$ such that $A(Y)$ and $E(N \otimes B(H_1))E$ are isomorphic.

PROOF. By 3.12 of [3], there is an index set J and a collection $\{P_j: j \in J\}$ of (not necessarily distinct) nonzero projections in N such that Y and UDS $\{P_j N: j \in J\}$ are isomorphic as inner product modules over N . Hence, we may regard Y as a submodule of the inner product module X defined above. Define a projection E in $A(X)$ by $E\{B_j\}_{j \in J} = \{P_j B_j\}_{j \in J}$ for $\{B_j\}_{j \in J} \in X$; E is the projection of X onto Y . It is clear that $A(Y)$ and $EA(X)E$ are isomorphic. If Q is a central projection in $A(X)$ majorizing E , then $Qy = y \forall y \in Y$ and hence for $y_1, y_2 \in Y$ and $x \in X$, we have $Qx \cdot \langle y_1, y_2 \rangle = Q(x \otimes y_2)(y_1) = (x \otimes y_2)(Qy_1) = (x \otimes y_2)(y_1) = x \cdot \langle y_1, y_2 \rangle$. Since $\langle Y, Y \rangle = N$, this implies that $Qx = x \forall x \in X$, showing that the central cover of E in $A(X)$ is the identity map. To conclude the proof, take $H_1 = l^2(J)$ and apply 2.6.

3. Compact automorphism groups. In this section, M will be a von Neumann algebra of operators on a separable Hilbert space H , and G a compact group acting as a strong-operator continuous group of *-automorphisms of M . Let $N = \{A \in M: g(A) = A \forall g \in G\}$ be the fixed-point subalgebra for G . Define $\Gamma: M \rightarrow M$ by $\Gamma(A) = \int_G g(A) dg$ for $A \in M$. (All integrals of this sort are taken in the sense of §1 of [1]. For $A \in M$, $\Gamma(A)$ is the unique operator in M such that $f(\Gamma(A)) = \int_G f(g(A)) dg$ for all ultraweakly continuous linear functionals f on M .) It is clear that Γ is an ultraweakly continuous faithful conditional expectation of M on N . If we define $\langle \cdot, \cdot \rangle: M \times M \rightarrow N$ by $\langle A, B \rangle = \Gamma(B^*A)$, it follows that $\langle \cdot, \cdot \rangle$ is an N -valued inner product on M (viewed as a right N -module). In general, M is not complete with respect to the norm obtained from this inner product, much less self-dual, but we can form the self-dual inner product module \overline{M} of bounded N -module maps of M into N . Here, "bounded" means "bounded with respect to the norm obtained from the inner product". We write \overline{M} instead of M' to avoid any possible confusion with the commutant of M . By 2.8 of [3], \overline{M} consists precisely of all (complex -) linear $\tau: M \rightarrow N$ for which there is a real $r \geq 0$ such that $\tau(A)^* \tau(A) \leq r \langle A, A \rangle \forall A \in M$. (For such maps, it turns out that N -linearity is automatic.)

Let K be the Hilbert space constructed from the tensor product $\overline{M} \otimes H$ as in §2, and θ the associated faithful normal *-representation of $A(\overline{M})$ on K .

By 2.4, the subspace of K spanned by $\{A \otimes \xi + Z : A \in M, \xi \in H\}$ is dense in K . For $A_1, A_2, \dots, A_n \in M$ and $\eta_1, \eta_2, \dots, \eta_n \in H$, we have

$$\begin{aligned} \left\| \sum_{i=1}^n A_i \otimes \eta_i + Z \right\|^2 &= \sum_{i,j} \langle A_i, A_j \rangle \eta_i, \eta_j \\ &= \sum_{i,j} \int_G (g(A_j^* A_i) \eta_i, \eta_j) dg \\ &= \sum_{i,j} \int_G (g^{-1}(A_i) \eta_i, g^{-1}(A_j) \eta_j) dg \\ &= \left\| \sum_{i=1}^n A_i \odot \eta_i \right\|^2 \end{aligned}$$

We conclude that there is an isometry V of K onto the subspace $M \odot H$ of $L^2(G, H)$ defined in §1 such that $V(A \otimes \eta + Z) = A \odot \eta \forall A \in M, \eta \in H$. If we define $\rho : A(\overline{M}) \rightarrow B(M \odot H)$ by $\rho(T) = V\theta(T)V^*$ for $T \in A(\overline{M})$, ρ will be a faithful normal $*$ -representation of $A(\overline{M})$ on $M \odot H$.

THEOREM 3.1. $\rho(A(\overline{M})) = M \times G|_{M \odot H}$.

PROOF. Notice that if $T \in A(\overline{M})$ is such that $TM \subseteq M$, then $\rho(T)(C \odot \eta) = T(C) \odot \eta \forall C \in M, \eta \in H$. For $A, B \in M$ we have $g(B^* A^* A B) \leq \|A\|^2 g(B^* B) \forall g \in G$, so $\langle AB, AB \rangle \leq \|A\|^2 \langle B, B \rangle$. It follows that for each $A \in M$, the map $B \rightarrow AB$ of M into M is a bounded N -module map, and hence extends uniquely to a map $m_A \in A(\overline{M})$. We have $\rho(m_A)(C \odot \eta) = AC \odot \eta = \tilde{A}(C \odot \eta) \forall C \in M, \eta \in H$, so $\rho(m_A) = \tilde{A}|_{M \odot H}$. Likewise each $g \in G$ gives rise to a map $k_g \in A(\overline{M})$ such that $k_g(B) = g(B) \forall B \in M$. We have $\rho(k_g)(C \odot \eta) = g(C) \odot \eta = L_g(C \odot \eta) \forall C \in M, \eta \in H$, so $\rho(k_g) = L_g|_{M \odot H}$. We conclude that $M \times G|_{M \odot H} \subseteq \rho(A(\overline{M}))$.

For the reverse inclusion, consider the projection $I \otimes I \in A(\overline{M})$. We have $\rho(I \otimes I)(C \odot \eta) = IC, I \odot \eta = \Gamma(C) \odot \eta = I \odot \Gamma(C)\eta$ (since $g(\Gamma(C)) = \Gamma(C) \forall g \in G) \forall C \in M, \eta \in H$, whence it follows that $\rho(I \otimes I)$ is the projection of $M \odot H$ onto the closed subspace $I \odot H = \{I \odot \xi : \xi \in H\}$. If S is an operator on $M \odot H$ commuting with $M \times G|_{M \odot H}$, then $S(I \odot \xi) = SL_g(I \odot \xi) = L_g S(I \odot \xi) \forall \xi \in H, g \in G$; we conclude that $S(I \odot H) \subseteq I \odot H$, which shows that $\rho(I \otimes I) \in M \times G|_{M \odot H}$. For $A, B, C \in M, \eta \in H$, we compute that

$$\begin{aligned} \rho(A \otimes B)(C \odot \eta) &= A\langle C, B \rangle \odot \eta = A\Gamma(B^* C) \odot \eta \\ &= \tilde{A}(\Gamma(B^* C) \odot \eta) = \tilde{A}\rho(I \otimes I)\tilde{B}^*(C \odot \eta). \end{aligned}$$

Hence, $\rho(A \otimes B) \in M \times G|_{M \odot H} \forall A, B \in M$. By 2.5, then, $\rho(A(\overline{M})) \subseteq M \times G|_{M \odot H}$, which concludes the proof.

COROLLARY 3.2. *Let G be a compact, strong-operator continuous group*

of automorphisms of M , with fixed-point algebra N . Then $M \times G$ is a factor if and only if N is a factor and $M \odot H = L^2(G, H)$.

PROOF. This follows from 3.1, 1.2, and the fact that Morita equivalent W^* -algebras have isomorphic centers (8.1 of [5]).

REMARK 3.3. If $M \odot H = L^2(G, H)$, then G must act faithfully on M , i.e. the homomorphism $g \rightarrow g(\cdot)$ of G into the group of $*$ -automorphisms of M must be injective. (If $g \in G$ is such that $g(A) = A \ \forall A \in M$, then $L_g \Phi = \Phi \ \forall \Phi \in M \odot H$, which forces $g = e$, the identity of G , if $M \odot H = L^2(G, H)$.) The example at the end of §1 shows that it is possible for G to act faithfully on M with N a factor without having $M \odot H = L^2(G, H)$.

4. Compact abelian automorphism groups. In this section we impose the additional requirement that the compact group G of automorphisms of M be abelian. One of our main purposes will be to shed some light in the abelian case on the condition $M \odot H = L^2(G, H)$ which insures the isomorphism of $A(\bar{M})$ with $M \times G$.

We let \hat{G} denote the dual group of G . For each character $\lambda \in \hat{G}$, define a subspace M_λ of M and a linear map $\Gamma_\lambda: M \rightarrow M$ by

$$M_\lambda = \{A \in M: g(A) = \overline{\lambda(g)}A \ \forall g \in G\},$$

$$\Gamma_\lambda(A) = \int_G \lambda(g)g(A) dg \quad (A \in M).$$

(It is not hard to see that, in the terminology of [1], M_λ is the *spectral subspace* of M corresponding to $\{\lambda\} \subseteq \hat{G}$ and that the *spectrum* of the action of G on M is precisely $\{\lambda \in \hat{G}: M_\lambda \neq 0\}$.)

REMARK 4.1. The following properties of the subspaces M_λ and the maps Γ_λ are all easily checked:

- (i) each M_λ is a two-sided N -module;
- (ii) $\langle A, B \rangle = B^*A$ for $A, B \in M_\lambda$;
- (iii) $M_\lambda^* = M_{\bar{\lambda}}$;
- (iv) $\langle M_{\lambda_1}, M_{\lambda_2} \rangle = 0$ for $\lambda_1 \neq \lambda_2$;
- (v) $M_{\lambda_1} M_{\lambda_2} \subseteq M_{\lambda_1 \lambda_2}$;
- (vi) Γ_λ is a projection of M onto M_λ and a bounded N -module map;
- (vii) $\Gamma_{\lambda_1} \Gamma_{\lambda_2} = 0$ for $\lambda_1 \neq \lambda_2$.

Notice in particular that the subspace of M spanned by the M_λ 's is a $*$ -subalgebra of M .

PROPOSITION 4.2. *The subspace spanned by $\{M_\lambda: \lambda \in \hat{G}\}$ is ultraweakly dense in M .*

PROOF. Let f be an ultraweakly continuous linear functional on M such that $f(M_\lambda) = 0 \ \forall \lambda \in \hat{G}$. Take $A \in M$ and define a complex-valued continuous

function ϕ on G by $\phi(g) = f(g^{-1}(A))$. For each $\lambda \in \hat{G}$, we have

$$\begin{aligned} \hat{\phi}(\lambda) &= \int_G \overline{\lambda(g)} f(g^{-1}(A)) dg \\ &= \int_G \lambda(g) f(g(A)) dg = f(\Gamma_\lambda(A)) = 0, \end{aligned}$$

so $\phi = 0$ and in particular $f(A) = 0$.

As noted above, each M_λ is an inner product module over N with N -valued inner product $\langle A, B \rangle = B^*A$ ($A, B \in M_\lambda$).

PROPOSITION 4.3. *Each M_λ is self-dual.*

PROOF. Let $\tau: M_\lambda \rightarrow N$ be a bounded N -module map and for simplicity assume that $\|\tau\| \leq 1$. We must produce an operator $B \in M_\lambda$ such that $\tau(A) = B^*A \forall A \in M_\lambda$. By 2.8 of [3] we have $\tau(A)^*\tau(A) \leq A^*A \forall A \in M_\lambda$, and it follows from 6.1 of [3] that for $A_1, A_2, \dots, A_n \in M_\lambda$, the matrices $[\tau(A_i)^*\tau(A_j)]$ and $[A_i^*A_j]$ in $N_{(n)}$, the W^* -algebra of $n \times n$ matrices with entries in N , satisfy

$$[\tau(A_i)^*\tau(A_j)] \leq [A_i^*A_j].$$

Hence for $\xi_1, \xi_2, \dots, \xi_n \in H$, we have

$$\begin{aligned} \left\| \sum_{i=1}^n \tau(A_i)\xi_i \right\|^2 &= \sum_{i,j} (\tau(A_i)^*\tau(A_j)\xi_j, \xi_i) \\ &\leq \sum_{i,j} (A_i^*A_j\xi_j, \xi_i) = \left\| \sum_{i=1}^n A_i\xi_i \right\|^2, \end{aligned}$$

so there is a bounded linear map $B_0: [M_\lambda H] \rightarrow H$ (where $[M_\lambda H]$ is the closed linear span of $\{A\xi: A \in M_\lambda, \xi \in H\}$) satisfying $B_0A\xi = \tau(A)\xi \forall A \in M_\lambda, \xi \in H$. Extend B_0 to a bounded operator on H by setting $B_0([M_\lambda H]^\perp) = 0$. Take $T \in M'$. Clearly, $[M_\lambda H]$ and $[M_\lambda H]^\perp$ are invariant subspaces for T , so $TB_0\eta = 0 = B_0T\eta$ for $\eta \in [M_\lambda H]^\perp$, and

$$TB_0A\xi = T\tau(A)\xi = \tau(A)T\xi = B_0AT\xi = B_0TA\xi \quad \forall A \in M_\lambda, \xi \in H.$$

We conclude that $B_0 \in M$. Set $B = \Gamma_\lambda(B_0^*)$, so $B \in M_\lambda$. For $A \in M_\lambda$, we have

$$B^*A = \int_G \overline{\lambda(g)} g(B_0)A dg = \int_G g(B_0A) dg = \tau(A),$$

since $B_0A = \tau(A)$ and $g(\tau(A)) = \tau(A) \forall g \in G$.

Form the self-dual inner product module $\text{UDS}\{M_\lambda: \lambda \in \hat{G}\}$ over N as in §3 of [3].

PROPOSITION 4.4. *\bar{M} and $\text{UDS}\{M_\lambda: \lambda \in \hat{G}\}$ are isomorphic as inner product modules over N .*

PROOF. For each $\tau \in \bar{M}$ and $\lambda \in \hat{G}$, 4.3 gives a unique operator $B(\tau)_\lambda \in M_\lambda$ such that $\tau(A) = B(\tau)_\lambda^* A \quad \forall A \in M_\lambda$. For $\lambda_1, \lambda_2, \dots, \lambda_n$ distinct characters in \hat{G} , we have

$$\begin{aligned} \left\| \sum_{i=1}^n B(\tau)_{\lambda_i}^* B(\tau)_{\lambda_i} \right\|^2 &= \left\| \tau \left(\sum_{i=1}^n B(\tau)_{\lambda_i} \right) \right\|^2 \\ &\leq \|\tau\|^2 \left\| \left\langle \sum_{i=1}^n B(\tau)_{\lambda_i}, \sum_{i=1}^n B(\tau)_{\lambda_i} \right\rangle \right\| \\ &= \|\tau\|^2 \left\| \sum_{i=1}^n B(\tau)_{\lambda_i}^* B(\tau)_{\lambda_i} \right\|, \end{aligned}$$

the last equality holding by virtue of (ii) and (iv) of 4.1. Hence, $\|\sum_{i=1}^n B(\tau)_{\lambda_i}^* B(\tau)_{\lambda_i}\| \leq \|\tau\|^2$. We conclude that the map $\tau \rightarrow \{B(\tau)_\lambda\}_{\lambda \in \hat{G}}$ is a bounded N -module map of \bar{M} into $\text{UDS}\{M_\lambda : \lambda \in \hat{G}\}$. If τ has the form $\langle \cdot, A \rangle$ for some $A \in M$, it is clear that $B(\tau)_\lambda = \Gamma_\lambda(A) \quad \forall \lambda \in \hat{G}$, so in consequence the \hat{G} -tuple $\{\Gamma_\lambda(A)\}_{\lambda \in \hat{G}} \in \text{UDS}\{M_\lambda : \lambda \in \hat{G}\} \forall A \in M_\lambda$. Let f be a normal positive linear functional on N , and take $A \in M$, $\tau \in \bar{M}$. Since $\tau(B)^* \tau(B) \leq \|\tau\|^2 B^* B \quad \forall B \in M$ (and hence $\|\tau(B)\xi\| \leq \|\tau\| \|B\xi\| \quad \forall B \in M, \xi \in H$), it follows that τ is continuous with respect to the strong-operator topology of M . Hence, the function ϕ on G defined by $\phi(g) = f(\tau(g(A)))$ is continuous. We have

$$\begin{aligned} \hat{\phi}(\lambda) &= \int_G \overline{\lambda(g)} f(\tau(g(A))) dg = f(\tau(\Gamma_\lambda(A))) \\ &= f(B(\tau)_\lambda^* \Gamma_\lambda(A)) \quad \forall \lambda \in \hat{G}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\lambda \in \hat{G}} |\hat{\phi}(\lambda)| &\leq \sum_{\lambda \in \hat{G}} f(B(\tau)_\lambda^* B(\tau)_\lambda)^{1/2} f(\Gamma_\lambda(A)^* \Gamma_\lambda(A))^{1/2} \\ &\leq f \left(\sum_{\lambda \in \hat{G}} B(\tau)_\lambda^* B(\tau)_\lambda \right)^{1/2} f \left(\sum_{\lambda \in \hat{G}} \Gamma_\lambda(A)^* \Gamma_\lambda(A) \right)^{1/2} \\ &< \infty \text{ because } \{B(\tau)_\lambda\}, \{\Gamma_\lambda(A)\} \in \text{UDS}\{M_\lambda : \lambda \in \hat{G}\}. \end{aligned}$$

We conclude that

$$\begin{aligned} f(\tau(A)) &= \sum_{\lambda \in \hat{G}} \hat{\phi}(\lambda) \\ &= \sum_{\lambda \in \hat{G}} f(B(\tau)_\lambda^* \Gamma_\lambda(A)) \\ &= f(\langle \{\Gamma_\lambda(A)\}, \{B(\tau)_\lambda\} \rangle_\lambda). \end{aligned}$$

This shows that $\tau(A) = \langle \{\Gamma_\lambda(A)\}, \{B(\tau)_\lambda\} \rangle \quad \forall \tau \in \bar{M}, A \in M$ and it follows from the construction of the extended inner product of \bar{M} that $\langle \tau, \psi \rangle = \langle \{B(\tau)_\lambda\},$

$\{B(\psi)\} \forall \tau, \psi \in \overline{M}$. Hence the map $\tau \rightarrow \{B(\tau)_\lambda\}$ is an inner product module isomorphism of \overline{M} into $\text{UDS}\{M_\lambda: \lambda \in \hat{G}\}$. But it is clear that if $\{B_\lambda\} \in \text{UDS}\{M_\lambda: \lambda \in \hat{G}\}$, then $\tau(A) = \langle \{\Gamma_\lambda(A)\}, \{B_\lambda\} (A \in M)$ defines a $\tau \in \overline{M}$ such that $B(\tau)_\lambda = B_\lambda \forall \lambda \in \hat{G}$. This completes the proof.

For $\lambda \in \hat{G}$, let $L(M_\lambda) = \{A \in M: AM_\lambda = 0\}$ and let Q_λ be the left-annihilating projection of M_λ in M , i.e. the projection in M such that $L(M_\lambda) = MQ_\lambda$.

PROPOSITION 4.5. Q_λ is the projection of H on $[M_\lambda H]^\perp$ and belongs to the center of N .

PROOF. Let P_λ be the projection of H on $[M_\lambda H]$. Since this latter subspace is invariant under M' , we have $P_\lambda \in M$, and $I - P_\lambda \in L(M_\lambda)$, so $Q_\lambda \geq I - P_\lambda$. But also $I - Q_\lambda \geq P_\lambda$, since $Q_\lambda P_\lambda = 0$, so $Q_\lambda = I - P_\lambda$, which proves the first statement. Next, we show that $Q_\lambda \in N$. For $A \in M_\lambda$ and $g \in G$, we have $g(Q_\lambda)A = \lambda(g)g(Q_\lambda A) = 0$, so $g(Q_\lambda) \leq Q_\lambda \forall g \in G$ and we conclude that $g(Q_\lambda) = Q_\lambda \forall g \in G$, i.e. $Q_\lambda \in N$. Finally, note that $[M_\lambda H]$ is an invariant subspace for N , so Q_λ lies in the center of N .

We can now give a characterization of the condition $M \odot H = L^2(G, H)$ when G is compact abelian.

PROPOSITION 4.6. *The following are equivalent:*

- (i) $M \odot H = L^2(G, H)$;
- (ii) $[M_\lambda H] = H \forall \lambda \in \hat{G}$;
- (iii) $L(M_\lambda) \cap \text{center}(N) = 0 \forall \lambda \in \hat{G}$.

PROOF. The equivalence of (ii) and (iii) follows from 3.5; we show that (i) and (ii) are equivalent. Suppose first that (i) holds, and take $\lambda \in \hat{G}$. If $\eta \in [M_\lambda H]^\perp$, define $\Phi \in L^2(G, H)$ by $\Phi(g) = \lambda(g)\eta (g \in G)$. For $A \in M, \xi \in H$, we have $(A \odot \xi, \Phi) = \int_G (g^{-1}(A)\xi, \lambda(g)\eta) dg = \int_G (\lambda(g)g(A)\xi, \eta) dg = (\Gamma_\lambda(A)\xi, \eta) = 0$, so $\eta = 0$ by (i). Now suppose (ii) holds, and take $\Phi \in (M \odot H)^\perp$. Given $\xi \in H$, define $\phi \in L^2(G)$ by $\phi(g) = (\Phi(g), \xi)$. We claim that $\hat{\phi}(\lambda) = 0 \forall \lambda \in \hat{G}$, which will show that $\Phi = 0$. Indeed, for $\lambda \in \hat{G}$ and $\epsilon > 0$, (i) gives $A_1, A_2, \dots, A_n \in M_\lambda$ and $\xi_1, \xi_2, \dots, \xi_n \in H$ such that $\|\xi - \sum_{i=1}^n A_i \xi_i\| \leq \epsilon$. Hence

$$\begin{aligned} |\hat{\phi}(\lambda)| &= \left| \int_G \overline{\lambda(g)} (\Phi(g), \xi) dg \right| \\ &\leq \left| \int_G \overline{\lambda(g)} \left(\Phi(g), \sum_{i=1}^n A_i \xi_i \right) dg \right| + \epsilon \|\Phi\|. \end{aligned}$$

For $i = 1, 2, \dots, n$, we have

$$\begin{aligned} \int_G \overline{\lambda(g)}(\Phi(g), A_i \xi_i) dg &= \int_G (\Phi(g), \lambda(g) A_i \xi_i) dg \\ &= \int_G (\Phi(g), g^{-1}(A_i) \xi_i) dg \\ &= (\Phi, A_i \circ \xi_i) = 0, \end{aligned}$$

so $|\hat{\phi}(\lambda)| \leq \epsilon \|\Phi\|$. As $\epsilon > 0$ was arbitrary, we have $\hat{\phi}(\lambda) = 0 \forall \lambda \in \hat{G}$, which completes the proof.

As a corollary, we obtain the following abelian version of 3.2.

COROLLARY 4.7. *Let G be a compact abelian strong-operator continuous group of automorphisms of M , with fixed-point algebra N . Then $M \times G$ is a factor if and only if N is a factor and G acts faithfully on M .*

PROOF. In light of 3.2, 3.3 and 4.6, we need only show that if G acts faithfully on M , and N is a factor, then $[M_\lambda H] = H \forall \lambda \in \hat{G}$, i.e. $Q_\lambda = 0 \forall \lambda \in \hat{G}$. Notice that since each Q_λ lies in the center of N , we have $Q_\lambda = 0$ or $I \forall \lambda \in \hat{G}$. Let $S = \{\lambda \in \hat{G}: Q_\lambda = 0\} = \{\lambda \in \hat{G}: Q_\lambda \neq I\} = \{\lambda \in \hat{G}: M_\lambda \neq 0\}$. We claim that S is a subgroup of \hat{G} . If $\lambda \in S$, then $\bar{\lambda} \in S$ by (iii) of 4.1.

If $\lambda_1, \lambda_2 \in S$, then $[M_{\lambda_1} H] = H$ and $[M_{\lambda_2} H] = H$, so $M_{\lambda_1} M_{\lambda_2} \neq 0$, so $M_{\lambda_1 \lambda_2} \neq 0$ by (v) of 4.1 and hence $\lambda_1 \lambda_2 \in S$. To see that $S = \hat{G}$, suppose that $g \in G$ is such that $\lambda(g) = 1 \forall \lambda \in S$. For $\lambda \in S$ and $A \in M_\lambda$ we have $g(A) = \overline{\lambda(g)} A = A$, and for $\lambda \notin S$, we have $M_\lambda = 0$. Hence, g fixes each M_λ , so by 4.2, $g(A) = A \forall A \in M$. Since G acts faithfully on M , g must be the identity element of G . We conclude that $S = \hat{G}$.

Takesaki's duality theorem for crossed products (4.5 of [8]) can be used in connection with 4.7 above in determining what sorts of von Neumann algebras can appear as fixed-point algebras for actions of compact abelian groups on M . As a sample application, we will prove the following proposition after establishing a preliminary lemma.

PROPOSITION 4.8. *Suppose that there is a faithful strong-operator continuous action of the circle group $T = \{z \in \mathbb{C}: |z| = 1\}$ as automorphisms on M for which the fixed-point algebra N is a type I factor. Then M must be of type I.*

The proof of 4.8 requires the following lemma.

LEMMA 4.9. *Let K be a separable Hilbert space and let the discrete group Z of integers act as automorphisms on $B(K)$. Then $B(K) \times Z$ is of type I.*

PROOF OF LEMMA. Let τ_n be the automorphism of $B(K)$ corresponding to $n \in Z$. Since all *-automorphisms of $B(K)$ are inner, there is a unitary $U \in B(K)$ such that $\tau_n(A) = U^n A U^{-n} \forall A \in B(K), n \in Z$. For $\phi \in l^2(Z), \xi \in K$, define a function $\phi \otimes \xi \in L^2(Z, K)$ by $\phi \otimes \xi(n) = \phi(n) U^n \xi$. It is easily checked that

the $\phi \otimes \xi$'s span a dense subspace of $L^2(Z, K)$ and that $(\phi_1 \otimes \xi_1, \phi_2 \otimes \xi_2) = (\phi_1, \phi_2)(\xi_1, \xi_2) \forall \phi_1, \phi_2 \in l^2(Z), \xi_1, \xi_2 \in K$, so we may identify $L^2(Z, K)$ in this way with the Hilbert space tensor product $l^2(Z) \otimes K$. For $A \in B(K), \phi \in l^2(Z), \xi \in K$, we have

$$\begin{aligned} \tilde{A}(\phi \otimes \xi)(n) &= \tau_n^{-1}(A)(\phi \otimes \xi)(n) = \phi(n)U^{-n}AU^nU^{-n}\xi \\ &= (\phi \otimes A\xi)(n) \quad (n \in Z), \end{aligned}$$

so $\tilde{A}(\phi \otimes \xi) = \phi \otimes A\xi$. If we let V_m (for $m \in Z$) denote the unitary operator on $l^2(Z)$ defined by $(V_m\phi)(n) = \phi(n - m)$, we have

$$\begin{aligned} L_m\tilde{U}^{-m}(\phi \otimes \xi)(n) &= L_m(\phi \otimes U^{-m}\xi)(n) \\ &= \phi(n - m)U^{m-n}U^{-m}\xi \\ &= \phi(n - m)U^{-n}\xi = (V_m\phi \otimes \xi)(n), \end{aligned}$$

so $L_m\tilde{U}^{-m}(\phi \otimes \xi) = V_m\phi \otimes \xi$. Since $B(K) \times Z$ is generated by the operators $\tilde{A}(A \in B(K))$ and $L_m\tilde{U}^{-m}$ ($m \in Z$), it follows that $B(K) \times Z$ is isomorphic with the tensor product of $B(K)$ by the (abelian) von Neumann algebra of operators on $l^2(Z)$ generated by the V_m 's. This proves the lemma.

PROOF OF 4.8. We have $M \times T$ isomorphic with $A(\bar{M})$ by 4.6 and 4.7. $M \times T$, being Morita equivalent to the type I factor N , must be a type I factor by 2.7, and hence isomorphic with $B(K)$ for some Hilbert space K , which must be separable because $M \times T$ acts on a separable Hilbert space. By the duality result of Takesaki alluded to above, there is an action of Z as automorphisms of $B(K)$ such that $B(K) \times Z$ is isomorphic with $M \otimes B(L^2(T))$. By the lemma, then, this tensor product must be of type I and hence M is of type I.

We remark that it is not clear whether this line of argument can be made to yield an analogous result for arbitrary compact abelian groups acting faithfully on M . The problem lies in generalizing Lemma 4.9, where one encounters the difficulty that an arbitrary abelian group of automorphisms of $B(K)$ need not be implemented by a *group* of unitary operators (although each individual automorphism will of course be inner).

5. A class of examples. In this section we construct a class of examples in which several of the features of the analysis developed above stand out in high relief. As a by-product, we obtain what appears to be a new method for constructing outer automorphisms (and indeed compact abelian groups of outer automorphisms) on many II_1 -factors.

Let D be a countable discrete group with identity e , and let H be the Hilbert space $l^2(D)$; elements of H will be written as complex-valued functions on D . Let $M = VN(D)$, the von Neumann algebra generated by the left regular

representation of D on H . Notice that for $\xi, \eta \in l^2(D)$, the convolution formula

$$(\xi * \eta)(s) = \sum_{t \in D} \xi(t)\eta(t^{-1}s) \quad (s \in D)$$

defines a complex-valued function $\xi * \eta$ on D (which may or may not belong to $l^2(D)$). We shall call a function $x \in l^2(D)$ a *left convolver* if $x * \xi \in l^2(D) \forall \xi \in l^2(D)$ and the operator $\xi \rightarrow x * \xi$ is bounded on $l^2(D)$. In II.5 of [7] it is shown that every operator in M has the form $\xi \rightarrow x * \xi$ for some left convolver x ; it is straightforward to verify that convolution on the left by any left convolver gives an operator in M . We shall henceforth identify M with the algebra of left convolvers in $l^2(D)$ (with convolution for multiplication and involution defined by $x^*(s) = x(s^{-1})$ ($s \in D$)). For $s \in D$, we write δ_s for the function (and left convolver) in $l^2(D)$ with value 1 at s and 0 elsewhere.

Let λ be a character of D , i.e. a homomorphism of D into the circle group $T = \{z \in \mathbb{C} : |z| = 1\}$. For $x \in M$, define $a_\lambda(x) \in l^2(D)$ by $a_\lambda(x)(s) = \lambda(s)x(s)$ ($s \in D$). For $\xi \in l^2(D)$, $s \in D$, we have

$$\begin{aligned} (a_\lambda(x) * \xi)(s) &= \sum_{t \in D} \lambda(t)x(t)\xi(t^{-1}s) \\ &= \lambda(s) \sum_{t \in D} x(t)\overline{\lambda(t^{-1}s)}\xi(t^{-1}s) \\ &= \lambda(s)(x * (\bar{\lambda}\xi))(s), \end{aligned}$$

so

$$a_\lambda(x) * \xi = \lambda(x * (\bar{\lambda}\xi)),$$

which shows that $a_\lambda(x) \in M$. Routine computations now show that a_λ is a *-automorphism of M .

PROPOSITION 5.1. *If D is an infinite-conjugacy-class group (so that M is a Π_1 -factor) and λ is a nontrivial character of D , then a_λ is an outer automorphism of M .*

PROOF. Suppose by way of contradiction that there is a unitary $u \in M$ such that $a_\lambda(x) = u * x * u^* \forall x \in M$. For $s \in D$, define $\xi_s \in l^2(D)$ by $\xi_s(t) = u(s^{-1}ts)$ ($t \in D$). We have

$$\begin{aligned} (u, \xi_s) &= \sum_{t \in D} u(t)\overline{u(s^{-1}ts)} \\ &= \sum_{t \in D} u(t)u^*(s^{-1}t^{-1}s) \\ &= \sum_{t \in D} u(t)(\delta_s * u^*)(t^{-1}s) \\ &= (u * \delta_s * u^*)(s) = a_\lambda(\delta_s)(s) \\ &= \lambda(s). \end{aligned}$$

Now u is a unit vector in $l^2(D)$ (because $u * u^* = \delta_e$) and hence so is ξ_s . Since $|(u, \xi_s)| = 1$, we conclude that ξ_s must be a scalar multiple of u , so $|u(t)| = |\mu(s^{-1}ts)| \forall s, t \in D$. By assumption, a_λ is not the identity automorphism of D , so there is a $t_0 \in D \setminus \{e\}$ such that $u(t_0) \neq 0$. Since the set $\{s^{-1}t_0s : s \in D\}$ is infinite, we have a contradiction.

REMARK 5.2. If u is as in the proof above, we have $a_\lambda(u) = u$ and hence $\text{supp}(u) \subseteq \ker(\lambda)$. We can therefore replace the assumption in 5.1 that D be an infinite-conjugacy-class group with the weaker condition that the conjugacy class in D of every $t \in \ker(\lambda) \setminus \{e\}$ be infinite.

In [2], R. Kallman shows that every outer automorphism of an infinite conjugacy class group induces an outer automorphism of the corresponding II_1 -factor, and points out that there exist infinite-conjugacy-class groups all of whose automorphisms are inner. The example he mentions is the semidirect product $D = Q_* \times Q$ (where Q is the additive group of rationals, Q_* is the multiplicative group of nonzero rationals, and multiplication in D is defined by $(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1b_2 + b_1)$ for $a_1, a_2 \in Q_*, b_1, b_2 \in Q$). Although it has no outer automorphisms, D has an abundance of nontrivial characters, since the abelian group Q_* is a homomorphic image of D ; hence $VN(D)$ has many outer automorphisms. It would be interesting to know whether every infinite-conjugacy-class group with no outer automorphisms must necessarily possess a nontrivial character.

Returning to the general situation, let G be a compact abelian group and suppose that there exists a homomorphism $s \rightarrow \lambda_s$ of D onto \hat{G} . We proceed to construct a faithful, strong-operator continuous action of G on $M = VN(D)$. For $g \in G, x \in M$, define $g(x) \in M$ by $g(x)(s) = \overline{\lambda_s(g)x(s)}$ ($s \in D$); it is apparent that $g(\cdot) = a_\lambda$, where λ is the character of D defined by $\lambda(s) = \overline{\lambda_s(g)}$, so $g(\cdot)$ is a *-automorphism of M . It is equally clear that $g \rightarrow g(\cdot)$ is an isomorphism (injective because $s \rightarrow \lambda_s$ is onto) of G into the group of *-automorphisms of M . To show that the action of G on M is strong-operator continuous, take $s \in D$ and let $\{g_n\}$ be a sequence in G with limit $g \in G$. (Note that \hat{G} must be countable, so the topology on G is metrizable.) For $x \in M$, we have

$$\begin{aligned} \|g_n(x) * \delta_s - g(x) * \delta_s\|^2 &= \sum_{t \in D} |g_n(x)(ts^{-1}) - g(x)(ts^{-1})|^2 \\ &= \sum_{t \in D} |\lambda_{ts^{-1}}(g_n) - \lambda_{ts^{-1}}(g)|^2 |x(ts^{-1})|^2. \end{aligned}$$

This sum goes to 0 as $n \rightarrow \infty$ because $\lambda_{ts^{-1}}(g_n) \rightarrow \lambda_{ts^{-1}}(g) \forall t \in D$. Since $\{\delta_s : s \in D\}$ spans a dense subset of $l^2(D)$, strong-operator continuity follows. The spectral subspaces M_λ for this action of G on M are immediately identifiable; for each $\lambda \in \hat{G}$, we have $M_\lambda = \{x \in M : \text{supp}(x) \subseteq \{s \in D : \lambda_s = \lambda\}\}$. In particular,

$N = \{x \in M: \text{supp}(x) \subseteq D_1\}$, where D_1 is the kernel of the homomorphism $s \mapsto \lambda_s$; thus N is isomorphic with $VN(D_1)$. Notice that for $\lambda \in \hat{G}$ and $s \in D$ such that $\lambda_s = \lambda$, we have $M_\lambda = \delta_s * N$, so M_λ and N are isomorphic as inner product modules over N , and $[M_\lambda H] = H$. By 4.6 and 3.1, $M \times G$ is isomorphic with $A(\bar{M})$, which in turn is isomorphic with $N \otimes B(l^2(\hat{G}))$ by 2.6.

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