TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 224, Number 2, 1976

SOME CONSEQUENCES OF THE ALGEBRAIC NATURE OF $p(e^{i\theta})$

BY

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ABSTRACT. For polynomial p of degree n, the curve $p(e^{i\theta})$ is a closed curve in the complex plane. We show that the image of this curve is a subset of an algebraic curve of degree $2n$. Using Bézout's theorem and taking into account imaginary intersections at infinity, we show that if p and q are polynomials of degree m and n respectively, then the curves $p(e^{i\theta})$ and $q(e^{i\theta})$ intersect at most $2mn$ times. Finally, let U_k be the set of points w, not on $p(e^{i\theta})$, such that $p(z) - w$ has exactly k roots in $|z| < 1$. We prove that if L is a line then $L \cap U_k$ has at most $n - k + 1$ components in L and in particular U_n is convex.

1. **Introduction.** Let p be a polynomial of degree n. The curve $p(e^{i\theta})$, $0 \leqslant \theta \leqslant 2\pi$, is a closed curve in the complex plane. We show that the image of the curve is a subset of an algebraic curve of degree $2n$, i.e., we can find a polynomial $h(w, \bar{w})$ of degree $2n$ in the variables jointly such that $h(p(z), \overline{p(z)}) = 0$ for $|z| = 1$. This fact used with Bézout's theorem shows that $p(e^{i\theta})$ intersects an algebraic curve of degree m at most $2mn$ times counting multiplicity, where the multiplicity is counted as in algebraic geometry. We will show that if p and q are polynomials of degree m and n respectively, then the curves $p(e^{i\theta})$ and $q(e^{i\theta})$ intersect at most $2mn$ times. This will follow again from Bézout's theorem, but this time certain imaginary intersections at infinity must be taken into account. Finally, let U_k be the set of points w, not on $p(e^{i\theta})$, such that $p(z) - w$ has exactly k roots in $|z| < 1$. We prove that if L is a line, then $L \cap U_k$ has at most $n - k + 1$ components in L and in particular U_n is convex.

2. **The algebraic nature of $p(e^{i\theta})$.** Let p be a polynomial of degree n. We look at the closed curve $p(e^{i\theta})$, $0 \leqslant \theta \leqslant 2\pi$. We may write $p(e^{i\theta}) = R(\theta) + iI(\theta)$ where R and I are trigonometric polynomials of degree n, with real coefficients. Let L be the line with equation $Ax + By + C = 0$ with A, B and C real. The intersections of $p(e^{i\theta})$ with L correspond to the real zeros of $AR(\theta) + BI(\theta) + C$, with the multiplicity of the intersection counted as the multiplicity of the zero. Now $AR(\theta) + BI(\theta) + C$ is a trigonometric polynomial of degree n with real coefficients and therefore has at most $2n$ zeros counting multiplicity

Received by the editors October 20, 1975.
AMS (MOS) subject classifications (1970). Primary 30A06; Secondary 30A90, 14H99.
Key words and phrases. Polynomials, Bézout's theorem, algebraic curves.

(Polya-Szegö [4, p. 77]). Thus $p(e^{i\theta})$ intersects L at most $2n$ times.

The property of $p(e^{i\theta})$ stated above is characteristic of algebraic curves of degree $2n$. Let $\sigma(\theta) = p(e^{i\theta})$ and let $[\sigma]$ denote the image of σ. We see that $[\sigma]$ is a subset of an algebraic curve of degree $2n$ as follows: Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ and for fixed w let $g(z) = p(z) - w$ and $g^*(z) = z^n \overline{g(1/\bar{z})}$. Let $R(g, g^*)$ be the resultant of g and g^* (see van der Waerden [6]). Indicating the dependence on w and $\bar{w}$, write $R(g, g^*) = h(w, \bar{w})$. The resultant can be written in the usual way as the determinant of a $2n \times 2n$ matrix:

$$\begin{vmatrix}
a_0 - w & a_1 & \cdots & a_n & & & \\
 & a_0 - w & a_1 & \cdots & a_n & & \\
 & & \ddots & & & \ddots & \\
 & & & a_0 - w & a_1 & \cdots & a_n \\
\bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_0 - \bar{w} & & & \\
 & \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_0 - \bar{w} & & \\
 & & \ddots & & & \ddots & \\
 & & & \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_0 - \bar{w}
\end{vmatrix}$$

We see that $h(w, \bar{w}) = |w - a_0|^{2n} + \cdots$, i.e., h is of degree $2n$ in w and $\bar{w}$ jointly. From the theory of resultants, we have another expression for h:

$$h(w, \bar{w}) = |a_0 - w|^{2n} \prod_{1 \leq k; l \leq n} (1 - z_k \bar{z}_l)$$

where $z_1, z_2, \ldots, z_n$ are the zeros of $p(z) - w$. We see from this that $h(w, \bar{w})$ is real. (For further information on the resultant of two polynomials whose zeros are symmetric in $|z| = 1$, see Cohn [1].) Now if $w = p(e^{i\theta})$, then $g(e^{i\theta}) = 0 = g^*(e^{i\theta})$ and therefore $R(g, g^*) = h(w, \bar{w}) = 0$. Thus $[\sigma]$ is a subset of $\{w \mid h(w, \bar{w}) = 0\}$, an algebraic curve of degree $2n$. Thus we have

THEOREM 1. *If p is a polynomial of degree n then there exists a polynomial $h(w, \bar{w})$ of degree $2n$ such that $h(p(z), \overline{p(z)}) = 0$ for $|z| = 1$.*

3. **Intersection with an algebraic curve.** By Bézout's theorem (see Walker [7, p. 111] and Fulton [2, p. 112]), an algebraic curve of order m which does not contain the curve $p(e^{i\theta})$ intersects $p(e^{i\theta})$ at most $2mn$ times. We measure the multiplicity of these intersections in the classical way (see Walker [7, p. 109]). In this case, the expansions $w = p(e^{i\theta})$ and $\bar{w} = \overline{p(e^{i\theta})}$ in the variable θ near $\theta = \phi$ is called a place or branch of the curve $h(w, \bar{w}) = 0$ with center $p(e^{i\phi})$. If $g(w, \bar{w}) = 0$ is another algebraic curve such that $g(p(e^{i\phi}), \overline{p(e^{i\phi})}) = 0$, then the order of the zero of $g(p(e^{i\theta}), \overline{p(e^{i\theta})})$ at $\theta = \phi$ is called the order of g at the place $w = p(e^{i\theta})$, $\bar{w} = \overline{p(e^{i\theta})}$. The multiplicity of the intersection of the two curves at $p(e^{i\phi})$ is defined to be the sum of the orders of g at the places of h with center $p(e^{i\phi})$. Thus we have

THEOREM 2. *If p is a polynomial of degree n, then an algebraic curve of degree m intersects $p(e^{i\theta})$ at most $2mn$ times counting multiplicity.*

Now we state as a corollary our starting observation

COROLLARY 1. *If p is a polynomial of degree n, then a line intersects the curve $p(e^{i\theta})$ at most $2n$ times counting multiplicity.*

We note here that Theorem 1 also indicates that the number of intersections of $p(e^{i\theta})$ with itself is limited. In general, the maximum number of such intersections is $(n-1)^2$ (see Quine [5]).

4. **Intersection of $p(e^{i\theta})$ and $q(e^{i\theta})$.** Let p and q be polynomials of degree m and n respectively. We wish to investigate the number of intersections of $p(e^{i\theta})$ and $q(e^{i\theta})$. Theorems 1 and 2 would indicate that the maximum number of intersections is $4mn$. In fact, this number is too large because the algebraic curves containing $p(e^{i\theta})$ and $q(e^{i\theta})$ pass through the same points at infinity and this reduces the number of finite intersection points possible. This situation also occurs in classical algebraic geometry in the case of two circles. Both being algebraic curves of degree 2, by Bézout's theorem they intersect exactly four times. Two of these times, however, are at imaginary points at infinity called the circular points, leaving only two intersections at most in the real finite plane.

We now discuss these intersections at infinity. As in §2, let p be a polynomial of degree n, let $g(z) = p(z) - w$, $g^*(z) = z^n \overline{g(1/\bar{z})}$, and $h(w, \bar{w}) = R(g, g^*)$. To investigate the points at infinity on the curve $h(w, \bar{w}) = 0$, we look at the curve in complex projective space $P_2(\mathrm{C})$. To do this, we change to homogeneous coordinates by writing $H(w, \bar{w}, \zeta) = \zeta^{2n} h(w/\zeta, \bar{w}/\zeta)$ and then we investigate the curve $H(w, \bar{w}, \zeta) = 0$ in $P_2(\mathrm{C})$. The points at infinity are points where $\zeta = 0$. From the determinant of §2 for h, we find that $H(w, \bar{w}, 0) = (w\bar{w})^n$. Thus we see that the points in $P_2(\mathrm{C})$ with homogeneous coordinates $(1, 0, 0)$ and $(0, 1, 0)$ are points of multiplicity n on the curve.

Now let p and q be polynomials of degree m and n respectively. Let $h(w, \bar{w}) = 0$ and $k(w, \bar{w}) = 0$ be the algebraic curves of degree $2m$ and $2n$ respectively containing the curves $p(e^{i\theta})$ and $q(e^{i\theta})$ respectively. In $P_2(\mathbf{C})$ both curves pass through the points with homogeneous coordinates (1, 0, 0) and (0, 1, 0) and these points are of multiplicity m on one and n on the other. These two points are then intersection points of multiplicity at least mn (Walker [7, p. 114]). Now from Bézout's theorem, we deduce

THEOREM 3. *If p and q are polynomials of degree m and n respectively, then the curves $p(e^{i\theta})$ and $q(e^{i\theta})$ intersect at most $2mn$ times. The theorem is sharp.*

To show that the theorem is sharp, we consider the polynomials $p(z) = \frac{1}{2} + z^m$ and $q(z) = -\frac{1}{2} + z^n$. We see that $p(e^{2\pi i\alpha/3m}) = q(e^{\pi i\beta/3n}) = (3^{1/2}/2)i$ for $\alpha = 1, \ldots, m, \beta = 1, \ldots, n$, and also $p(e^{-2\pi i\alpha/3m}) = q(e^{-\pi i\beta/3n}) = (-3^{1/2}/2)i$ for $\alpha = 1, \ldots, m, \beta = 1, \ldots, n$. The determinant of §2 gives the algebraic curves $(|w - \frac{1}{2}|^2 - 1)^n = 0$ and $(|w + \frac{1}{2}|^2 - 1)^n = 0$ containing $p(e^{i\theta})$ and $q(e^{i\theta})$ respectively. Now $(|p(e^{i\theta}) + \frac{1}{2}|^2 - 1)^n$ has zeros of order n at $e^{\pm 2\pi i\alpha/3m}$, for $\alpha = 1, \ldots, m$. By our previous remark on multiplicity of intersections, the curves $p(e^{i\theta})$ and $q(e^{i\theta})$ intersect a total of $2mn$ times.

5. **Oriented intersection with a line.** We want to define oriented intersection multiplicity of $p(e^{i\theta})$ with a line. Let p be a polynomial of degree n and let L be a line with parametric equation $\gamma(t) = w_0 + ta$, $|a| = 1$. Write $\sigma(\theta) = p(e^{i\theta})$. Suppose $p'(z) \neq 0$ for $|z| = 1$ and let w be a point of intersection of $p(e^{i\theta})$ and L. Let $e^{i\theta_1}, \ldots, e^{i\theta_k}$ be roots of $p(z) - w$ on $|z| = 1$. We say L intersects $p(e^{i\theta})$ transversely at w if $\sigma'(\theta_j)$ and a are linearly independent for $j = 1, \ldots, k$. We assign an oriented intersection number $\lambda(e^{i\theta_j}) = +1$ if the ordered basis $\{\sigma'(\theta_j), a\}$ is positively oriented with respect to the usual orientation on $\mathbf{C}$ as a vector space. We set $\lambda(e^{i\theta_j}) = -1$ otherwise. We call $\Sigma_{j=1}^k \lambda(e^{i\theta_j})$ the oriented intersection multiplicity of $p(e^{i\theta})$ and L at w. This will clearly be less than or equal to k, the intersection multiplicity.

Now suppose $w_0 \notin [\sigma]$. We denote the winding number of σ about the point w_0 by $\omega(\sigma, w_0)$. Let L be a line with equation $\gamma(t)$ as before. Let $T = \gamma^{-1}([\sigma])$ and for $T \in S$ let $\delta(t)$ be the oriented intersection multiplicity of L and $p(e^{i\theta})$ at $\delta(t)$.

LEMMA 1.

$$\omega(\sigma, w_0) = \sum_{t \in T; t > 0} \delta(t) = - \sum_{t \in T; t < 0} \delta(t).$$

PROOF. Let S^1 denote the circle $|z| = 1$ in the complex plane considered as an oriented 1-manifold with positive orientation in the counterclockwise direction. Let $f\colon S^1 \longrightarrow S^1$ be defined by $f(z) = (p(z) - w_0)/|p(z) - w_0|$. At each point $z \in f^{-1}(a)$ we have an intersection number $+1$ if f preserves orientation there and -1 otherwise. Clearly this number is just $\lambda(z)$ as defined earlier. Now $\Sigma_{z\in f^{-1}(a)}\lambda(z)$ is just the topological degree of the map f, i.e., $\omega(\sigma, w_0)$ (see Guillemin and Pollack [3, p. 109]). Also by definition $\Sigma_{z\in f^{-1}(a)}\lambda(z) = \Sigma_{t\in T;t>0}\delta(t)$. This proves the first equation. By considering $f^{-1}(-a)$ we get the second equation. This completes the proof of the lemma.

We remark that if $\gamma(t_0) \notin [\sigma]$ we have the equation

$$\omega(\sigma, \gamma(t_0)) = \sum_{t\in T;t>t_0} \delta(t) = -\sum_{t\in T;t<t_0} \delta(t).$$

This follows by writing $\gamma(t) = \gamma(t_0) + a(t - t_0)$ and applying the lemma.

Now $\mathbf{C} - [\sigma]$ decomposes into a number of components on which $\omega(\sigma, w)$ is constant as a function of w. By the argument principle, for $w \in \mathbf{C} - [\sigma]$ we have $n(\sigma, w)$ is the number of zeros of $p(z) - w$ in $|z| < 1$; therefore in this case $0 \leqslant \omega(\sigma, w) \leqslant n$.

6. **Intersection of L and U_k.** Let $U_k = \{w \in \mathbf{C} - [\sigma] \mid n(\sigma, w) = k\}$ for $k = 0, 1, 2, \ldots, n$. Let $\gamma(t)$ be the equation of a line L. We wish to find a bound on the number of components on $U_k \cap L$ using Theorem 1 and Lemma 1. Let us suppose that $\gamma^{-1}(U_k \cap L)$ consists of l disjoint intervals $I_1, \ldots, I_l$, ordered from left to right along the real line. (We know from Corollary 1 that the number of intervals is finite and $\leqslant 2n - 1$.) Let $I_j = (t_j, t_j^*)$, $j = 1, \ldots, l$. We have $t_1 < t_1^* \leqslant t_2 < t_2^* \leqslant \cdots \leqslant t_l < t_l^*$. Clearly for $k = 1, \ldots, l$, t_j and t_j^* are elements of $T = \gamma^{-1}([\sigma])$. Now if $\tau \in I_j$, by the remark following Lemma 1, we have

$$\sum_{t>\tau;t\in T} \delta(t) = k$$

and therefore

$$\sum_{t>t_j;t\in T} \delta(t) = k \quad \text{and} \quad \sum_{t\geqslant t_j^*;t\in T} \delta(t) = k.$$

From this

$$\sum_{t_{j-1}^*<t<t_j;t\in T} \delta(t) = 0 \quad \text{for } j = 2, \ldots, n.$$

Therefore we can find z_1 and z_2 on $|z| = 1$ such that $p(z_1)$ and $p(z_2)$ are in $\{w \mid w = \gamma(t), t_{j-1}^* \leqslant t \leqslant t_j\}$ and $\lambda(z_1) = 1$, $\lambda(z_2) = -1$. Thus $p(e^{i\theta})$ intersects L at least twice counting multiplicity between $\gamma(I_{j-1})$ and $\gamma(I_j)$. Also since $\Sigma_{t<t_1;t\in T}\delta(t) = k$, we can argue similarly that $p(e^{i\theta})$ intersects L at least k times counting multiplicity to the left of $\gamma(I_1)$. Likewise $p(e^{i\theta})$ intersects L at least k times counting multiplicity to the right of $\gamma(I_l)$. Thus the total number of intersections of $p(e^{i\theta})$ and L is at least $2(l - 1) + 2k$. By Theorem 1,

$2(l-1)+2k \leqslant 2n$ and therefore $l \leqslant n-k+1$. Thus we get

THEOREM 4. *Let p be a polynomial of degree n and let U_k be the set of points w not on $p(e^{i\theta})$ such that $p(z)-w$ has k roots in $|z|<1$. If L is a line, then $L \cap U_k$ has at most $n-k+1$ components in L.*

PROOF. Our previous discussion proves the assertion when $p'(z) \neq 0$ for $|z|=1$. Now, for an arbitrary polynomial p, the theorem will hold if and only if it holds for polynomials arbitrarily close to p. Therefore without loss of generality we may assume that $p'(z) \neq 0$ for $|z|=1$. Also the theorem holds for a line L if and only if it holds for lines arbitrarily close to L. By a modification of the transversality theorem (Guillemin and Pollack [3, p. 68]) we can show that given a polynomial p such that $p'(z) \neq 0$ for $|z|=1$ and any line L, we can find a line arbitrarily close to L which intersects $p(e^{i\theta})$ transversely. Thus without loss of generality, we may assume that $p(e^{i\theta})$ and L intersect transversely. Now the theorem follows from our previous discussion.

COROLLARY. *U_n is convex.*

We remark that the corollary also follows from a theorem of Cohn [1, p. 130].

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