TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 226, 1977

SOME SMOOTH MAPS WITH INFINITELY MANY HYPERBOLIC PERIDOIC POINTS

BY

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ABSTRACT. If a smooth map of the two-disk to itself has only hyperbolic periodic points and has no source or sink whose period is a power of two then it has infinitely many periodic points. This and similar results are proved.

In this note we give a Lefschetz fixed point type argument to show that certain smooth maps with only hyperbolic peridoic points must have infinitely many periodic points. Surprisingly, this works even for such homologically simple spaces as the two-dimensional disk and the unit interval.

A C^1 map f: $M \to M$ is said to have *hyperbolic peridoic points* provided that, for each $x \in M$ and each integer $n > 0$ such that $f^n(x) = x$, the derivative map Df_x^n: $TM_x \to TM_x$ has no eigenvalues of absolute value one. For our purposes below it is quite acceptable that 0 be an eigenvalue of Df_x^n, i.e. that there be periodic singular points, as long as there are no eigenvalues of absolute value one.

A periodic point x for which all eigenvalues of Df_x^n are less than one in absolute value (including perhaps 0) will be called a *sink*. If all eigenvalues of Df_x^n are greater than one in absolute value then x is called a *source*. In the following theorem S^1 and S^2 denote the circle and two-sphere respectively, while D^2 is the two-dimensional disk and I is the unit interval $[0, 1]$. For D^2 and I we assume that no boundary points are periodic.

THEOREM. *If the following types of C^1 maps have only hyperbolic periodic points then they have infinitely many periodic points*:

(a) *Any f: $S^2 \to S^2$ of degree* 1 *which has at most one source or sink with least period a power of* 2.

(b) *Any f: $S^2 \to S^2$ of degree* 0 *or* -1, *or any g: $D^2 \to D^2$ which has no source or sink with least period a power of* 2.

Received by the editors November 3, 1975.

AMS (MOS) subject classifications (1970). Primary 58F20; Secondary 55C20.

(1) The author would like to thank the Instituto de Matematica Pura e Aplicada do C. N. de Pq of Brazil and the Institut des Hautes Etudes Scientifiques for their hospitality during the preparation of this article. Research supported in part by NSF Grant GP 42329X.

(c) *Any* $f: S^1 \to S^1$ *of degree* 0 *or* -1, *or any* $g: I \to I$ *which has no sink with least period a power of* 2.

We remark that in the above theorem fixed points are, of course, considered to be points whose least period is a power of 2, namely $2^0 = 1$. It is also worth noting that a generic set of C^1 self-maps of M will have only hyperbolic periodic points, i.e. a set of second category in the Baire space $C^1(M, M)$, (see Shub [4] for this).

An example of a smooth embedding $f: D^2 \to D^2$ with a single sink of least period 3 and infinitely many hyperbolic periodic points can be found in [5, pp. 786–787]. On the other hand D. Pixton has shown me an example of an embedding $g: D^2 \to D^2$ with a single sink of least period 2^n (any $n \geqslant 0$) and finitely many hyperbolic periodic points.

The proof of the theorem above will depend heavily on a function $\eta(f)$ which is sometimes called the homology zeta function (see [2], [3] or [5]) and which we now define.

DEFINITION. *For any* $f: M \to M$ *we define*

$$\eta(f) = \exp\left(\sum_{m=1}^{\infty} m^{-1} L(f^m) t^m\right),$$

where $L(f^m)$ *is the Lefschetz number of* f^m.

For a general reference on the Lefschetz number and the Lefschetz fixed point theorem see [1]. We remark that *a priori* $\eta(f)$ is a formal power series in t, but it is not difficult to show it is always a rational function of t (see [5]). Moreover, since $L(f^m)$ depends only on the homotopy type of f, it follows that if f and g are homotopic then $\eta(f) = \eta(g)$.

If γ is a hyperbolic periodic orbit of a map f and has least period p, then for each point $x \in \gamma$ we will denote by E_x^u the subspace of TM_x spanned by generalized eigenspaces of Df_x^p which correspond to eigenvalues which are greater than one in absolute value. Since $Df_x^n: E_x^u \to E_{f^n(x)}^u$ is an isomorphism, it is clear that $\dim E_x^u = \dim E_y^u$ if $x, y \in \gamma$. Also it is clear that if $Df_x^p: E_x^u \to E_x^u$ preserves (reverses) orientation then also $Df_y^p: E_y^u \to E_y^u$ preserves (reverses) orientation if $x, y \in \gamma$.

LEMMA 1. *Let* $f: M \to M$ *be a* C^1 *map with finitely many periodic points, all hyperbolic, then*

$$\eta(f) = \prod_{\gamma} (1 - \Delta_\gamma t^{p(\gamma)})^{(-1)^{u(\gamma)+1}},$$

where the product is over all periodic orbits γ, $p(\gamma)$ *is the least period of* γ, Δ_γ *is* ± 1 *depending on whether* $Df_x^{p(\gamma)}: E_x^u \to E_x^u$ *preserves or reverses orientation for* $x \in \gamma$, *and* $u(\gamma) = \dim E_x^u$ *for* $x \in \gamma$.

REMARK. If the formula above is interpreted in terms of formal power series, it remains correct when f has infinitely many periodic points provided they are hyperbolic (see [3, 2.6]).

PROOF. We know that

$$\eta(f) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} L(f^m) t^m\right) \quad \text{and} \quad L(f^m) = \sum_{x \in \operatorname{Fix}(f^m)} I(x, f^m)$$

where $I(x,f^m)$ denotes the index of the fixed point (of f^m) x (see [1]). By a result of Smale [5, p. 767],

$$I(x,f^m) = (-1)^{u(\gamma)} \Delta_\gamma \quad \text{when } x \in \gamma.$$

If we define $L_m(\gamma)$ to be $\sum_{x \in \gamma} I(x,f^m)$ (setting $I(x,f^m) = 0$ if $f^m(x) \neq x$), then $L(f^m) = \sum_\gamma L_m(\gamma)$. So it follows that

$$\eta(f) = \prod_\gamma \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} L_m(\gamma) t^m\right). \tag{1}$$

Now it is easily seen that

$$L_m(\gamma) = \begin{cases} 0 & \text{if } m \not\equiv 0 \bmod p(\gamma), \\ p(\gamma)(-1)^{u(\gamma)} \Delta^{m/p(\gamma)} & \text{if } m \equiv 0 \bmod p(\gamma). \end{cases}$$

On the other hand, if $\Delta = \pm 1$ then

$$\begin{aligned} (1 - \Delta t^p)^{-1} &= 1 + \Delta t^p + \Delta^2 t^{2p} + \cdots + (\Delta t^p)^n + \cdots \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} (\Delta t^p)^n\right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{m} p \Delta^{m/p} t^m\right) \quad \text{where } m = np. \end{aligned}$$

Thus it follows that

$$(1 - \Delta_\gamma t^{p(\gamma)})^{(-1)^{u(\gamma)+1}} = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} L_m(\gamma) t^m\right),$$

and hence from (1) that

$$\eta(f) = \prod_\gamma (1 - \Delta_\gamma t^{p(\gamma)})^{(-1)^{u(\gamma)+1}}.$$

Q.E.D.

We will consider polynomials of the form

$$\prod_i (1 \pm t^{p_i}), \tag{2}$$

where the p_i are a finite set of positive integers. We need to factor these polynomials as much as possible while still preserving the form (2). Thus $(1 - t^{2n})$ factors into $(1 - t^n)(1 + t^n)$, while if m is odd $(1 - t^m)$ cannot be further factored. The polynomial $(1 + t^n)$ can never be further factored while retaining the form (2). A factorization of a polynomial of the form (2) which cannot be further factored preserving this form will be called a *special factorization.*

LEMMA 2. *If $p(t)$ is a polynomial of the form* (2), *then its special factorization is unique up to sign and order of the factors.*

PROOF. The proof is by induction on the degree of $p(t)$. The result is clearly true if the degree is 1. Suppose now the result holds for polynomials of degree $k - 1$ and $p(t)$ has degree k. Then if there are two special factorizations

$$\prod_i (1 \pm t^{p_i}) = p(t) = \prod_j (1 \pm t^{q_j}), \tag{3}$$

we let $r = \max(\{p_i\} \cup \{q_i\})$ and consider two cases:

(A) One of the two factorizations has a factor $(1 + t^r)$. In this case $p(t)$ has a primitive $2r$th root of unity and hence both factorizations must have a factor of $(1 + t^r)$ since this is the only possible factor which could contribute such a root. Thus if we divide both sides by this factor and apply the induction hypothesis we are done.

(B) There are no factors of $(1 + t^r)$, but one side of (3) has a factor $(1 - t^r)$. In this case we know that r is odd since otherwise $(1 - t^r) = (1 - r^{r/2}) \cdot (1 + t^{r/2})$ would not be fully factored. Also we know that $p(t)$ has a primitive rth root of unity and hence both sides of (3) have a factor $(1 - t^r)$ since no other allowable factor could contribute an rth root of unity. Dividing both sides by $(1 - t^r)$ and applying the induction hypothesis we again obtain the desired result. Q.E.D.

PROOF OF THEOREM. We consider first (a) and notice that by considering simple examples and Lemma 1 it follows that $\eta(f) = 1/(1 - t)^2$ if f has degree one. (Recall $\eta(f)$ depends only on the homotopy type of f.)

Now, if f has only finitely many periodic points, then Lemma 1 says

$$\frac{1}{(1 - t)^2} = \prod_i (1 \pm t^{p_i}) \Big/ \prod_j (1 \pm t^{q_j})$$

where the p_i are the periods of hyperbolic periodic points and q_j are the periods of sources and sinks. Thus at most one q_j is a power of 2.

By multiplying to clear fractions we obtain

$$\prod_j (1 \pm t^{q_j}) = (1 - t)^2 \prod (1 \pm t^{p_i}).$$

Now taking a special factorization of both sides (which is unique by Lemma 2) we arrive at a contradiction since a special factorization of the right-hand side of the above equation will contain the factor $(1 - t)$ at least twice and the special factorization of the left-hand side can contain the factor at most once since at most one of the q_i is a power of 2. (Recall that $(1 - t^s)$ cannot be further factored in a special factorization unless s is even.)

Thus we have contradicted the assumption that f has finitely many periodic points.

For (b) we note that one finds easily that if $f: S^2 \to S^2$ has degree -1 then $\eta(f) = 1/(1 - t^2)$, and if $\deg f = 0$ then $\eta(f) = 1/(1 - t)$. Also for any $g: D^2 \to D^2$, $\dot{\eta}(g) = 1/(1 - t)$.

Thus as in part (a) we obtain $\prod (1 \pm t^{q_j}) = (\eta)^{-1} \prod (1 \pm t^{p_i})$ by assuming f (or g) has finitely many periodic points. From this we derive a contradiction since $(\eta)^{-1}$ is either equal to $(1 - t)$ or $(1 - t)(1 + t)$, and since now no q_j is a power of 2, the factor $(1 - t)$ cannot occur in a special factorization of $\prod_j (1 \pm t^{q_j})$. Thus, in this case also, f (or g) has infinitely many periodic points.

The proof for (c) is essentially the same. We simply remark that if $f: S^1 \to S^1$ has degree -1 then $\eta(f) = (1 + t)/(1 - t)$ and if $f: S^1 \to S^1$ has degree 0 or $g: I \to I$ is any map, then

$$\eta(f) = \eta(g) = 1/(1 - t). \qquad \text{Q.E.D.}$$

ADDED IN PROOF. It has come to the author's attention that for maps of the interval A. N. Sharkovskiy [6] has proven a much stronger result than part (c) of the theorem above.

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